On inclusive and non-inclusive vertex irregular d-distance vertex labelings

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Abstract

In this paper, we generalise the notion of distance irregular labeling introduced by Slamin to vertex irregular d-distance vertex labeling, for any distance d up to the diameter. We also define the inclusive vertex irregular d-distance vertex labeling. We give the lower bound of the inclusive vertex irregular 1-distance vertex labeling for general graphs and a better lower bound on caterpillars. The inclusive labelings for paths $P_n, n \equiv 0 \mod 3$, stars S_n, double stars $S(m,n)$, cycles C_n and wheels W_n are provided. From the inclusive vertex irregular 1-distance vertex labeling on cycles, we derive the vertex irregular 1-distance vertex labeling on prisms.

Keyword: Vertex irregular d-distance vertex labeling, inclusive vertex irregular d-distance vertex labeling, distance irregularity strength, inclusive distance irregularity strength.

1 Introduction

Let $G = (V, E)$ be a simple, finite and undirected graph with vertex set V and edge set E. The *order* of the graph is $|V| = n$ and the *size* of the graph is $|E| = m$. Let u, v be two vertices of G. The *distance* $d(u, v)$ is the minimum of the lengths of the $u - v$ paths of G. For a connected graph G of diameter k, a *distance-d graph* G_d for $d = 1, \ldots, k$ is a graph with the same vertex set $V(G)$ and the edge set consists of the pairs of vertices that lie at distance d apart. A *labeling* is a mapping from the set of elements in a graph (vertices, edges, or both) to a set of numbers (usually positive integers). There are many types of labelings that have been studied (see [2] for the complete survey on labelings.).

Slamin [4] introduced the distance irregular labeling by combining the distance labeling by Mirka et al. [3] and the irregular labeling by Chartrand et al. [1]. In the paper, the motivation of this labeling was discussed. In this labeling, the *weight* of a vertex x, $wt(x)$, in G is defined as the sum of the labels of all the vertices adjacent to x, i.e. vertices at distance 1 from x. Formally,

$$wt(x) = \sum_{y \in N(x)} \lambda(y).$$

Definition 1.1. *[4] Let k be a positive integer. A distance irregular vertex labeling of the graph G with V vertices is an assignment $\lambda : V \to \{1, 2, \ldots, k\}$ so that the weights at each vertex are distinct.*

The *distance irregularity strength* of G, denoted by $dis(G)$, is the minimum value of the largest label k over all such irregular assignments.

In the paper, Slamin provided the distance irregularity strength for complete graphs K_n and paths P_n for all n, also cycles C_n and wheels W_n where $n \geq 5, n \in \{0, 1, 2, 5\} \mod 8$. In the same paper, Slamin also proposed an open problem, which is to generalise the distance irregular labeling of graphs where the weight sum of a vertex includes the label of the vertex itself.

In this paper, we generalise the distance irregular labeling to (inclusive) vertex irregular d-distance vertex labeling for all d up to the diameter. For the non-inclusive labelings on graphs, the label of the vertex is not included in its weight, while the inclusive one includes the vertex label in its weight.

Definition 1.2. *A* (non-inclusive) vertex irregular d-distance vertex labeling λ *is an irregular labeling of vertices of a graph G where the weight of a vertex $v \in V(G)$ is the sum of all labels of distance up to d from v, i.e.*

$$wt(v) = \sum_{\{u: 1 \leq d(u,v) \leq d\}} \lambda(u).$$

For simplicity, the (non-inclusive) vertex irregular d-distance-vertex labeling will be called vertex irregular d-distance-vertex labeling. The minimum value of the largest label used over all such irregular labelings is called the d-distance irregularity strength of G, denoted by $dis_d^1(G)$.

We define the weight of a vertex in a vertex irregular d-distance vertex labeling to be the sum of all labels up to distance d, because if we consider the weight of a vertex v to be the sum of all labels at distance d from v, then, constructing the distance-d graph of G turns the d-distance labeling problem into 1-distance labeling problem.

Definition 1.3. *An* inclusive vertex irregular d-distance vertex labeling λ' *is an irregular labeling of vertices in a graph G where the weight of a vertex $v \in V(G)$ is the sum of the label of v and all labels up to distance d from v.*

$$wt(v) = \lambda'(v) + \sum_{\{u:1 \leq d(u,v) \leq d\}} \lambda'(u).$$

The minimum value of the largest label used over all such irregular labelings is called the inclusive d-distance irregularity strength of G, denoted by $dis_d^0(G)$. The distance irregular labeling given in Definition 1.1 is now called *vertex irregular* 1*-distance vertex labeling* and its distance irregularity strength will be denoted as $dis_1^1(G)$.

Notice that when d equals the diameter of the graph G, then there is no inclusive vertex irregular d-distance vertex labeling of G. This is because the weight of each vertex is the sum of all labels of vertices in the graph, hence, it is impossible to have distinct weight for the vertices. Consequently, there is no inclusive vertex irregular 1-distance vertex labeling of complete graphs K_n for any n (diameter of K_n is 1).

Furthermore, graphs that admit the vertex irregular d-distance vertex labeling do not necessary admit the inclusive vertex irregular d-distance vertex labeling and vice versa. For example, complete graphs K_n admit the vertex irregular 1-distance vertex labeling but not the inclusive one. Moreover, there exists an inclusive vertex irregular 1-distance vertex labeling on stars S_n, but there is no vertex irregular 1-distance vertex labeling on stars.

Figure 1 shows an example of distance irregular vertex labeling and inclusive distance irregular vertex labeling of a cycle on 12 vertices. The 1-distance irregularity strength and the inclusive 1-distance irregularity strength of C_{12} are 7 and 5, respectively. Throughout this paper, in all examples, the number on a vertex is the label of the vertex and the number outside the vertex circle shows the weight of the corresponding vertex.

2 Inclusive vertex irregular 1-distance vertex labeling

In inclusive vertex irregular 1-distance vertex labeling, the label of a vertex is included in its weight.

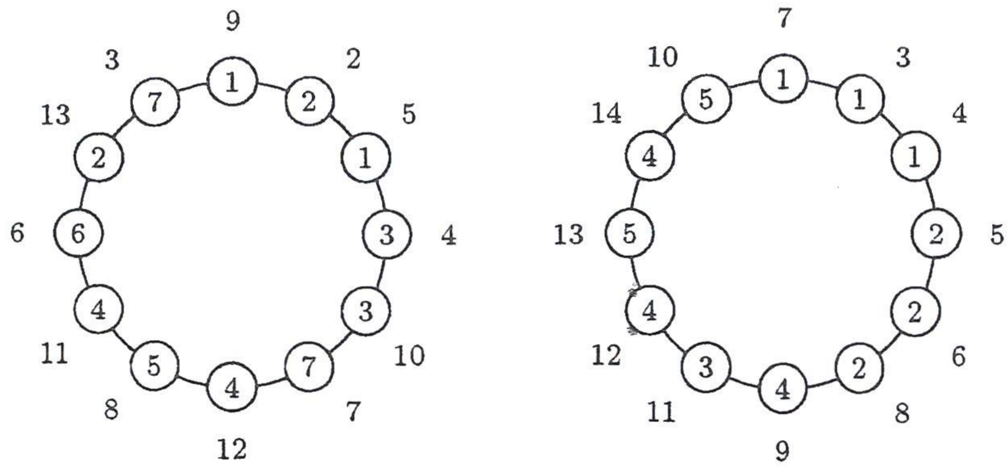

Figure 1: Vertex irregular and inclusive vertex irregular 1-distance vertex labeling on C_{12} with $dis_1^1(C_{12}) = 7$ and $dis_1^0(C_{12}) = 5$.

Lemma 2.1. *Let G be a connected graph on n vertices with minimum degree δ and maximum degree Δ, then* $dis_1^0(G) \geq \lceil \frac{n+\delta}{\Delta+1} \rceil$.

Proof. The smallest possible weight is $\delta + 1$ (when we label a vertex of degree δ and all its neighbour with label 1). Since there are n vertices, the largest weight is at least $n+\delta$. Hence the largest label is at least $\lceil \frac{n+\delta}{\Delta+1} \rceil$. □

This lower bound is shown to be sharp for the inclusive vertex irregular 1-distance vertex labeling on path P_n, $n \equiv 0 \mod 3$ as in Theorem 2.2.

Theorem 2.2. *Let P_n be a path on n vertices, $n \equiv 0 \mod 3$, then* $dis_1^0(P_n) = \lceil \frac{n+1}{3} \rceil = \frac{n}{3} + 1$.

Proof. Define λ' as follow.

$$\lambda'(v_i) = \begin{cases} \frac{n}{3} + 1 & \text{if } i = n-1, n \\ \lceil \frac{i}{3} \rceil + 1 & \text{if } i = \frac{2n}{3} + 2 + 3k, 0 \leq k < \frac{n}{9} - 1 \\ \lceil \frac{i}{3} \rceil & \text{otherwise} \end{cases}$$

With this labeling, the weight of the vertices are

$$wt(v_i) = \begin{cases} \frac{2n}{3} + 2 & \text{if } i = n \\ i + 1 & \text{if } i = 1, \ldots, \frac{2n}{3} \\ i + 2 & \text{if } i \geq \frac{2n}{3} + 1 \end{cases}$$

$wt(v_i) = i + 1$ for $i = 1, \ldots, \frac{2n}{3}$ □

The inclusive 1-distance irregularity strength of path P_n, where $n \equiv 1, 2 \mod 3$ remain open. Paths can be considered as tree of maximum degree 2. Other types of trees with larger maximum degree that will be considered in this paper are stars, double stars and caterpillars.

Theorem 2.3. *Let S_n be a star on $n+1$ vertices, i. e. a star with n leaves, then $dis_1^0(S_n) = n$.*

Proof. The vertex weight of each leaf is the sum of its label and the label of the central vertex. Therefore, in order to obtain distinct weights for the leaves, all the leaves must have distinct labels. Label the leaves from 1 up to n and label the central vertex with 1 (note that the central vertex can actually be labeled with any positive integer at most n). The weight of all vertices are $\{2, \ldots, n+1\} \cup \{\frac{n(n+1)}{2} + 1\}$. □

A double star $S(m, n), 0 \leq m \leq n$, is the graph consisting of the union of two stars S_n and S_m together with a line joining their centers.

Theorem 2.4. *Let $S(m, n), m \leq n$ be a double star, then*

$$dis_1^0(S(m,n)) = \begin{cases} n & \text{if } m < n \\ n+1 & \text{if } m = n. \end{cases}$$

Proof. Denote the central vertices with n and m leaves by v_1 and v_2, respectively. Let v_{1i} be the leaves of v_1 and v_{2j} be the leaves of v_2. Define the inclusive irregular 1-distance vertex labeling λ' as follow:

$$\begin{aligned} \lambda'(v_1) &= 1 \\ \lambda'(v_2) &= n \\ \lambda'(v_{1i}) &= i, & i = 1, \ldots, n \\ \lambda'(v_{2j}) &= j+1, & j = 1, \ldots, m. \end{aligned}$$

The weights set of the leaves of v_1 is $\{2, 3, \ldots, n+1\}$ and of v_2 is $\{n+2, n+2, \ldots, n+m+1\}$. Furthermore, $wt(v_1) = \frac{n(n+1)}{2} + n + 1$ and $wt(v_2) = \frac{(m+1)(m+2)}{2} + n > (n+m+1)$.

Case 1. $m < n$
The largest label is n and $wt(v_2) \leq \frac{n(n+1)}{2} + n$, thus all the weights are distinct and $dis_1^0(S_{m,n}) = n$ in this case.

Case 2. $m = n$
The largest label is $\lambda'(v_{2m}) = m + 1 = n + 1$ and

$$wt(v_2) = \frac{(n+1)(n+2)}{2} + n > \frac{n(n+1)}{2} + n + 1 = wt(v_1).$$

Since all the weights are different, hence the theorem. □

A *caterpillar* $S(n_1, n_2, \ldots, n_k)$ is generalisation of double star $S(m, n)$. It is a tree with the property that the removal of its endpoints leaves a path. Theorem 2.3 and 2.4 have shown that the general lower bound given in Lemma 2.1 is not sharp for this type of graphs. Lemma 2.5 gives a better lower bound for the inclusive vertex irregular 1-distance vertex labeling on caterpillar.

Lemma 2.5. *Let $S(n_1, n_2, \ldots, n_k)$ be a caterpillar, $n_1 \leq n_2 \leq \ldots \leq n_k$, then $dis_1^0(S(n_1, n_2, \ldots, n_k)) \geq \max\{n_k, \lceil \frac{n_1+n_2+\ldots+n_k+1}{2} \rceil\}$.*

Proof. From Theorem 2.3, all the leaves of a star must have different labels, thus, the largest label of a caterpillar is at least n_k. Furthermore, the caterpillar has $n_1 + n_2 + \ldots + n_k$ leaves and in this labeling, all of them must have distinct weights. Note that the weight of a leaf is the sum of the label of the leaf and its center vertex. The smallest weight of a leaf is 2 (when the label of the leaf and the center are both 1) and the largest weight of a leaf is at least $n_1 + n_2 + \ldots + n_k + 1$. Hence, the largest label is at least $\lceil \frac{n_1+n_2+\ldots+n_k+1}{2} \rceil$. □

Figure 2 shows the examples of inclusive vertex irregular 1-distance vertex labeling on caterpillars where the lower bound is sharp. The numbers outside the circle are the weight of the corresponding vertices. The inclusive

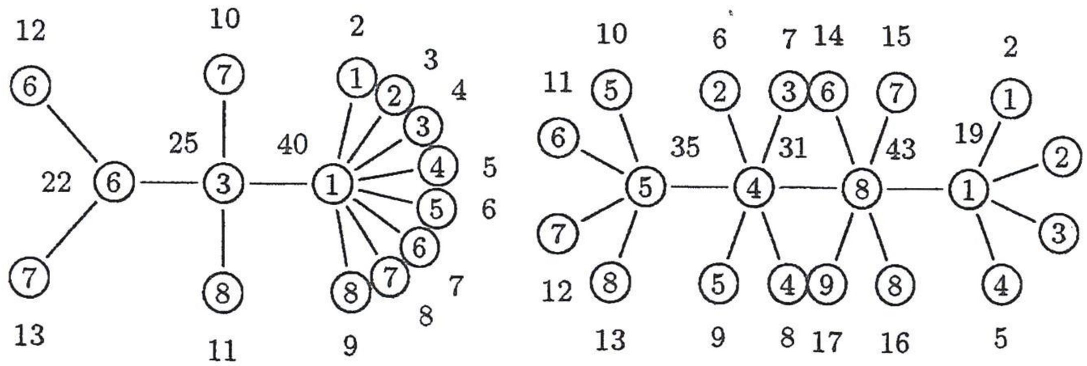

Figure 2: Inclusive distance-irregular vertex labeling on caterpillars with $dis_1^0(S(2,2,8)) = 8$ and $dis_1^0(S(4,4,4,4)) = 9$.

vertex irregular 1-distance vertex labeling becomes more complicated for tree in general. It remains an open problem to determine the inclusive distance irregularity strength of trees.

Theorem 2.6 provides the inclusive vertex irregular 1-distance vertex labeling for cycles C_n for all $n > 13$.

Figure 3 shows the labeling of C_{14} and C_{17} using Algorithm 1.

Table 1 summarises the weights of each vertex when applying Algorithm 1 to label C_n, where $n \equiv 0 \mod 3$ and $\frac{n}{3} \equiv 0 \mod 3$. In step 3, it shows that all weights are distinct, starting from 3 to $n+2$.

For $n \equiv 0 \mod 3$ and $\frac{n}{3} \equiv 2 \mod 3$, the table of weights is slightly different to Table 1. For the weight of v_{n-2}, it is n in step 1 and 2, but changes to $n+1$ in step 3. The rest are the same as Table 1.

Subcase 2. $\lceil \frac{n}{3} \rceil \equiv 1 \mod 3$.

In this case, modifying Algorithm 1 can be used to label C_n only when $n \equiv 0 \mod 3$, however, when $n \equiv 2 \mod 3$ we define a new algorithm.

- $n \equiv 0 \mod 3$.

  We can use the similar algorithm as Algorithm 1 to label cycle when $n \equiv 0 \mod 3$. The changes in the algorithm are

  * The initial label of v_{n-1} is $\frac{n}{3}$ (the initial weight of v_n is $\frac{2n}{3}+2$).
  * In step 3, the modification of vertices label starts from $i = \frac{2n}{3}+1+3k$.

  In this case, the weight of vertex v_{n-1} increases by 1 after Step 2.

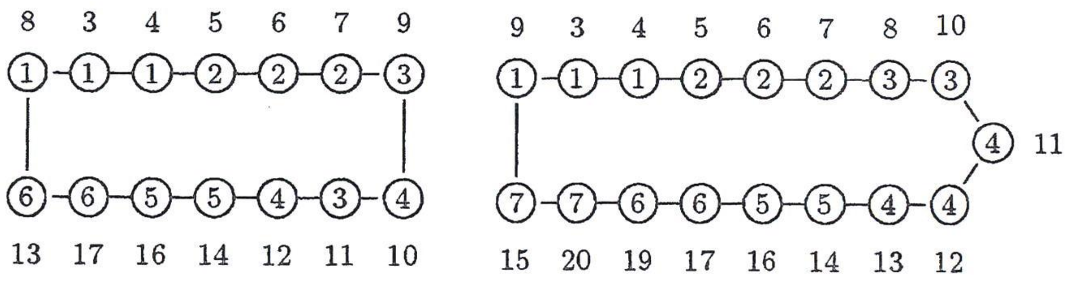

Figure 3: $dis_1^0(C_{14}) = 6$ and $dis_1^0(C_{19}) = 7$

Vertices	Weights Step 1	Weights Step 2	Weights Step 3
v_1	$\frac{n}{3}+3$	$\frac{n}{3}+3$	$\frac{n}{3}+3$
$v_2,\ldots,v_{\frac{n}{3}+1}$	$3,4,\ldots,\frac{n}{3}+2$	$3,4,\ldots,\frac{n}{3}+2$	$3,4,\ldots,\frac{n}{3}+2$
$v_{\frac{n}{3}+2}$	$\frac{n}{3}+3$	$\frac{n}{3}+4$	$\frac{n}{3}+4$
$v_{\frac{n}{3}+3},\ldots,v_{\frac{2n}{3}}$	$\frac{n}{3}+4,\ldots\frac{2n}{3}+1$	$\frac{n}{3}+5,\ldots\frac{2n}{3}+2$	$\frac{n}{3}+5,\ldots\frac{2n}{3}+2$
$v_{\frac{2n}{3}+1}$	$\frac{2n}{3}+2$	$\frac{2n}{3}+3$	$\frac{2n}{3}+4$
$v_{\frac{2n}{3}+2}$	$\frac{2n}{3}+3$	$\frac{2n}{3}+4$	$\frac{2n}{3}+5$
$v_{\frac{2n}{3}+3},\ldots,v_{n-3}$	$\frac{2n}{3}+4,\ldots,n-2$	$\frac{2n}{3}+5,\ldots,n-1$	$\frac{2n}{3}+6,\ldots,n$
v_{n-2}	n	$n+1$	$n+1$
v_{n-1},v_n	$n+2,\frac{2n}{3}+3$	$n+2,\frac{2n}{3}+3$	$n+2,\frac{2n}{3}+3$

Table 1: Table of weights of vertices based on the Algorithm 1 for $n\equiv 0 \mod 3$ and $\frac{n}{3}\equiv 0 \mod 3$.

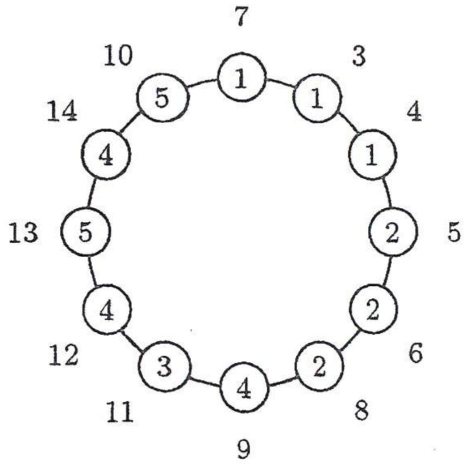

Figure 4: Inclusive vertex irregular 1-distance vertex labeling on C_{12} with $dis_1^0(C_{12}) = 5$.

Figure 4 shows an example of inclusive vertex irregular 1-distance vertex labeling on C_{12} with the largest label is 5.

- $n \equiv 2 \mod 3$

For this case, we define a new algorithm as follows.

Algorithm 2

1. Label all vertices v_i with $\lceil \frac{i+1}{3} \rceil, i = 1, \ldots, n-4$, label v_{n-3}, v_{n-2} with $\lceil \frac{n}{3} \rceil$ and label v_{n-1}, v_n with $\lceil \frac{n}{3} \rceil + 1$. (The weights of $v_2, v_3, \ldots, v_{n-5}$ are $4, 5, \ldots, n-3$ respectively. Other weights are $wt(v_1) = wt(v_{\lceil \frac{n}{3} \rceil + 1}) = \lceil \frac{n}{3} \rceil + 3$, $wt(v_n) = wt(v_{2\lceil \frac{n}{3} \rceil + 1}) = 2\lceil \frac{n}{3} \rceil + 3$, $wt(v_{n-1}) = n+3$, $wt(v_{n-2}) = n+2$, $wt(v_{n-3}) = n$, and $wt(v_{n-4}) = n - 1$.)
2. Add 1 to the label of vertices v_i of $i = \lceil \frac{n}{3} \rceil + 2 + 3k$, where k are non negative integers, $0 \le k < \frac{2n-15}{9}$. (This modification increases the vertex weight of every vertex $v_i, i = \lceil \frac{n}{3} \rceil + 1, \ldots, n-4$ by 1. With this new label, the weight of $\lceil \frac{n}{3} \rceil + 3$ is no longer repeated, but $wt(v_n) = wt(v_{2\lceil \frac{n}{3} \rceil}) = 2\lceil \frac{n}{3} \rceil + 3$.)
3. Add 1 to the label of vertices v_i of $i = 2\lceil \frac{n}{3} \rceil + 1 + 3k$, k are non negative integers, $0 \le k < \frac{n-12}{9}$. (This modification increases the vertex weight of every vertex $v_i, i = 2\lceil \frac{n}{3} \rceil, \ldots, n-4$ by 1.)

With this algorithm, we obtain the weight set is $\{4, 5, \ldots, n+3\}$. Figure 5 shows the labeling of C_{20} using Algorithm 2.

- **Case 2.** $n \equiv 1 \mod 3$
  We label the cycle C_n by modifying the label of C_{n-2}. Since $n \equiv 1 \mod 3$, $m = (n-2) \equiv 2 \mod 3$. Let v_i denote the vertices of cycle C_n and w_i denote the vertices of cycle $C_{m=n-2}$.
  Subcase 1: If $\lceil \frac{n}{3} \rceil \equiv 0, 2 \mod 3$.

$$\lambda'(v_i) = \begin{cases} \lambda'(w_i) & \text{if } i = 1, \ldots, n-4 \\ \lceil \frac{n}{3} \rceil & \text{if } i = n-3 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } i = n-2, n-1, n. \end{cases}$$

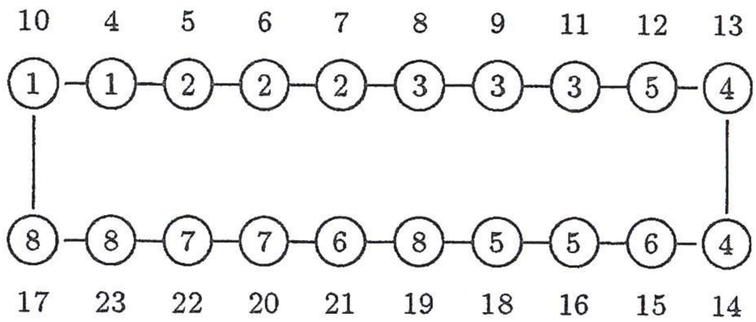

Figure 5: $dis_1^0(C_{20}) = 8$.

The weights of the extra two vertices are $n-1$ ($wt(v_{n-4})$) and $n+2$ ($wt(v_{n-1})$). Figure 6 shows the labeling of C_{19} obtained from the labeling C_{17} in Figure 3.

Theorem 2.6. *Let C_n be a cycle on n vertices, $n > 13$, then $dis_1^0(C_n) = \lceil \frac{n}{3} \rceil + 1$.*

Proof. Let $v_i, i = 1, \ldots, n$ be the vertices of C_n. Based on Lemma 2.1, $dis_1^0(C_n) \geq \lceil \frac{n}{3} \rceil + 1$. We are going to construct a labeling λ' that achieves that lower bound. We divide this proof into two main cases, when $n \equiv 0, 2 \mod 3$ and when $n \equiv 1 \mod 3$.

- **Case 1.** $n \equiv 0, 2 \mod 3$
  When $n \equiv 0, 2 \mod 3$, the way to label C_n depends on whether $\lceil \frac{n}{3} \rceil \equiv 0, 2 \mod 3$ (see Subcase 1) or $\lceil \frac{n}{3} \rceil \equiv 1 \mod 3$ (see Subcase 2).

Subcase 1: $\lceil \frac{n}{3} \rceil \equiv 0, 2 \mod 3$.
For $\lceil \frac{n}{3} \rceil \equiv 0, 2 \mod 3$, use the following algorithm to label the vertices.

Algorithm 1

1. Label all vertices v_i with $\lceil \frac{i}{3} \rceil, i = 1, \ldots, n-2$ and label v_{n-1}, v_n with $\lceil \frac{n}{3} \rceil + 1$. (The weights of $v_2, v_3, \ldots, v_{n-3}$ are $3, 4, \ldots, n-2$ respectively and the other weights are $wt(v_1) = wt(v_{\lceil \frac{n}{3} \rceil + 2}) = \lceil \frac{n}{3} \rceil + 3$, $wt(v_n) = wt(v_{2\lceil \frac{n}{3} \rceil + 2}) = 2\lceil \frac{n}{3} \rceil + 3$, $wt(v_{n-1}) = n+2$ and $wt(v_{n-2}) = n$.)
2. Add 1 to the label of vertices v_i of $i = \lceil \frac{n}{3} \rceil + 3k$, where k are positive integers, $1 \leq k < \frac{2n-3}{9}$. (This modification increases the vertex weight of every vertex $v_i, i = \lceil \frac{n}{3} \rceil + 2, \ldots, n-2$ by 1. With this new label, the weight of $\lceil \frac{n}{3} \rceil + 3$ is no longer repeated, but $wt(v_n) = wt(v_{2\lceil \frac{n}{3} \rceil + 1}) = 2\lceil \frac{n}{3} \rceil + 3$.)
3. Add 1 to the label of vertices v_i of $i = 2\lceil \frac{n}{3} \rceil + 2 + 3k$, k are non negative integers, $0 \leq k < \frac{n}{9} - 1$. (This modification increases the vertex weight of every vertex $v_i, i = 2\lceil \frac{n}{3} \rceil + 1, \ldots, n-3$ by 1.)

Corollary 2.7. *Let W_n be a wheel on $n+1$ vertices, then $dis_1^0(W_n) = dis_1^0(C_n)$.*

Proof. Suppose we have an inclusive vertex irregular 1-distance labeling on a wheel W_n with $dis_1^0(W_n) = s$, then taking away the central vertex will give an inclusive vertex irregular 1-distance labeling on C_n with $dis_1^0(C_n) \leq s$, so, $dis_1^0(W_n) \geq dis_1^0(C_n)$. On the other hand, suppose we have a distance irregular labeling on a cycle C_n with $dis_1^0(C_n) = t$. Add a central vertex to the cycle and connect the central vertex to every other vertex to form a wheel. Label the central vertex with 1, then the weight of each non-central vertex increases by 1 and the weight of the central vertex is the sum of all labels on the cycle, but the largest label remain the same. Thus, $dis_1^0(W_n) = dis_1^0(C_n)$. □

3 Non-inclusive vertex irregular 1-distance vertex labeling

In this section, we give the non-inclusive vertex irregular 1-distance vertex labeling on prisms $C_n \times P_2$, for $n \equiv 0, 1, 3, 4, 5 \mod 6$. This labeling is obtained by utilising the inclusive vertex irregular 1-distance vertex labeling on cycles in Section 2.

Lemma 3.1. *Let $n \equiv 2 \mod 4$, C_n be a cycle on n vertices and $C_{\frac{n}{2}} \times P_2$ be a prism on n vertices. Let λ' be the inclusive vertex irregular 1-distance vertex labeling on C_n. If the $wt_{\lambda'}(C_n) = \{3, 4, \ldots, n+2\}$, then λ' induces*

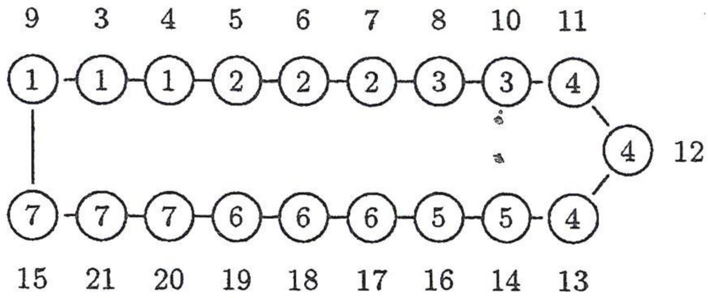

Figure 6: Labeling of C_{19} obtained from the labeling of C_{17} with $dis_1^0(C_{19}) = 7$

Subcase 2: If $\lceil \frac{n}{3} \rceil \equiv 1 \mod 3$.

$$\lambda'(v_i) = \begin{cases} 1 & \text{if } i = 1 \\ \lambda'(w_{i-1}) & \text{if } i = 2, \ldots, n-1 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } i = n. \end{cases}$$

From previous case, we know that the weight set of C_m is $\{4, 5, \ldots, m + 3 = n + 1\}$. The weights for the extra 2 vertices are 3 ($wt(v_2)$) and $n + 2 = m + 4$ ($wt(v_{n-1})$). Therefore, the weight set of C_n is $\{3, 4, 5, \ldots, m + 3, m + 4 = n + 2\}$.

Figure 7 shows the labeling of C_{22} obtained from the labeling C_{20} in Figure 5.

□

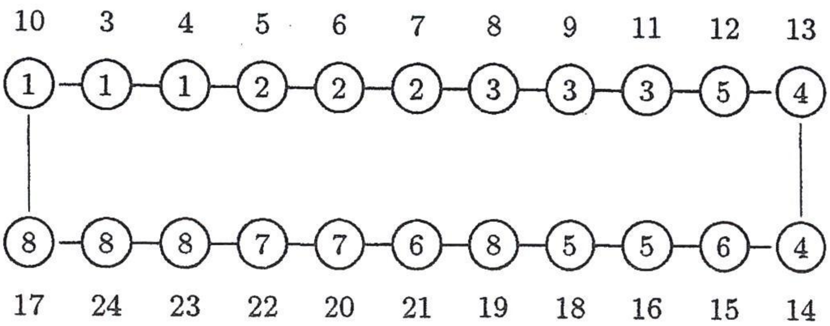

Figure 7: Inclusive vertex irregular 1-distance vertex labeling of C_{22} obtained from the labeling of C_{20} with $dis_1^0(C_{22}) = 8$.

a vertex irregular 1-distance vertex-labeling λ *on a prism* $C_{\frac{n}{2}} \times P_2$ *with* $dis_1^1(C_{\frac{n}{2}} \times P_2) = \frac{n+2}{3}$.

Proof. Using the label of λ' on cycle C_n and place those label on prism $C_{\frac{n}{2}} \times P_2$ as shown in Figure 8. It is easy to see that the weights sets of the two graphs are the same. □

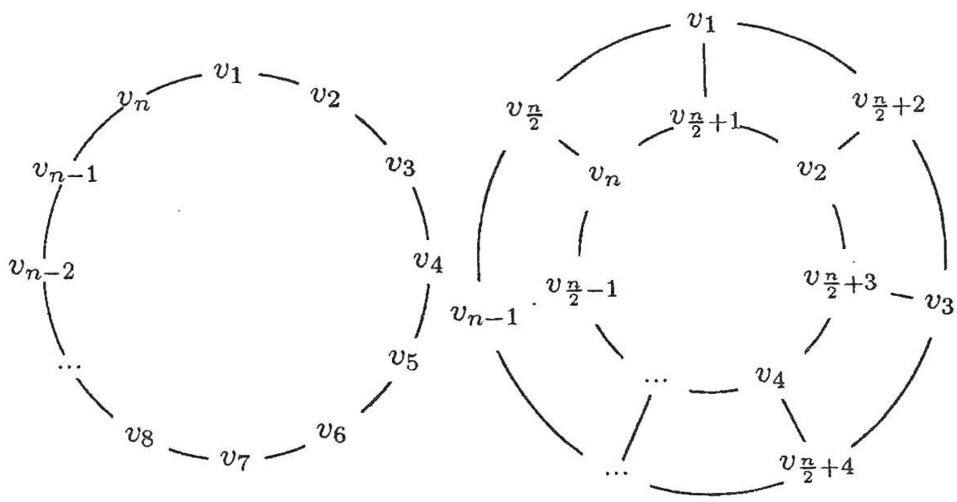

Figure 8: Labeling λ' on a cycle C_n induces labeling λ on a prism $C_{\frac{n}{2}} \times P_2$

Theorem 3.2. *Let* $C_n \times P_2$ *be a prism on* $2n$ *vertices,* $n \equiv 0, 1, 3, 4, 5 \mod 6$, *then* $dis_1^1(C_n \times P_2) = \lceil \frac{2n+2}{3} \rceil = \frac{2n}{3} + 1$.

Proof. When n is odd, we use the inclusive vertex irregular 1-distance vertex labeling of cycles on $2n$ vertices ($\lambda'(C_{2n})$) obtained in Theorem 2.6 and apply Lemma 3.1.

When n is even, we can decompose $C_n \times P_2$ to two independent copies of C_n in term of vertex weights (see Figure 9 for the illustration). The weight of a vertex either only depends on labels from first cycle or only from the second cycle.

Thus, we can use the label of C_n from Theorem 2.6. Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be the vertices of the first and second copy of the cycles, C_n^1 and C_n^2, respectively. Let λ' be an inclusive vertex irregular 1-distance vertex labeling of cycles on C_n. For v_i, we use the label obtained in Theorem 2.6 and $\lambda'(w_i) = \lambda'(v_i) + \lceil \frac{n}{3} \rceil$. The weights set of the C_n^1 is $\{3, \ldots, n+2\}$. The addition of $\frac{n}{3}$ at each vertex label increases each vertex weight of C_n^2 by n. So the smallest weight in C_n^2 is $n+3$ and the weight set of $C_n^2 = \{n+3, \ldots, 2n+2\}$. The largest label is $\lambda'(w_n) = \lambda'(v_n) + \frac{n}{3} = \frac{2n}{3} + 1$. □