On the Ramsey Numbers $r(S_n, K_6 - 3K_2)$

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Abstract

For every connected graph F with n vertices and every graph G with chromatic surplus $s(G) \le n$ the Ramsey number $r(F,G)$ satisfies $r(F,G) \ge (n-1)(\chi(G)-1)+s(G)$, where $\chi(G)$ denotes the chromatic number of G. If this lower bound is attained, then F is called G-good. For all connected graphs G with at most six vertices and $\chi(G) \ge 4$, every tree T_n of order $n \ge 5$ is G-good. In case of $\chi(G) = 3$ and $G \ne K_6 - 3K_2$ every non-star tree T_n is G-good except for some small n, whereas $r(S_n, G)$ for the star $S_n = K_{1,n-1}$ in a few cases differs by at most 2 from the lower bound. In this note we prove that the values of $r(S_n, K_6 - 3K_2)$ are considerably larger for sufficiently large n. Furthermore, exact values of $r(S_n, K_6 - 3K_2)$ are obtained for small n.

KEYWORDS: Ramsey number, Ramsey goodness, star, small graph

1 Introduction

Let G be a graph with chromatic number $\chi(G)$. The chromatic surplus $s(G)$ is defined to be the smallest number of vertices in a color class under any $\chi(G)$-coloring of the vertices of G. It is well-known (cf. [3]) that for any connected graph F with n vertices and any graph G with $s(G) \le n$ the Ramsey number $r(F,G)$ satisfies

$$r(F,G) \ge (n-1)(\chi(G)-1) + s(G). \tag{1}$$

When equality occurs in (1), F is said to be G-good. The concept of G-goodness generalizes a classical result of Chvátal [2] who proved that

$r(T_n, K_m) = (n-1)(m-1)+1$ for any tree T_n with n vertices. Results concerning the G-goodness of trees have been obtained for various classes of graphs G and also for small graphs G. The Ramsey number $r(T_n, G)$ for connected graphs G with at most 5 vertices was studied in [3], $r(T_n, G)$ for connected graphs with six vertices was investigated in [5] and [6]. These results show that every tree T_n with $n \geq 5$ is G-good if G is a connected graph with at most six vertices and $\chi(G) \geq 4$. In case of $\chi(G) = 3$ and $G \neq K_6 - 3K_2$ every non-star tree T_n is G-good except for some small n, whereas $r(S_n, G)$ for the star $S_n = K_{1,n-1}$ in a few cases differs by at most 2 from the lower bound (1). For graphs G with $\chi(G) = 2$ and at most six vertices the values of $r(T_n, G)$ are not completely determined, but it is known that for some G, especially for non-star complete bipartite graphs, they differ considerably from the lower bound (1) (see [1, 7, 8]). Here we will prove that also the values of $r(S_n, K_6 - 3K_2)$ are much larger. We present a lower bound for $r(S_n, K_6-3K_2)$ depending on $r(S_n, C_4)$ which implies that $r(S_n, K_6-3K_2) \geq 2n + \lfloor\sqrt{n-1}\rfloor - 1$ if $n = q^2+1$ or $n = q^2+2$ where q is any prime power and that $r(S_n, K_6-3K_2) > 2n-2+\lfloor\sqrt{n-1} - 6(n-1)^{11/40}\rfloor$ for all sufficiently large n. For $n \leq 10$, our lower bound matches the exact value of $r(S_n, K_6 - 3K_2)$ or differs from it by at most 1.

Some specialized notation will be used. A coloring of a graph always means a 2-coloring of its edges with colors red and green. An (F_1, F_2)-coloring is a coloring containing neither a red copy of F_1 nor a green copy of F_2. We use V to denote the vertex set of K_n and define $d_r(v)$ to be the number of red edges incident to $v \in V$ in a coloring of K_n. Moreover, $\Delta_r = \max_{v\in V} d_r(v)$. The set of vertices joined red to v is denoted by $N_r(v)$. Similarly we define $d_g(v)$, Δ_g and $N_g(v)$. For $U \subseteq V(K_n)$, the subgraph induced by U is denoted by $[U]$. Furthermore, $[U]_r$ and $[U]_g$ denote the red and the green subgraph induced by U. We write $G' \subseteq G$ if G' is a subgraph of G. For disjoint subsets $U_1, U_2 \subseteq V(K_n)$, $q_r(U_1, U_2)$ denotes the number of red edges between U_1 and U_2, and $q_g(U_1, U_2)$ is defined similarly.

2 Results

The following theorem establishes a general lower bound for $r(S_n, K_6-3K_2)$ depending on $r(S_n, C_4)$.

Theorem 1 *Let $n \geq 2$. Then*

$$r(S_n, K_6 - 3K_2) \geq r(S_n, C_4) + \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $m = r(S_n, C_4) - 1$. Take an (S_n, C_4)-coloring of K_m. For n odd, add a red K_{n-1}, and, for n even, a K_n with $n/2$ independent green edges and all other edges colored red. Join the vertices of the K_m and the vertices of the K_{n-1} or K_n, respectively, by green edges. Obviously, no red S_n occurs. Now consider any subgraph H of order six. If at least four vertices of H belong to the K_m, then a green $K_6 - 3K_2 \subseteq H$ is impossible since deleting any two vertices of a $K_6 - 3K_2$ leaves a graph of order four containing a C_4. Otherwise, at least three vertices of H belong to the K_{n-1} or K_n. Then adjacent red edges occur in H and again a green $K_6 - 3K_2 \subseteq H$ is impossible. Thus, the lower bound is established. ■

Exact results on the values of $r(S_n, C_4)$ are known only in special cases. Parsons [7] proved that $r(S_n, C_4) = n + \lfloor\sqrt{n-1}\rfloor$ if $n = q^2+1$ or $n = q^2+2$ where q is any prime power. Burr, Erdös, Faudree, Rousseau and Schelp [1] showed that $r(S_n, C_4) > n - 1 + \lfloor\sqrt{n-1} - 6(n-1)^{11/40}\rfloor$ for all sufficiently large n. From these results and Theorem 1 we obtain the following lower bounds on $r(S_n, K_6 - 3K_2)$.

Corollary 1

(i) Let $n = q^2 + 2$ where q is any power of 2 or $n = q^2 + 1$ where q is any odd prime power. Then

$$r(S_n, K_6 - 3K_2) \geq 2n + \lfloor\sqrt{n-1}\rfloor .$$

(ii) Let $n = q^2 + 1$ where q is any power of 2 or $n = q^2 + 2$ where q is any odd prime power. Then

$$r(S_n, K_6 - 3K_2) \geq 2n - 1 + \lfloor\sqrt{n-1}\rfloor .$$

(iii) If n is sufficiently large, then

$$r(S_n, K_6 - 3K_2) > 2n - 2 + \left\lfloor\sqrt{n-1} - 6(n-1)^{11/40}\right\rfloor .$$

Using recent results on $r(S_n, C_4)$ of Wu Yali, Sun Yongqi, Zhang Rui and Radziszowski [8], further lower bounds on $r(S_n, K_6 - 3K_2)$ can be obtained from Theorem 1. The next theorem shows that the lower bound for $r(S_n, K_6 - 3K_2)$ given in Theorem 1 matches the exact value of the Ramsey number if $n \leq 6$ or $n = 8$ and differs by at most 1 from it if $n = 7$ or $9 \leq n \leq 10$. The value of $r(S_5, K_6 - 3K_2)$ has already been obtained by Gu Hua, Song Hongxue and Liu Xiangyang [4] using a different method.

Theorem 2

n	2	3	4	5	6	7	8	9	10
$r(S_n, K_6 - 3K_2)$	6	6	10	11	14	15/16	19	20/21	23/24

The proof of Theorem 2 is based on the following lemmas.

Lemma 1 *The red subgraph of an $(S_4, K_6 - 3K_2)$-coloring of K_9 is isomorphic to $K_1 \cup 2C_4$ or to $C_4 \cup C_5$.*

Proof. $S_4 \nsubseteq [V]_r$ implies $\Delta_r \leq 2$. Thus, every component of $[V]_r$ has to be a path or a cycle. If the union of all paths in $[V]_r$ contains at least three vertices, then it is a subgraph of a cycle. Moreover, $2K_1 \subseteq K_2$. Hence, $[V]_r \subseteq H$ where $H \in \{C_9,\ C_3 \cup C_6,\ C_4 \cup C_5,\ 3C_3,\ K_2 \cup C_7,\ K_2 \cup C_3 \cup C_4,\ K_1 \cup C_8,\ K_1 \cup C_3 \cup C_5,\ K_1 \cup 2C_4\}$. Except for $[V]_r = H = K_1 \cup 2C_4$ or $[V]_r = H = C_4 \cup C_5$ we find a forbidden $K_6 - 3K_2$ in $[V]_g$. ∎

Lemma 2 $r(S_4, K_6 - 3K_2) \leq 10$.

Proof. Assume that an $(S_4, K_6 - 3K_2)$-coloring of K_{10} exists. Delete one vertex $v \in V$. By Lemma 1, the red subgraph of $[V \setminus \{v\}]$ has to be isomorphic to $K_1 \cup 2C_4$ or to $C_4 \cup C_5$. Moreover, $\Delta_r \leq 2$ forces only green edges from v to the vertices of $V \setminus \{v\}$ belonging to a red cycle. Thus, in both cases we find a green $K_6 - 3K_2$, a contradiction. ∎

Lemma 3 $r(S_6, K_6 - 3K_2) \leq 14$.

Proof. Assume that we have an $(S_6, K_6 - 3K_2)$-coloring of K_{14}. This implies $\Delta_r \leq 4$ and $W_4 = K_5 - 2K_2 \subseteq [V]_g$ because $r(S_6, W_4) = 13$ (see [3]). We distinguish three cases.

Case 1. $K_5 \subseteq [V]_g$. For any $K_5 \subseteq [V]_g$ with vertex set U and any two vertices $w, w' \in V \setminus U$ joined green with $q_r(w, U) = q_r(w', U) = 2$ the following property $pr(w, w', U)$ must be fulfilled: $|N_r(w) \cap N_r(w') \cap U| \in \{0, 2\}$. Otherwise w and w' would have exactly one common red neighbor $u \in U$ and this would yield $K_6 - 3K_2 \subseteq [(U \setminus \{u\}) \cup \{w, w'\}]_g$, a contradiction. We distinguish two subcases.

Case 1.1. $2K_5 \subseteq [V]_g$. Let U_1 and U_2 be the vertex sets of two vertex-disjoint green copies of K_5 and let $W = V \setminus (U_1 \cup U_2)$. Then $K_6 - 3K_2 \not\subseteq [V]_g$ forces $q_r(w, U_1) \geq 2$ and $q_r(w, U_2) \geq 2$ for every $w \in W$. Using $\Delta_r \leq 4$, we obtain that $q_r(w, U_1) = q_r(w, U_2) = 2$ for every $w \in W$, $q_r(W, U_1) = q_r(W, U_2) = 8$ and $[W]_g = K_4$. Moreover, $K_6 - 3K_2 \not\subseteq [V]_g$ forces $q_r(u, U_1) \geq 2$ for every $u \in U_2$ and $q_r(u, U_2) \geq 2$ for every $u \in U_1$. Thus, $\Delta_r \leq 4$ implies $q_r(u, W) \leq 2$ for every $u \in U_1 \cup U_2$. If there are vertices $u_1 \in U_1$ and $u_2 \in U_2$ such that $q_r(u_1, W) = q_r(u_2, W) = 0$, then $K_6 - 3K_2 \subseteq [W \cup \{u_1, u_2\}]_g$, a contradiction. Thus we may assume that $q_r(u, W) \geq 1$ for every $u \in U_1$. Since $q_r(u, W) \leq 2$ for every $u \in U_1$ and $q_r(W, U_1) = 8$ there must be two vertices u_1 and u_2 in U_1 with $q_r(u_1, W) = q_r(u_2, W) = 1$, and $q_r(u, W) = 2$ for every $u \in U_1 \setminus \{u_1, u_2\}$. Hence, the bipartite graph $[W \cup U_1]_r$ is isomorphic to $C_6 \cup P_3$, to $C_4 \cup P_5$ or to P_9. In all three cases we find two vertices $w_1, w_2 \in W$ with exactly one common red neighbor $u \in U_1$, contradicting $pr(w_1, w_2, U_1)$.

Case 1.2. $K_5 \subseteq [V]_g$ and $2K_5 \not\subseteq [V]_g$. Let $U = \{u_1, \ldots, u_5\}$ be the vertex set of a green K_5 and let $W = V \setminus U$. Since $K_6 - 3K_2 \not\subseteq [V]_g$, $q_r(w, U) \geq 2$ for every $w \in W$. Thus, $\Delta_r \leq 4$ forces only vertices of degree less or equal 2 in $[W]_r$. As $K_5 \not\subseteq [W]_g$, we obtain $[W]_r = C_4 \cup C_5$ by Lemma 1. Moreover, $q_r(w, U) = 2$ for every $w \in W$. Let $W_1 = \{w_1, w_2, w_3, w_4\}$ and $W_2 = \{w_5, w_6, w_7, w_8, w_9\}$ be the vertex sets of the red C_4 and the red C_5 in $[W]$, where $w_i w_{i+1}$ for $i = 1, 2, 3, 5, 6, 7, 8$, $w_4 w_1$ and $w_9 w_5$ are red. We may assume that $w_1 u_1$ and $w_1 u_2$ are red and use that $pr(w, w', U)$ holds for any two vertices $w, w' \in W_2$ joined green.

First let $|N_r(w_1) \cap N_r(w) \cap U| = 0$ for every $w \in W_2$. Thus, $N_r(w) \cap U \subseteq \{u_3, u_4, u_5\}$ for every $w \in W_2$. We may assume that $w_5 u_3$ and $w_5 u_4$ are red. From $pr(w_5, w, U)$ for $w \in \{w_7, w_8\}$ we derive $N_r(w) \cap U = \{u_3, u_4\}$ for $w \in \{w_7, w_8\}$. Now apply $pr(w_6, w_8, U)$ and $pr(w_7, w_9, U)$. Hence, also $N_r(w) \cap U = \{u_3, u_4\}$ for $w \in \{w_6, w_9\}$. It follows that $d_r(u_3) \geq 5$, contradicting $\Delta_r \leq 4$.

The remaining case is that $|N_r(w_1) \cap N_r(w) \cap U| = 2$ for some $w \in W_2$, say $w = w_5$. Then $\{w_1, w_5, u_3, u_4, u_5\}$ induces a green K_5. Consequently, $K_6 - 3K_2 \not\subseteq [V]_g$ implies $q_r(w, \{u_3, u_4, u_5\}) = 2$ for every $w \in \{w_3, w_7, w_8\}$ and $q_r(w, \{u_3, u_4, u_5\}) \geq 1$ for $w \in \{w_6, w_9\}$. Because of $pr(w_1, w, U)$ for $w \in \{w_6, w_9\}$, we obtain $q_r(w, \{u_3, u_4, u_5\}) = 2$ also for $w \in \{w_6, w_9\}$. Moreover, we may assume that $w_3 u_3$ and $w_3 u_4$ are red. Note that $pr(w_3, w, U)$ holds for every $w \in \{w_6, w_7, w_8, w_9\}$. Thus, $N_r(w) \cap U = \{u_3, u_4\}$ for ever $w \in \{w_6, w_7, w_8, w_9\}$. This implies $d_r(u_3) \geq 5$ contradicting $\Delta_r \leq 4$.

Case 2. $K_5 - e \subseteq [V]_g$ and $K_5 \not\subseteq [V]_g$. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be th vertex set of a $K_5 - e \subseteq [V]_g$. We may assume that u_1u_5 is red. If a verte $w \in W = V \setminus U$ exists such that $q_r(w, U) \leq 1$, then we either find a gree $K_6 - 3K_2$ or a green K_5, both a contradiction. Thus, $q_r(w, U) \geq 2$ for ever $w \in W$. Note that $\Delta_r \leq 4$ and $K_5 \not\subseteq [V]_g$. Hence, $[W]_r = C_4 \cup C_5$ b Lemma 1. Again let $W_1 = \{w_1, w_2, w_3, w_4\}$ and $W_2 = \{w_5, w_6, w_7, w_8, w_9$ be the vertex sets of the red C_4 and the red C_5 in $[W]$, where w_iw_{i+} for $i = 1, 2, 3, 5, 6, 7, 8$, w_4w_1 and w_9w_5 are red. From $\Delta_r \leq 4$ we obtai that u_1 must have a green neighbor in W_2, say w_5. Now consider the tw green copies of $K_5 - e$ induced by $W_3 = \{w_1, w_3, w_5, w_6, w_8\}$ and $W_4 =$ $\{w_2, w_4, w_5, w_7, w_9\}$. Mind that $W_3 \cap W_4 = \{w_5\}$. If $q_r(u_1, W_3) \leq 1$ o $q_r(u_1, W_4) \leq 1$, then a green $K_6 - 3K_2$ or a green K_5 would occur i $[W_3 \cup \{u_1\}]$ or $[W_4 \cup \{u_1\}]$. Otherwise $d_r(u_1) \geq 5$, contradicting $\Delta_r \leq 4$.

Case 3. $K_5 - 2K_2 \subseteq [V]_g$ and $K_5 - e \not\subseteq [V]_g$. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ b the vertex set of a $K_5 - 2K_2 \subseteq [V]_g$. We may assume that u_1u_5 and u_2u_4 ar red. If a vertex $w \in W = V \setminus U$ exists such that $q_r(w, U) \leq 1$ we either fin a green $K_6 - 3K_2$ or a green $K_5 - e$, a contradiction. Thus, $q_r(w, U) \geq 2$ fo every $w \in W$. Note that $\Delta_r \leq 4$. Hence, $[W]_r = K_1 \cup 2C_4$ or $[W]_r = C_4 \cup C$ by Lemma 1. But then $K_5 - e \subseteq [W]_g \subseteq [V]_g$, a contradiction. ∎

Lemma 4 *Let $n \geq 2$. Then*

$$r(S_{n+2}, K_6 - 3K_2) \leq r(S_n, K_6 - 3K_2) + 5.$$

Proof. Let $m = r(S_n, K_6 - 3K_2) + 5$. By (1), $r(S_n, K_6 - 3K_2) \geq 2n$, and this implies $m \geq 2n + 5$. Assume that an $(S_{n+2}, K_6 - 3K_2)$-coloring of K_m exists. Since $r(S_n, W_4) = 2n + 1$ if n is even and $r(S_n, W_4) = 2n - 1$ if n is odd (see [3]) we obtain $r(S_{n+2}, W_4) \leq 2n + 5 \leq m$. Thus, $S_{n+2} \not\subseteq [V]_r$ forces $W_4 \subseteq [V]_g$. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of a green $W_4 = K_5 - 2K_2$ and $W = V \setminus U$. Note that $|W| = r(S_n, K_6 - 3K_2)$. Hence, $S_n \subseteq [W]_r$ and a vertex $w^* \in W$ exists with degree at least $n - 1$ in $[W]_r$. From $S_{n+2} \not\subseteq [V]_r$ it follows that $q_r(w^*, U) \leq 1$, i.e. $q_g(w^*, U) \geq 4$.

If $[U]_g = K_5$, then $K_6 - 3K_2 \subseteq [\{w^*\} \cup U]_g$, a contradiction, and we may assume that $K_5 \not\subseteq [V]_g$. Now let $[U]_g = K_5 - e$ assuming that the edge u_1u_5 is red. If w^* is joined green to u_1 and u_5, then a green $K_6 - 3K_2$ is contained in $[\{w^*\} \cup U]$. Otherwise w^* is joined red to u_1 or to u_5, say to u_1, but this implies that $[\{w^*\} \cup \{u_2, u_3, u_4, u_5\}]$ is a green K_5. Again we have obtained a contradiction and we may assume that $K_5 - e \not\subseteq [V]_g$. It remains that $[U]_g = K_5 - 2K_2$. Here we may assume that the edges u_1u_2 and u_4u_5 are red. If w^* is joined red to u_3, then a green $K_6 - 3K_2$ is contained in $[\{w^*\} \cup U]$. Otherwise w^* is joined green to u_3 and to at least three vertices in $\{u_1, u_2, u_4, u_5\}$, say to u_1, u_2 and u_4. But this gives a forbidden green $K_5 - e$ in $[\{w^*\} \cup \{u_1, u_2, u_3, u_4\}]$, and we are done. ■

Now we will use the results obtained in Theorem 1 and Lemmas 2, 3 and 4 to prove Theorem 2.

Proof of Theorem 2. At first we will show that the given values are lower bounds for $r(S_n, K_6 - 3K_2)$. The exact results of $r(S_n, C_4)$ for $n \leq 10$ can be found in [1], namely

n	2	3	4	5	6	7	8	9	10
$r(S_n, C_4)$	4	4	6	7	8	9	11	12	13

Applying Theorem 1, we obtain the desired lower bounds. It remains to establish the given values as upper bounds for $r(S_n, K_6 - 3K_2)$. Obviously, $r(S_n, K_6 - 3K_2) \leq 6$ for $2 \leq n \leq 3$. The other cases are settled by Lemmas 2, 3 and 4. ■

From Lemma 4 and the exact results for $5 \leq n \leq 6$ in Theorem 2 we obtain a general upper bound for $r(S_n, K_6 - 3K_2)$.

Theorem 3 *Let $n \geq 5$. Then*

$$r(S_n, K_6 - 3K_2) \leq \left\lfloor \frac{5n-2}{2} \right\rfloor .$$

Proof. For $5 \leq n \leq 6$ the upper bound matches the exact values in Theorem 2. For $n \geq 7$, induction on n using Lemma 4, separately for n even and n odd, yields the desired upper bound. ■.

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