On the Stronger Reconstruction of Nearly Acyclic Graphs

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Abstract

A graph is called *set-reconstructible* if it is determined uniquely (up to isomorphism) by the set of its vertex-deleted subgraphs. The maximal subgraph of a graph H that is a tree rooted at a vertex u of H is the *limb* at u. It is shown that two families $\mathscr{F}_1$ and $\mathscr{F}_2$ of nearly acyclic graphs are set-reconstructible. The family $\mathscr{F}_1$ consists of all connected cyclic graphs G with no end vertex such that there is a vertex lying on all the cycles in G and there is a cycle passing through at least one vertex of every cycle in G. The family $\mathscr{F}_2$ consists of all connected cyclic graphs H with end vertices such that there are exactly two vertices lying on all the cycles in H and there is a cycle with no limbs at its vertices.

Key words: Isomorphism, Harary's Conjecture, Set-reconstruction.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. We use the terminology in Harary [6]. The *degree* of a vertex v of a graph G is denoted by $deg\ v$ (or $deg_G v$). A vertex v with $deg\ v = m$ is referred to as an m *-vertex.* A 1-vertex is an *end vertex* and the unique neighbour of a 1-vertex is its *base.* The maximal subgraph that is a tree rooted at a vertex u of a graph H is the *limb* at u. A vertex-deleted unlabeled subgraph

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$G-v$ of a graph G is a *card* of G. A graph H is a *set-reconstruction* of G if H has the same set of cards as G. A graph is *set-reconstructible* if it is isomorphic to all its set-reconstructions. Equivalently, a graph G is set-reconstructible if it is determined uniquely (up to isomorphism) from the set $\mathscr{S}(G)$ of its (non-isomorphic) cards. A family $\mathscr{G}$ of graphs is *set-recognizable* if, for each $G \in \mathscr{G}$, every set-reconstruction of G is also in $\mathscr{G}$, and *weakly reconstructible* if, for each graph $G \in \mathscr{G}$, all set-reconstructions of G that are in $\mathscr{G}$ are isomorphic to G. A family $\mathscr{G}$ of graphs is *set-reconstructible* if, for any graph $G \in \mathscr{G}$, G is set-reconstructible (that is, if $\mathscr{G}$ is both set-recognizable and weakly set-reconstructible). If a property (parameter) Q of a graph G is uniquely determined by the set of cards of G, then Q is a *set-recognizable property* (*set-reconstructible parameter*). The well-known Ulam's Conjecture asserts that every graph with at least 3 vertices is reconstructible [14]. For a survey on results concerning this conjecture and its variants, the reader may consult [2, 3, 12]. In this paper, we study the following strong form of Ulam's Conjecture.

Harary's Conjecture [5]. All graphs with at least four vertices are set-reconstructible.

It is known [8, 9, 11] that many parameters and several classes of graphs like graphs with less than 12 vertices, disconnected graphs, trees and separable graphs without end vertices are set-reconstructible. Arjomandi and Corneil [1] have proved that unicyclic graphs are set-reconstructible. Outerplanar graphs have been set-reconstructed by Giles [4]. Ramachandran and Monikandan [13] have proved that all graphs are set-reconstructible if and only if all 2-connected graphs are set-reconstructible. Manvel and Weinstein [10] have reconstructed nearly acyclic graphs (graphs G with a vertex u such that $G-u$ is acyclic). Their proof invoked Kelly's Principle, Lemma 1 [7] for counting the number of subgraphs of a coloured graph G isomorphic to a coloured graph K, where $|V(K)| < |V(G)|$. Therefore, it cannot be extended directly to set-reconstruction as the principle does not work for a set of cards.

In this paper, we address the set-reconstructibility of connected graphs of order at least twelve. It is shown that two disjoint families $\mathscr{F}_1$ and $\mathscr{F}_2$ of nearly acyclic graphs are set-reconstructible (Sections 3 and 4). The family $\mathscr{F}_1$ consists of all connected cyclic graphs G with no end vertex such that there is a vertex lying on all the cycles in G and there is a cycle passing through at least one vertex of every cycle in G. The family $\mathscr{F}_2$ consists of all connected cyclic graphs H with end vertices such that there are exactly two vertices lying on all the cycles in H and there is a cycle with no limbs at its vertices.

2 Notation and Terminology

By $DS(G)$ and $NDS(v)$, we mean, respectively, the *degree sequence* of a graph G and the sequence of degrees of the neighbours (*neighbourhood degree sequence*) of v in G. The *pruned graph* $P(H)$ of a graph H is obtained by successively deleting the end vertices of H. A tree (possibly trivial) is a *path-tree* rooted at the vertex u if all the branches at u are paths and it is denoted by T_u. A (u,v)*-path-limb tree* , denoted by $P_{u,v}$, is a tree obtained from a graph that is a path $P: u, x_1, x_2, ..., x_k, v$ of order $k+2$ and k trees $T_1, T_2, ..., T_k$ (T_i may be trivial) by identifying the vertex x_i and a vertex of T_i for $i = 1, 2, ..., k$. A *caterpillar* is a tree in which a single path is incident to every edge. Let $P: v_0, v_1, ..., v_{r+1}$ $(r > 1)$ be a graph that is a path. Let $T_{w_1}, T_{w_2}, ..., T_{w_r}$ be r path-trees rooted at $w_1, w_2, ..., w_r$ respectively. A graph obtained from $P, T_{w_1}, T_{w_2}, ..., T_{w_r}$ by identifying the vertices v_i and w_i, $i = 1, 2, ..., r$ is called an *alien caterpillar* (henceforth abbreviated by a.c) and it is denoted by G_{ac}; the path P is the *spine* of the a.c. A *quasi alien caterpillar* (henceforth abbreviated by q.a.c), denoted G_{qac}, is a graph obtained from a G_{ac} by adding a new vertex s and joining it to all the end vertices of G_{ac} and possibly to the vertices of P.

Among the vertices lying on all the cycles in $G \in \mathscr{F}_1 \cup \mathscr{F}_2$, let s be a one with the maximum degree. Among all the cycles passing through at least one vertex of every cycle in $G \in \mathscr{F}_1$ (among all the cycles with no limbs in $G \in \mathscr{F}_2$,), let C be a one with the maximum length. Then the vertex s lies on C and we label the vertices of C by $s, v_1, ..., v_m, s$ in this order, where $m \geq 2$ (Figure 1). For a graph $G \in \mathscr{F}_1 \cup \mathscr{F}_2$ and any k in $\{1, 2, ..., m\}$, let $P_G(s, v_k)$ be the collection (possibly empty) of all (s, v_k)-path-limb trees, where $P_G(s, v_k) = \{P^1_{sv_k}, P^2_{sv_k}, ..., P^{r_k}_{sv_k}\}$. Let a (b) be the least (largest) integer such that $P_G(s, v_a) \neq \phi$ ($P_G(s, v_b) \neq \phi$). Note that $1 \leq a \leq b \leq m$. In particular, for a graph $G \in \mathscr{F}_1$, by the maximality of C, it is clear that $1 < a \leq b < m$, indeed, the vertices v_1 and v_m are 2-vertices in G. Clearly, $a = b$ for $G \in \mathscr{F}_2$.

Let $l_G(P^i_{sv_k})$ be the sum of the number of edges of all the limbs at the internal vertices of $P^i_{sv_k}$. For a vertex v_j with $P_G(s, v_j) \neq \phi$, we define $l_G(s, v_j)$ to be $\sum_{i=1}^{r_j} l_G(P^i_{sv_j})$. Throughout this paper, we use the notation G, $S(G)$, s, $P_G(s, v_j)$ and $P^i_{sv_k}$ in the sense of the above definitions.

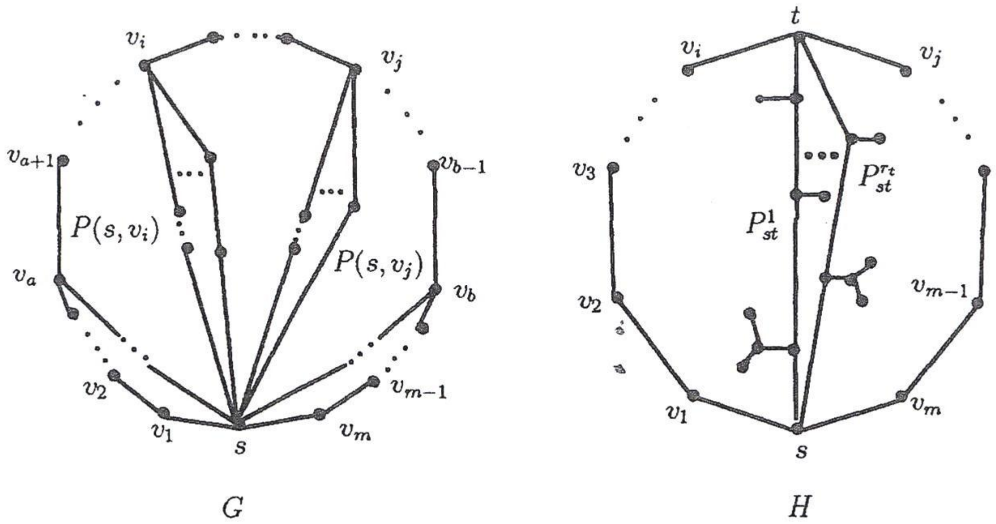

Figure 1. A graph G in $\mathscr{F}_1$ and a graph H in $\mathscr{F}_2$.

Manvel [9] proved the following two theorems.

Theorem 1. *The $DS(G)$ of any graph G with minimum degree at most 3 is set-reconstructible.*

Theorem 2. *Separable graphs without end vertices are set-reconstructible.*

For any graph G with $\delta(G) \leq 3$, we can determine the $NDS(v)$ in G using the degree sequences of G and the card $G - v$. The next simple lemma will be used while proving the family $\mathscr{F}_1$ is set-reconstructible.

Lemma 3. *Let G be a graph with $\delta(G) \leq 3$. If G has a vertex w whose neighbours are all k-vertices for some $k > 0$ and G has no $(k-1)$-vertices, then G is set-reconstructible.*

Proof. From $DS(G)$ and $DS(G-w)$, the set-recognizability of G follows. Now G can be obtained uniquely from $G - w$ by adding a new vertex v and joining it to all the $deg w$ vertices of degree $k-1$. □

3 Set-Reconstruction of $\mathscr{F}_1$

An *extension* of a card $G - v$ of G is a graph obtained from the card by adding a new vertex w and joining it to $deg\ v$ vertices of the card. The next lemma shows that the family $\mathscr{F}_1$ is set-recognizable.

Lemma 4. *A graph G is in $\mathscr{F}_1$ if and only if it satisfies the following conditions.*

(i) The minimum degree in G is at least two.

(ii) It has a card, say $G - x$ that is an a.c.

(iii) It has no card $G - y$ containing three blocks of order at least three with a common cut vertex, where $deg\ y \geq 3$.

Proof. Necessity: Since G is connected without end vertex and the card $G - s$ is an a.c, conditions (i) and (ii) follow. Suppose that G has a card $G - y$, where $deg\ y \geq 3$, containing three blocks, say A, B, C of order at least three with a common cut vertex, which is clearly s. Since G is 2-connected, in the graph G, the vertex y must be adjacent to at least one vertex other than the cut vertex in each of the three branches at s containing the blocks A, B and C, respectively. Consequently, the graph G cannot have a cycle passing through at least one vertex of every other cycle in G, giving a contradiction and so condition (iii) follows.

Sufficiency: In $G - x$, let $P : w_0, w_1, ..., w_{r+1}$ $(r > 1)$ be the spine and let $T_{w_1}, T_{w_2}, ..., T_{w_r}$ be the path-trees rooted at $w_1, w_2, ..., w_r$, respectively. To get an extension of the card $G - x$, we must add a new vertex x and join it to all the end vertices (because of (i)) and possibly to some non-end vertices. If x is joined only to the end vertices or x is joined both to the end vertices and the vertices on the spine P, then clearly x lies on all the cycles in G and the cycle $x, w_0, w_1, ..., w_{r+1}, x$ passing through a vertex on every cycle in G. Thus, the graph G belongs to $\mathscr{F}_1$. Otherwise, the vertex x is joined to all the end vertices and to a non-end vertex, say z other than the root vertex in T_{w_i} for some i. Again, if i is the least or largest integer such that $|T_{w_i}| \neq 1$, then the graph G must belong to $\mathscr{F}_1$; otherwise, the card $G - w_i$ has three blocks of order at least three with a common cut vertex (namely x), giving a contradiction to (iii) and completing the proof. □

Theorem 5. *The family $\mathscr{F}_1$ is set-reconstructible.*

Proof. Since a graph G in $\mathscr{F}_1$ has 2-vertices, its degree sequence $DS(G)$ is set-reconstructible by Theorem 1. Therefore the family $\mathscr{F}_1$ is set-recognizable by Lemma 4.

In view of Lemma 3, we can assume that G contains no induced path of order five or more. We proceed by two cases as below.

Case 1. The set $P_G(s, v_j) \neq \phi$ for exactly one vertex v_j.

A graph G in $\mathscr{F}_1$ satisfies Case 1 if and only if it has exactly two vertices of degree greater than two.

Weak reconstruction: Now, $DS(G) = [2, 2, \ldots, 2, \Delta, \Delta]$, where $\Delta > 2$. If a Δ-vertex-deleted card, say $G - v$, is not a path, then G can be obtained uniquely from $G - v$ by adding a new vertex and joining it to all the end vertices and to the $(\Delta - 1)$-vertex (if any). So, we assume that both the Δ-vertex-deleted cards are paths (this happens when $\Delta = 3$ and the two 3-vertices are adjacent in G). Now, since G has order at least twelve, each of these paths has order eleven, which implies there must be a 2-vertex-deleted card $G - x$ containing precisely two end vertices. Hence G is set-reconstructible by Lemma 3.

Case 2. The set $P_G(s, v_j) \neq \phi$ for at least two vertices v_j.

A graph $G \in \mathscr{F}_1$ satisfies Case 2 if and only if it does not satisfy Case 1.

Weak reconstruction: The vertex lying on all the cycles (namely s) is the only Δ-vertex in G. In view of Lemma 3, we can assume that the order of each element (path) in every nonempty $P_G(s, v_j)$ is at most four. Depends upon the existence of $(\Delta - 1)$-vertices, we have two more subcases.

Case 2.1. The graph G has no $(\Delta - 1)$-vertex.

If G has a 2-vertex, say w, adjacent to a Δ-vertex and a 2-vertex (this situation happens only if G contains a path $P^i_{sv_j}$ of order four for some i and j), then such a card $G - w$ can be identified in the set of cards $S(G)$ and it can be uniquely augmented to G by adding a new vertex w to $G - w$ and joining it to the unique $(\Delta - 1)$-vertex and the unique end vertex. Therefore, we assume that the order of each path in every nonempty $P_G(s, v_j)$ is at most three. Consequently, we have $P_G(s, v_2) \neq \phi$ and $P_G(s, v_{m-1}) \neq \phi$. Clearly, every path $P^i_{sv_j}$ of order three in G will become a leaf of a limb isomorphic to $K_{1,n}$ $(n \geq 1)$ in the card $(\Delta, G - s)$ and no path $P^i_{sv_j}$ of order two will not become so in the card $(\Delta, G - s)$. Thus, from the card $(\Delta, G - s)$, we can set-recognize whether there is a $P^i_{sv_j}$ of order two or three. Since v_1 and v_m are 2-vertices, it follows that not every path $P^i_{sv_j}$ in G has order two. On the other hand, if all the paths $P^i_{sv_j}$ have order three in G, then the neighbours of the Δ-vertex s are all 2-vertices and hence G is set-reconstructible by Lemma 3. So, we can assume that some $P^p_{sv_k}$ has order two and some other path $P^q_{sv_l}$ has order three. Then the order of C in G can be easily determined from $S(G)$ as the maximum among the orders of all the cycles in all the cards.

Suppose that only the two sets $P_G(s, v_2)$ and $P_G(s, v_{m-1})$ are nonempty (this case can be set-recognized as there are only three entries strictly greater than two in $DS(G)$). Then, by our assumption on Case 2.1, we have $deg\ v_2 \geq 4$ and $deg\ v_{m-1} \geq 4$ (because, if one of them were three, then the other would be $\Delta - 1$, contradicting). Let d_{min} and d_{max} be the minimum and the maximum of $deg\ v_2$ and $deg\ v_{m-1}$, respectively. Now, the graph G can be obtained uniquely from a 2-vertex-deleted card $G - u$ such that $DS(G - u) = [2,\ 2,\ \ldots,\ 2,\ d_{min} - 1,\ d_{max},\ \Delta - 1]$, by adding a new vertex and joining it to the $(\Delta - 1)$-vertex and the unique d_{min}-vertex.

We now assume that $P_G(s, v_j) \neq \phi$ for $2 \neq j \neq m - 1$. The card $G - v_i$, where $i = 2$ or $m - 1$, can be identified in $S(G)$ as the only r-vertex-deleted card, where $r > 2$, with a cut vertex such that all but one block re K_2. Hence $deg\ v_2$ and $deg\ v_{m-1}$ are known (Note that there may be nly one such card in $S(G)$; in that case $G-v_2 \cong G-v_{m-1}$ and $deg\ v_2 =$ $leg\ v_{m-1}$). Set d_{min} and d_{max} as before. If $d_{min} \geq 4$, then in each card $\tilde{}- u$ obtained by deleting a 2-vertex u, $u \notin V(C)$, the vertices v_2 and $'_{m-1}$ are identifiable as a set since the card $G-u$ belongs to the family $\mathscr{F}_1$. Ience, in this case, consider one such card $G-u$ with the unique $(\Delta-1)$-ertex such that $\{deg_{G-u}v_2,\ deg_{G-u}v_{m-1}\} = \{d_{min}-1,\ d_{max}\}$. Then the nique $(\Delta-1)$-vertex and the vertex, among v_2 and v_{m-1}, of degree $l_{min}-1$ are the neighbours of u in G. Similarly, if $d_{max}-d_{min} \geq 2$, then he card $G-w$, obtained by deleting a 2-vertex w adjacent with s and the ertex, among v_2 and v_{m-1}, of degree d_{max}, can be identified in $S(G)$ as he only 2-vertex-deleted card with the unique $(\Delta-1)$-vertex such that it elongs to the family $\mathscr{F}_1$, the length of the longest cycle in $G-w$ is same s that of C and $\{deg_{G-w}v_2,\ deg_{G-w}v_{m-1}\} = \{d_{min},\ d_{max}-1\}$. Now he unique $(\Delta-1)$-vertex and the vertex, among v_2 and v_{m-1}, of degree $l_{max}-1$ in $G-w$ are the neighbours of w in G. The only remaining ase to reconstruct in Case 2.1 is that the degree of one of v_2 and v_{m-1}, ay v_2 is three and the degree of v_{m-1} is three or four. If there exists an -vertex-deleted card $G-w$ containing exactly $r-1$ end vertices and a $\Delta-1)$-vertex, where $r>2$, then these r vertices are the neighbours of w in G. So, we can assume, in particular, that both v_3 and v_{m-2} are not 2-vertices (otherwise, v_3 or v_{m-2} is a 2-vertex. Then, in view of Lemma 3, one of the paths in $P_G(s,v_j)$ must be P_2, where $j=2$ or $m-1$, which mplies $G-v_3$ or $G-v_{m-2}$ is isomorphic to the card $G-w$ as defined ust above). Hence G is set-reconstructible.

Case 2.1.1. The degree of v_{m-1} in G is four.

Since $deg\ v_2$ is three, we have $|P_G(s,v_2)| = 2$; sv_1v_2 is a path in $P_G(s,v_2)$. The other path in $P_G(s,v_2)$ has order three if and only if there exists a 2-vertex-deleted card $G-u$ belonging to $\mathscr{F}_1$ such that it contains only one $(\Delta-1)$-vertex (namely, s) and an induced path P' of order four starting from s. Every extension of $G-u$ by adding a new vertex joining it to s and one of the two adjacent vertices in P' is either isomorphic to G or else contains a cycle of length $|C|+1$, giving a contradiction. We have therefore $P_G(s,v_2) = \{P_3, P_2\}$. We can now assume that $P_G(s,v_{m-1}) = \{P_3,P_3,P_2\}$ (otherwise in $G-v_m$, the vertex v_{m-1} can be distinguished from v_2 and thus G can be reconstructed from the identifiable card $G-v_m$ in $S(G)$). Similarly, we can assume that $P_G(s,v_3) = \{P_3,P_2\}$ (otherwise in $G-v_1$, the vertex v_3 can be distinguished from v_{m-1} and thus G can be reconstructed from the identifiable card $G-v_1$ in $S(G)$). Consider a 2-vertex deleted card $G-w$ belonging to $\mathscr{F}_1$ such that it contains only one $(\Delta-1)$-vertex (namely, s), $P_{G-w}(s,v_{m-1}) = \{P_3,P_3,P_2\}$, $P_{G-w}(s,v_2) = \{P_3,P_2\}$ and $P_{G-w}(s,v_3) = \{P_2\}$. Clearly, the vertices s and v_3 are the neighbours of

w in G. Hence G is set-reconstructible.

Case 2.1.2. The degree of v_{m-1} in G is three.

It is known that one of the two paths in $P_G(s, v_j)$ is P_3, where $j = 2$ or $m-1$. If the other path in $P_G(s, v_j)$ is also P_3, then G can be reconstructed by proceeding as in the beginning of Case 2.1.1. We have therefore $P_G(s, v_2) = P_G(s, v_{m-1}) = \{P_3, P_2\}$. We can assume that $P_G(s, v_3) = \{P_2\}$ (otherwise in $G - v_1$, the vertex v_3 can be distinguished from v_{m-1} and so G can be reconstructed from the card $G - v_1$). Similarly, we will have the assumption $P_G(s, v_{m-2}) = \{P_2\}$. If $P_G(s, v_4) \neq \{P_2\}$, then in the card $G - v_3$, the vertex v_2 can be identified uniquely, the vertex v_5 can be distinguished from v_{m-1} and so the vertex v_4 can be identified uniquely. Hence G can be reconstructed from the identifiable card $G - v_3$ in $S(G)$. We have therefore $P_G(s, v_4) = P_G(s, v_{m-3}) = \{P_2\}$. Proceeding like this, we shall get $P_G(s, v_j) = \{P_2\}$ for all $j = 3, 4, \ldots, m-2$, which implies s is adjacent to all other vertices in G and hence G is set-reconstructible.

Case 2.2. The graph G has a $(\Delta - 1)$-vertex.

Now $|P_G(s, v_k)| = 2$ for exactly one vertex v_k, the set $|P_G(s, v_l)| \geq 2$ for exactly one vertex v_l, $l \neq k$ and $P_G(s, v_j) = \phi$ for all $j \neq k, l$. Then $deg\ v_k = 3$ and $deg\ v_l = \Delta - 1$, which implies $DS(G) = [2, 2, \ldots, 2, 3, \Delta - 1, \Delta]$. Without loss of generality, we assume that $k < l$. Two cases arise depending on the value of Δ.

Case 2.2.1. The value of Δ is at least 5.

Let P be the (v_k, v_l)-path not containing s in G. In view of Lemma 3, we can assume that $2 \leq \nu(P) \leq 4$. The card $G - v_k$ is identifiable in $S(G)$ as the only 3-vertex-deleted card and hence the $NDS(v_k)$ is set-reconstructible. We shall use this to set-reconstruct the value of $\nu(P)$. It is two if and only if $NDS(v_k) = [2, 2, \Delta - 1]$ or $[2, \Delta - 1, \Delta]$. It is three if and only if there exists a 2-vertex-deleted card $G - v_j$ such that $NDS(v_j) = [3, \Delta - 1]$. If $\nu(P) = 2$, then G can be obtained uniquely from the card $G - v_k$ by adding a new vertex w and joining it either to "the two end vertices and the unique $(\Delta - 2)$-vertex" (if $NDS_G(v_i) = [2, 2, \Delta - 1]$ holds) or to "the unique end vertex, the unique $(\Delta - 2)$-vertex, and the unique $(\Delta - 1)$-vertex" (otherwise). If $\nu(P) = 3$, then again we shall use the card $G - v_k$. Clearly, $NDS_G(v_k) = [2, 2, 2]$ or $[2, 2, \Delta]$ and if the first holds, then G is set-reconstructible by Lemma 3. If the latter holds, then all graphs, obtained from $G - v_k$ by adding a new vertex w and joining it to the two end vertices and to a $(\Delta - 1)$-vertex, are isomorphic and they are G. Otherwise $\nu(P) = 4$. In this case, consider a 2-vertex-deleted card $G - w$ with $NDS_G(w) = [2, \Delta - 2]$ and a unique end vertex adjacent to a 3-vertex. Now every extension of $G - w$, by adding a new vertex w and joining it to the unique end vertex and to the unique $(\Delta - 2)$-vertex not adjacent to the end vertex, is isomorphic to G.

Case 2.2.2. The value of Δ is four.

Now $|P_G(s, v_k)| = |P_G(s, v_l)| = 2$ for some k, l with $k < l$ and $P_G(s, v_j) = \phi$ for all $j \neq k, l$. The degree sequence of G is clearly $[2, 2, \ldots, 2, 3, 3, 4]$. Let P be the (v_k, v_l)-path not containing s in G. In view of Lemma 3, we have the order of P and the order of each path in $P_G(s, v_k) \cup P_G(s, v_l)$ are at most four, which imply that $\nu(G)$ must be at most 13. Hence $\nu(G) = 12$ or 13. Now at least one of the three vertices v_k, v_l and s must be adjacent only to 2-vertices and hence G is set-reconstructible by Lemma 3. □

4 Set-Reconstruction of $\mathscr{F}_2$

As before, by s and t, we mean the two common vertices lying on all the cycles in $G \in \mathscr{F}_2$. Since there is no limb rooted at these two vertices, it follows that $deg\ s = deg\ t$. These two vertices are identifiable as a set in any 1-vertex-deleted card of G, as the only vertices of degree at least three without limbs and so we use the same label to refer such vertices in any 1-vertex-deleted card. For the sake of clarity in proofs, we shall partition the family $\mathscr{F}_2$ into two subfamilies $\mathscr{F}_{21}$ and $\mathscr{F}_{22}$ as follows: Let $\mathscr{F}_{21} = \{G \in \mathscr{F}_2 : deg_G\ s = deg_G\ t = 3\}$ and $\mathscr{F}_{22} = \{G \in \mathscr{F}_2 : deg_G\ s = deg_G\ t \geq 4\}$. We shall reconstruct each of them separately.

Lemma 6. *A graph G is in $\mathscr{F}_{21}$ if and only if it satisfies one of the following:*

(i) The maximum degree $\Delta(G)$ is 3 and G has only one 1-vertex-deleted card which is a union of three cycles passing through two common vertices.

(ii) Every 1-vertex-deleted card $G - x$ is in $\mathscr{F}_{21}$ and it contains only one end vertex and only one cycle without limbs, and G satisfies one of the following four conditions:

(α) It has a 4-vertex.

(β) There is a disconnected 2-vertex-deleted card.

(γ) There is a unicyclic 2-vertex-deleted card with a limb at exactly one vertex and the limb is not a path (with the root occurs as an end vertex).

(δ) There is a disconnected 3-vertex-deleted card such that one of its components is unicyclic with a limb at exactly one vertex and the limb is either P_3 or not a path.

(iii) Every 1-vertex-deleted card $G-y$ is in $\mathscr{F}_{21}$ such that $l_{G-y}(P_{uv}) \geq 2$, where u and v are the vertices lying on all the cycles in the card.

Proof. Necessity: For a graph G in $\mathscr{F}_{21}$, we have $l_G(P_{st}) = 1, 2$ or ≥ 3. If $l_G(P_{st}) \geq 3$, then every 1-vertex-deleted card $G - x$ is in $\mathscr{F}_{21}$ with $l(P_{uv}) \geq 2$, where u and v are the vertices lying on all the three cycles in

$G-x$, (iii) follows. If $l_G(P_{st})=1$, then the unique 1-vertex-deleted card $G-x$ is a union of three cycles passing through two common vertices and so $\Delta(G)=3$, (i) follows. So, assume that $l_G(P_{st})=2$. Now, there are at most two 1-vertex-deleted cards $G-x$, each belongs to $\mathscr{F}_{21}$, indeed, it contains exactly one end vertex and one cycle with no limbs. The graph G is now one of the five types shown in Figure 2. A graph $G\in\mathscr{F}_{21}$ with $l_G(P_{st})=2$ has a 4-vertex if and only if it is of type 2. A graph $G\in\mathscr{F}_{21}$ with $l_G(P_{st})=2$ has a disconnected 2-vertex-deleted card if and only if it is of type 1. A graph $G\in\mathscr{F}_{21}$ with $l_G(P_{st})=2$ has a unicyclic 2-vertex-deleted card such that only one vertex has a limb different from paths if and only if it is of type 3. A graph $G\in\mathscr{F}_{21}$ with $l_G(P_{st})=2$ is of type 4 (or type 5) if and only if there exists a disconnected 3-vertex-deleted card with a unicyclic component in which only one vertex has a limb different from paths (or P_3), (ii) follows.

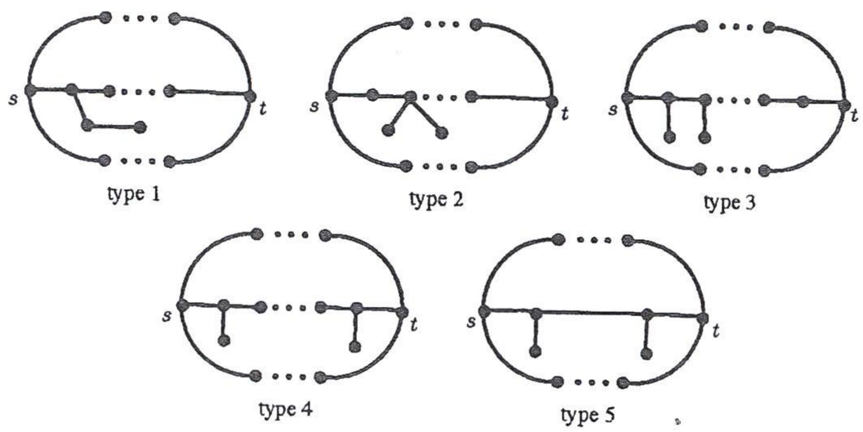

Figure 2. Five types of graphs arising under (ii)

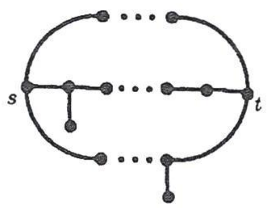

Figure 3. An extension H

Sufficiency: If (i) holds, then every extension of the 1-vertex-deleted card $G-x$ either belongs to $\mathscr{F}_{21}$ or must contain a 4-vertex, but which is excluded in (ii). If (ii) holds, then every extension H of an 1-vertex-deleted card $G-x$ either belongs to $\mathscr{F}_{21}$ or is isomorphic to the graph shown in Figure 3. If the later holds, then the extension satisfies none of the conditions $(\alpha),(\beta),(\gamma)$ and (δ), which is a contradiction. Finally, if (iii) holds, then every extension of a 1-vertex-deleted card must belong to $\mathscr{F}_{21}$.

□

Lemma 7. *A graph G is in $\mathscr{F}_{22}$ if and only if it satisfies one of the following:*

(i) It has only one 1-vertex-deleted card, which is a union of at least two cycles, all of them passing through two common vertices, and it has only two Δ-vertex-deleted cards, which are connected.

(ii) Every 1-vertex-deleted card is in $\mathscr{F}_{22}$ such that it has exactly one end vertex and at least three distinct cycles with no limbs.

(iii) Every 1-vertex-deleted card $G-x$ is in $\mathscr{F}_{22}$ such that $l_{G-x}(P_{w_1w_2}) \geq 2$, where w_1, w_2 are the vertices lying on all the cycles in $G-x$. There is no disconnected 3-vertex-deleted card $G-y$ in $\mathscr{F}_{22}$ with a component K such that $l_K(P^i_{uv}) = 1$ for all the elements P^i_{uv} in $P_K(u,v)$ except one P^q_{uv} for which $l_K(P^q_{uv}) = 0$, where u, v are the vertices lying on all the cycles in K.

Proof. Necessity: We proceed by three cases depending on the value of $l_G(P_{st})$ (> 0).

Case 1: $l_G(P_{st}) = 1$.

The graph G has only one 1-vertex-deleted card $G-x$, which is a union of cycles (at least two), all of them passing through both s and t. Since $deg_G s = deg_G t > 3$, it follows that no other vertex can have degree more than $deg\ s$. Since s and t have no limbs, the cards $G-s$ and $G-t$ are connected, (i) follows.

Case 2: $l_G(P_{st}) = 2$.

Now G has at most two end vertices. Since $|P_G(s,t)| \geq 4$ and at least one cycle in G has no limbs, it follows that each 1-vertex-deleted card is a graph in $\mathscr{F}_{22}$ such that it has exactly one end vertex and at least three distinct cycles without limbs, (ii) follows.

Case 3: $l_G(P_{st}) \geq 3$.

Every 1-vertex-deleted card $G-x$ is a graph in $\mathscr{F}_{22}$ with $l_{G-x}(P_{w_1w_2}) \geq 2$, where w_1, w_2 are the vertices lying on all the cycles in $G-x$. In this case, we shall show that the card $G-y$ does not exist. Suppose, to the contrary, that the card $G-y$ exists. Then, since exactly one P^q_{uv} has zero limb size, every extension of $G-z$ has at most one (u,v)-path without limbs and so the extension cannot have a cycle without limbs, which implies, in particular, the graph G does not belong to $\mathscr{F}_{22}$, giving a contradiction and completing the necessary part.

Sufficiency: If (i) holds, then every extension of an 1-vertex-deleted card $G-x$ is either belongs to $\mathscr{F}_{22}$ or a Δ-vertex-deleted card of the extension is disconnected (the later case happens when the newly added vertex, say w to $G-x$, is joined to one of the common vertices lying on all the cycles in $G-x$). Similarly, if (ii) holds, then every extension of a 1-vertex-deleted card $G-x$ contains a cycle without limbs and so it belongs to $\mathscr{F}_{22}$. Hence, we assume that (iii) holds. Now consider an extension H of a 1-vertex-deleted card $G-x$. Then w_1, w_2 are the two common vertices lying on all the cycles in H. Clearly, at least one path in $P_H(w_1, w_2)$ has no limbs. If at least two paths in $P_H(w_1, w_2)$ have no limbs, then H belongs to $\mathscr{F}_{22}$. So, we assume that exactly one path in $P_H(w_1, w_2)$ has no limbs. If a path in $P_H(w_1, w_2)$ has a limb, say L of size at least two, then the 1-vertex-deleted card of H, obtained by deleting an end vertex in L, has no cycles without limbs and so it does not belong to $\mathscr{F}_{22}$, contradicting (iii). Therefore all but one path in $P_H(w_1, w_2)$ have a limb of size one and the exceptional path has no limbs, which imply any 3-vertex-deleted card of H, corresponding to the base of an end vertex, will satisfy the properties of $G-y$, again contradicting (iii) and completing the proof. □

We denote the limb at u in G by $L_G(u)$ and the number of edges in $L_G(u)$ by $l_G(u)$.

Theorem 8. *The family $\mathscr{F}_{21}$ is set-reconstructible.*

Proof. Recognition: Follows by Theorem 1 and Lemma 6.
Weak reconstruction: Clearly G has only one (s,t)-path with limbs; let it be P_{st}. Let f, h be the vertices (not necessarily distinct) having limbs in G such that their distance from s, t are as small as possible, respectively. Let $\eta(G)$ be the number of vertices having limbs in G. Clearly $\eta(G)$ is equal to the number of disconnected cards with a unicyclic component. Let $\{r_1, r_2\}$ be the set of lengths of the (s,t)-path in both directions of the unique cycle without limbs in G. For a graph G in $\mathscr{F}_{21}$ with $l_G(s,t) \neq 1$, the set of values $\{r_1, r_2\}$ can be determined from any 1-vertex-deleted card.
Case 1. $\eta(G) = 1$.

Now $f = h$ and $l_G(f) = l_F(f) + 1$ for any $F \in S_1(G)$, where $S_1(G)$ is the set of all 1-vertex-deleted cards of G. We proceed by two cases depending on $l_G(f)$.
Case 1.1. $l_G(f) \geq 2$.

Since G is simple, the cycle without limbs contains a 2-vertex. Every connected 2-vertex-deleted card of the graph G considered under Case 1.1 has a limb different from paths if and only if $L_G(f)$ is not a path. This shows that whether $L_G(f)$ is a path or not can be set-recognized. Also, if the later holds, then $L_G(f)$ can be identified as the only limb of size $l_G(f)$ different from paths in a 2-vertex-deleted card containing the unique cycle with a limb of size $l_G(f)$ different from paths. If the former holds, then $L_G(f) \cong P_{l_G(f)+1}$. Now G can be obtained uniquely from an 1-vertex-deleted card by just replacing the unique limb with $L_G(f)$.
Case 1.2. $l_G(f) = 1$.

If the unique disconnected card $G-f$ has two end vertices, then G

can be obtained uniquely from $G-f$ by adding a new vertex and joining it to the isolated vertex and to the two end vertices. If the unique disconnected card $G-f$ has only one end vertex, then the extension H of $G-f$, by adding a new vertex and joining it to the isolated vertex, the end vertex and a 2-vertex in the unique cycle, is either isomorphic to G or does not have the 1-vertex-deleted card. Consequently, the vertex f has no 2-vertex neighbour in G. Now the extension H of $G-f$, by adding a new vertex and joining it to the isolated vertex and to two 2-vertices in the unique cycle, is either isomorphic to G or does not have the unique 1-vertex-deleted card.

In the remaining two cases below, we shall first find $\{L_G(f), L_G(h)\}$ and then we prove that G is set-reconstructible.

Case 2. $\eta(G) \geq 3$.

Let $\mathscr{A} = \{\{d_F(s,w), d_F(t,q)\} : F \in S_1(G)$ and w, q are vertices having limbs in F such that their distance from s, t along the (s,t)-path with limbs are as small as possible, respectively$\}$. Clearly $\mathscr{A}$ has at most three elements. If $|\mathscr{A}| = 3$, then clearly $L_G(f) = L_G(h) \cong P_2$; let $\mathscr{A} = \{\{c_i, d_i\} : c_i \leq d_i$ and $i = 1,2,3\}$. Then $\{d_G(s,f),\ d_G(t,h)\} = \{c_r, d_r\}$, where $c_r + d_r$ is minimum. If $|\mathscr{A}| = 2$ and $\mathscr{A} = \{\{c_i, d_i\} : c_i \leq d_i$ and $i = 1,2\}$. Now consider the card $G - x'$ in $S_1(G)$ such that $\{d_{G-x'}(s,w), d_{G-x'}(t,q)\} = \{c_r, d_r\}$, where $c_r + d_r$ is minimum. Then $\{L_G(f), L_G(h)\} = \{L_{G-x'}(w), L_{G-x'}(q)\}$. If $|\mathscr{A}| = 1$ and $\mathscr{A} = \{\{c_r, d_r\} : c_r \leq d_r\}$, then $\{L_G(f), L_G(h)\} = \{L_{G-x'}(w), L_{G-x'}(q)\}$, where $G-x'$ is a card in $S_1(G)$ such that $l_{G-x'}(w) + l_{G-x'}(q)$ is maximum.

Now we reconstruct G by three cases depending upon the values of $l_G(f)$ and $l_G(h)$. Without loss of generality, let us assume that $l_G(f) \leq l_G(h)$. If both $l_G(f)$ and $l_G(h)$ are at least two, then all graphs obtained, from a 1-vertex-deleted card F such that the size of the limb at one vertex, say z is $l_G(f) - 1$, by replacing the limb at z with $L_G(f)$ are isomorphic and they are G. If $l_G(f)$ is one and $l_G(h)$ is at least three, then G can be obtained uniquely, from a 1-vertex-deleted card F with limbs at two vertices, by replacing the limb of size at least two with $L_G(h)$.

Finally, consider the case that $l_G(f)$ is one and $l_G(h)$ is two. If $c_r = d_r$, then G can be obtained uniquely from a card $F \in S_1(G)$ such that $d_F(t,q) = c_r$, $l_F(q) = 2$ and $d_F(s,w) = c_1 (> c_r)$, by adding a new vertex and joining it to the 2-vertex, say u such that $d_F(s,u) = c_r$ and $l_F(u) = 0$. Otherwise, that is, if $c_r \neq d_r$, then $d_F(s,w) = c_r$ with $l_F(w) = 2$, where $F \in S_1(G)$. Now G can be obtained uniquely, from a card $F \in S_1(G)$ with $l_F(w) = l_F(q) = 1$, by adding a new vertex and joining it to the unique end vertex of the limb at w with $d_F(s,w) = c$. The only remaining case is that both $l_G(f)$ and $l_G(h)$ are one. If $c_r = d_r$, then G can be obtained uniquely from a card $F \in S_1(G)$ such that $d_F(t,q) = c_r$ with $l_F(q) = 1$ and $d_F(s,w) = c_1 (> c_r)$, by adding a new vertex and joining it to the 2-vertex, say u, with $d_F(s,u) = c$ and $l_F(u) = 0$. Otherwise, that is, if $c_r \neq d_r$, then we consider a card $F \in S_1(G)$ such that $\{d_F(s,w_1),\ d_F(t,q_1)\} = \{c,\ d_1\}$, where $d_1 > d_r$ and w_1, q_1 are vertices with limbs in F such that their distance from s and t along the (s,t)-path with limbs are as small as possible, respectively. Without loss of generality, $d_F(t,q_1) = d_1$. Now all graphs obtained from F, by adding a new vertex and joining it to a 2-vertex, say q such that $d_F(t,q) = d$, are isomorphic and they are G.

Case 3. $\eta(G) = 2$.

Let $\mathscr{L} = \{l_{G-x}(w) : G - x \in S_1(G)$ and w is a vertex with limbs in $G - x\}$. If $|\mathscr{L}| = 4$, then, by observation, the four numbers must be two pairs of consecutive numbers in $\mathscr{L}$ and the maximum number in each such pair must be $l_G(f)$ and $l_G(h)$, respectively. If $|\mathscr{L}| = 3$, then $l_G(f)$ and $l_G(h)$ must be the first two maximum numbers in $\mathscr{L}$. If $\mathscr{L} = 2$, then $l_G(f) = l_G(h) = \max \mathscr{L}$.

Suppose that there exists a unicyclic 2-vertex-deleted card E containing three vertices with limbs; denote the limbs by L_1, L_2, L_3. Without loss of generality, let us take that L_1 be the path of length either r_1-2 or r_2-2. Then $\{L_G(f),\ L_G(h)\} = \{L_2,\ L_3\}$. Suppose that there exists a unicyclic 2-vertex-deleted card E containing two vertices with limbs, say w and q such that $\{l_E(w),\ l_E(q)\} = \{l_G(f),\ l_G(h)\}$. Then $\{L_G(f),\ L_G(h)\} = \{L_E(w),\ L_E(q)\}$.

As in Case 2, we proceed now by three cases depending upon the values of $l_G(f)$ and $l_G(h)$. The two cases that 'both $l_G(f)$ and $l_G(h)$ are at least two' and '$l_G(f)$ is one and $l_G(h)$ is at least three' are just similar to Case 2. We now consider the case that $l_G(f)$ is one and $l_G(h)$ is two. From a 1-vertex-deleted card F with limbs at only one vertex, say z, find the distance of z from each of the two 3-vertices; let them be d_1, d_2. Now G can be obtained uniquely (up to isomorphism) from a 1-vertex-deleted card E with limbs at two vertices, by replacing the limb at a vertex whose distances from the two 3-vertices are d_1, d_2 with $L_G(h)$ (if both the vertices with limbs satisfy the distance conditions, then there is an automorphism of E interchanging the end vertices and so the resulting graphs are isomorphic).

The only remaining case is that both $l_G(f)$ and $l_G(h)$ are one. If G has a 2-vertex-deleted card E with a limb at only one vertex, then the extension H of E, by adding a new vertex and joining it to a 2-vertex in the unique cycle and to a vertex of degree one or two in the unique limb, satisfies one of the following conditions:

(i) The extension is isomorphic to G.

(ii) The set of lengths of the (s,t)-path in both directions of the unique cycle without limbs in the extension is not equal to $\{r_1, r_2\}$, contradicting.

One of the limb in the extension has order at least two, again contradicting.

ce G is set-reconstructible. If G has no such card E exists, then ider a disconnected 3-vertex-deleted card F. The extension H of F, idding a new vertex and joining it to the isolated vertex, a 2-vertex ne unique cycle and a vertex of degree one or two in the unique limb, sfies one of the above three conditions (i), (ii) and (iii). Hence G is reconstructible, which completes the proof. □

eorem 9. *The family $\mathscr{F}_{22}$ is set-reconstructible.*

of. Recognition: Follows by Theorem 1 and Lemma 7.
ak Reconstruction: As before, the vertices s and t are identifiable as et in any 1-vertex-deleted card of G as the only vertices of degree at st four without limbs. Therefore $l_G(P_{st}) = l_E(P_{st}) + 1$ for any 1-vertex-eted card E. Since G is a simple graph containing a cycle without ibs, it must contain a 2-vertex. In fact, we need a 2-vertex adjacent with or t in G. Consider a connected 2-vertex-deleted card F such that $(P_{st}) = l_G(P_{st})$ and at least one of the vertices s, t lost its degree by e in the card. Now, all graphs, obtained from E by adding a new vertex and joining it either to both s and t (if both s and t lost their degree one) or to the unique wounded vertex among s, t and to the end vertex the unique limb at the unwounded vertex among s, t (otherwise), are omorphic and they are G. □

oncluding Remarks

The *pruned graph* $P(H)$ of a graph H is obtained by successively deleting the end vertices of H. The following problems suggest ways in which e results in this paper can be improved. The three families $\mathscr{F}_1$, $\mathscr{F}_2$ nd $\mathscr{F}_3$, where $\mathscr{F}_3 = \{G \notin \mathscr{F}_2 : \delta(G) = 1 \text{ and } P(G) \in \mathscr{F}_1\}$, form a artition of graphs with a vertex s such that $G - s$ is a tree. Since the rst two families are proved to be set-reconstructible and graphs with a ertex s such that $G - s$ is a tree are set-recognizable, the family $\mathscr{F}_3$ is et-recognizable. If all graphs G with a vertex s such that $G - s$ is a tree re set-reconstructible, then it may not be difficult to prove that all graphs I with a vertex s such that $H - s$ is acyclic (that is, all nearly acyclic raphs) are set-reconstructible because H is a union of graphs G with a nique common vertex s.

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