Planarity, duality and Laplacian congruence

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Abstract

We discuss the connections tying Laplacian matrices to abstract duality and planarity of graphs.

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1   Introduction

The purpose of this paper is to discuss the relationships between Laplacian matrices and three important properties of graphs: planarity for individual graphs, 2-isomorphism (isomorphism of cycle matroids) for pairs of graphs, and duality for pairs of planar graphs. These ideas have long histories, and have been studied carefully and thoroughly for the better part of a century. The Laplacian matrix is motivated by Kirchoff's laws, which were formulated in the mid 1800s, before matrices were introduced. Planarity was studied by Kuratowski and Whitney in the 1920s and 1930s; Whitney also introduced the 2-isomorphism and duality relations at that time. The existence of a connection between Laplacian matrices and graph matroids is implicit in the matrix-tree theorem, which was formulated before World War II. (See [6] for some notes on the history of the matrix-tree theorem.) However it was not until the early 1990s that Watkins [11, 12] showed that the cycle matroid of a graph is determined by the Laplacian matrix, except for the fact that the Laplacian ignores loops.

Theorem 1 *(Watkins [11, 12]) If G_1 and G_2 are graphs with the same number of loops, then G_1 and G_2 are 2-isomorphic if and only if their reduced Laplacian matrices are congruent over $\mathbb{Z}$.*

To keep the introduction readable, we delay technical definitions (2-isomorphism, congruence, reduced Laplacian, etc.) until later in the paper.

In 1997, Bacher, de la Harpe and Nagnibeda [1] proved that the cycle matroid of a graph is related to the lattices of cuts and flows. (See also the account of Godsil and Royle [4, Chapter 14].) Here are several of their results. The statements are augmented with obvious requirements involving bridges and loops.

Theorem 2 *(Bacher, de la Harpe and Nagnibeda [1]) Let G_1 and G_2 be graphs.*

1. *If G_1 and G_2 are 2-isomorphic, then they have the same number of loops and their cut lattices are isomorphic.*

2. *If G_1 and G_2 are 2-isomorphic, then they have the same number of bridges and their flow lattices are isomorphic.*

3. *If G_1 and G_2 are dual planar graphs, then G_1 has the same number of loops as G_2 has bridges, and the cut lattice of G_1 is isomorphic to the flow lattice of G_2.*

Bacher, de la Harpe and Nagnibeda observed that the reduced Laplacians of a graph are Gram matrices for the cut lattice; see [1, p. 183], although the term "Laplacian" does not appear there. This implies that two graphs have isomorphic cut lattices if and only if they have congruent reduced Laplacians, and this in turn implies that part 1 of Theorem 2 is equivalent to one direction of Theorem 1, and the converse of part 1 of Theorem 2 is equivalent to the other direction of Theorem 1.

Not realizing that Watkins had already proven a result equivalent to the converse of part 1, Bacher, de la Harpe and Nagnibeda left the converses of the three parts of Theorem 2 as open problems in [1]. In 2010, Su and Wagner [10] verified the converses of all three parts, and also extended the theory to the cut and flow lattices of regular matroids.

The present paper grew out of our work on the problem of extending Theorem 1 to describe the relationship between Laplacian matrices of dual graphs. By the time we appreciated the connection with the work of Bacher, de la Harpe and Nagnibeda [1] and Su and Wagner [10], we had developed a set of ideas that includes our own proof of a result equivalent to part 3 of Theorem 2, and the converse. Our arguments are focused on matrices associated with graphs, rather than lattices associated with regular matroids. Of course a lattice may be identified with the collection of its Gram matrices, so the two approaches are not incompatible in theory; but the arguments are quite different.

The notion we develop is that in addition to having an unreduced Laplacian matrix that is uniquely defined up to simultaneous permutation of the

rows and columns, a graph G has "unreduced dual Laplacian matrices," which are not uniquely defined but are congruent to each other over $\mathbb{Z}$. (The definition appears in Section 4.) There are also reduced dual Laplacians, just as there are reduced Laplacians. Some properties of these matrices are analogues of properties of Laplacians; for instance the reduced dual Laplacian matrices of G are Gram matrices for the flow lattice of G.

Dual Laplacian matrices also have some properties that are rather different from properties of Laplacians. Two of these properties involve planarity and abstract duality.

Theorem 3 *Suppose G is a graph with m edges and b bridges.*

1. *The trace of an unreduced dual Laplacian matrix of G is an even integer, greater than or equal to $2(m - b)$.*

2. *G is planar if and only if G has an unreduced dual Laplacian matrix whose trace is equal to $2(m - b)$.*

Theorem 3 can be restated using the lattice terminology of Conway [3]: G is planar if and only if its flow lattice has a superbase with a Gram matrix of the smallest possible trace.

Theorem 4 *Let G be a planar graph. Then the following statements about a graph G^* are equivalent.*

1. *G and G^* are abstract duals.*

2. *The number of loops in G^* is the same as the number of bridges in G, and a reduced Laplacian matrix of G^* is a reduced dual Laplacian matrix of G.*

Here is an outline of the paper. In Section 2 we summarize the connections tying congruence of Laplacian matrices to row equivalence of incidence matrices, and 2-isomorphism of graphs. A small example is presented in Section 3. Reduced and unreduced dual Laplacian matrices are defined in Section 4, and the analogies between them and ordinary Laplacians are discussed. In Section 5 we verify some properties of dual Laplacians. Theorems 1, 3 and 4 are proven in Section 6; we also discuss a version of Theorem 1 for unreduced Laplacians, and we relate Theorem 3 to a famous planarity criterion of MacLane [7]. A couple of illustrative examples are presented in Section 7.

Before proceeding we should thank an anonymous reader for good advice, which significantly improved the readability of the paper.

2 Some properties of Laplacian matrices

We use standard notation and terminology for graphs. A graph G has a finite set $E(G)$ of edges, and a finite set $V(G)$ of vertices; we write $m = |E(G)|$ and $n = |V(G)|$. If $e \in E(G)$ is incident on $v, w \in V(G)$ then we write $e = vw$. If $e = vv$ then e is a *loop* at v; the number of loops in G is denoted ℓ. Two edges $e \neq e'$ are *parallel* if $e = vw$ and $e' = vw$. A graph is *simple* if it has neither loops nor parallels.

Definition 5 *Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$. Then the* Laplacian matrix *of G is the $n \times n$ matrix with entries given by*

$$L(G)_{ij} = \begin{cases} -|\{e \in E(G) \mid e = v_i v_j\}|, & \text{if } i \neq j \\ |\{e \in E(G) \mid e = v_i v_k \text{ and } k \neq i\}|, & \text{if } i = j \end{cases}.$$

Six elementary properties of the Laplacian are immediately apparent from Definition 5; we number them for ease of reference.

Property I $L(G)$ is a symmetric matrix with integer entries.

Property II $L(G)$ is not changed if loops are added to G or removed from G.

Property III If G and G' are graphs then $L(G) = L(G')$ up to simultaneous permutation of the rows and columns if, and only if, we obtain isomorphic graphs when we remove all loops from G and G'.

Property IV The sum of the columns of $L(G)$ is $\mathbf{0}$; and the same for the rows.

Here the bold $\mathbf{0}$ denotes a matrix or vector whose entries all equal 0.

Property V The trace $Tr(L(G))$ is $2(m - \ell)$.

Property VI If G is a disconnected graph with connected components $C_1, \ldots, C_{c(G)}$ then

$$L(G) = \begin{pmatrix} L(C_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & L(C_{c(G)}) \end{pmatrix}.$$

Properties IV and VI tell us that in each connected component of G, the row of $L(G)$ corresponding to one vertex is the negative of the sum of the remaining rows. The same holds for the columns, of course.

Definition 6 *Let V_0 be a subset of $V(G)$, which contains precisely one vertex from each connected component of G. The submatrix of $L(G)$ obtained by removing all rows and columns corresponding to elements of V_0 is a* reduced Laplacian *of G, denoted $L_{V_0}(G)$.*

Reduced Laplacian matrices inherit properties I, II and VI directly from $L(G)$. The reduced version of property V is an inequality: $Tr(L_{V_0}(G)) < 2(m - \ell)$ unless $\ell = m$, in which case $L_{V_0}(G)$ is the empty 0×0 matrix. The reduced version of property IV is the famous matrix-tree theorem: $\det L_{V_0}(G)$ is the number of maximal forests of G. Details are given in many standard references, e.g. [4, Theorem 13.2.1].

The reduced version of property III is complicated by the arbitrary choice of V_0. If G and G' are graphs then these two statements are equivalent: (a) when we adjoin a row and column to each of $L_{V_0}(G), L_{V_0'}(G')$ so that the row and column sums of both matrices are $\mathbf{0}$, we obtain matrices that are equal up to simultaneous permutation of the rows and columns; and (b) when we remove all loops from G and G', identify all the vertices from V_0 to each other, and identify all the vertices from V_0' to each other, we obtain isomorphic connected graphs.

We also use fairly standard terminology when discussing matrices associated with graphs. Recall that a square matrix of integers U is *unimodular* or *invertible over* $\mathbb{Z}$ if $\det U = \pm 1$.

Definition 7 *Two matrices B and B' are* strictly row equivalent *over $\mathbb{Z}$ if and only if $B' = UB$, where U is unimodular.*

Definition 8 *Two matrices B and B' are* loosely row equivalent *over $\mathbb{Z}$ if and only if*

$$\begin{pmatrix} B' \\ \mathbf{0} \end{pmatrix} = U \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix},$$

where U is unimodular and the two $\mathbf{0}$ submatrices may be of different sizes.

Row equivalence can also be described using elementary operations. Two matrices are strictly row equivalent over $\mathbb{Z}$ if and only if one can be obtained from the other using some finite sequence of elementary row operations over $\mathbb{Z}$, i.e., multiplying a row by -1, adding a nonzero multiple of one row to another and permuting rows. For loose row equivalence, it is also permissible to adjoin $\mathbf{0}$ rows, or remove them. A third way to describe row equivalence is that two k-column matrices are loosely row equivalent if and only if their rows generate the same subgroup of $\mathbb{Z}^k$. If two loosely row equivalent matrices have the same number of rows, then the matrices are strictly row equivalent. (The last assertion follows from properties of the Smith normal form of matrices with entries in $\mathbb{Z}$, cf. [5, Chapter 3] for instance.)

Definition 9 *Two matrices B and B' are* congruent *over $\mathbb{Z}$ if and only if $B' = UBU^T$, where U is unimodular and U^T is the transpose of U.*

Definition 10 *Let $\vec{G}$ denote an arbitrary directed version of G. Then the incidence matrix $N(\vec{G})$ is the $n \times m$ matrix whose entries are given by the following.*

$$N(\vec{G})_{ve} = \begin{cases} -1, & \text{if } v \text{ is the initial vertex of } e, \text{ and } e \text{ is not a loop} \\ 1, & \text{if } v \text{ is the terminal vertex of } e, \text{ and } e \text{ is not a loop} \\ 0, & \text{if } e \text{ is not incident on } v, \text{ or } e \text{ is a loop} \end{cases}$$

Definition 11 *Let V_0 be a subset of $V(G)$, which contains one vertex from each connected component of G. Then the submatrix of $N(\vec{G})$ obtained by removing all rows corresponding to elements of V_0 is a reduced incidence matrix of G, denoted $N_{V_0}(\vec{G})$.*

The following equalities are immediate.

Property VII If $V_0 \subseteq V(G)$ contains one vertex from each connected component of G, then

$$N(\vec{G}) \cdot N(\vec{G})^T = L(G) \text{ and } N_{V_0}(\vec{G}) \cdot N_{V_0}(\vec{G})^T = L_{V_0}(G).$$

If V_0' is another such subset of $V(G)$ then $N_{V_0'}(\vec{G})$ can be obtained from $N_{V_0}(\vec{G})$ as follows. For every connected component of G where V_0 contains a vertex v and V_0' contains a vertex $v' \neq v$, (a) add all the other rows of $N_{V_0}(\vec{G})$ corresponding to vertices from this connected component to the v' row, (b) multiply the new row by -1, and (c) label the new row with v rather than v'. If U is the product of elementary matrices corresponding to the row operations mentioned in (a) and (b), then

$$U \cdot N_{V_0}(\vec{G}) = N_{V_0'}(\vec{G}) \text{ and hence } U \cdot L_{V_0}(G) \cdot U^T = L_{V_0'}(G). \qquad (1)$$

We deduce the following elementary properties of the reduced matrices. Let V_0 and V_0' be two subsets of $V(G)$, each of which contains precisely one vertex from each connected component of G.

Property VIII $N_{V_0}(\vec{G})$ and $N_{V_0'}(\vec{G})$ are strictly row equivalent over $\mathbb{Z}$.

Property IX $L_{V_0}(G)$ and $L_{V_0'}(G)$ are congruent over $\mathbb{Z}$.

Notice that $\vec{G}$ appears in property VIII, while G appears in property IX. The difference is that changing the direction of an edge e does not affect $L_{V_0}(G)$, but it multiplies the e column of $N_{V_0}(\vec{G})$ by -1.

Formula (1) makes it clear that the row equivalence class of $N_{V_0}(\vec{G})$ determines the congruence class of $L_{V_0}(G)$. A natural question is this: does the congruence class of $L_{V_0}(G)$ also determine the row equivalence class of $N_{V_0}(\vec{G})$? Property X tells us that the answer is "yes."

Property X Let G_1 and G_2 be graphs with the same number of loops, and for $i \in \{1, 2\}$ let $V_{0i} \subseteq V(G_i)$ be a subset that contains one vertex from each connected component of each graph. Then any of the following conditions implies the others.

1. $L_{V_{01}}(G_1)$ and $L_{V_{02}}(G_2)$ are congruent over $\mathbb{Z}$.

2. There are oriented versions $\vec{G}_1, \vec{G}_2$ and a bijection $\beta : E(G_1) \to E(G_2)$ such that $N_{V_{01}}(\vec{G}_1)$ and $N_{V_{02}}(\vec{G}_2)$ are strictly row equivalent over $\mathbb{Z}$, when their columns are matched by β.

3. There are oriented versions $\vec{G}_1, \vec{G}_2$ and a bijection $\beta : E(G_1) \to E(G_2)$ such that $N(\vec{G}_1)$ and $N(\vec{G}_2)$ are loosely row equivalent over $\mathbb{Z}$, when their columns are matched by β.

4. G_1 and G_2 are 2-isomorphic. (I.e., their cycle matroids are isomorphic.)

The equivalence among conditions 2, 3 and 4 of property X is a famous theorem of Whitney [14], and there are many expositions in the literature. For instance, a thorough discussion is provided by Oxley [9, Chapter 5]. Note that the phrase "over $\mathbb{Z}$" is not important in conditions 2–4; these conditions remain equivalent if $\mathbb{Z}$ is replaced by a field. In fact most textbook presentations of the theory of incidence matrices, like those in [2, Chapter 2], [4, Chapter 8] and [9, Chapter 5], are formally restricted to fields; however the presentations are easily modified to work over $\mathbb{Z}$.

The fact that condition 1 of property X is equivalent to the other conditions is due to Watkins [11, 12]; this is Theorem 1 of the introduction. It is important to realize that "over $\mathbb{Z}$" is crucial in condition 1. In fact, condition 1 is not equivalent to the other conditions for any nontrivial graph over any field. For if F is a field, G is a nontrivial graph, $a > 1$ is an integer not divisible by the characteristic of F and $a^2 G$ is the graph obtained by replacing each edge of G with a^2 parallel edges, then G and $a^2 G$ are certainly not 2-isomorphic. However

$$L_{V_0}(a^2 G) = a^2 L_{V_0}(G) = (aI)L_{V_0}(G)(aI) = (aI)L_{V_0}(G)(aI)^T$$

and as aI is invertible over F, it follows that $L_{V_0}(a^2 G)$ is congruent to $L_{V_0}(G)$ over F. More details of property X, including a proof of the equivalence of condition 1 with conditions 2–4, are discussed in Section 6.

There are several equivalent ways to describe 2-isomorphisms. Two of them are stated in Definition 12. We refer to Oxley [9] for other equivalent descriptions, and a thorough account of their properties.

Definition 12 *Let G_1 and G_2 be graphs. Then a bijection $\beta : E(G_1) \to E(G_2)$ is a 2-isomorphism if it defines an isomorphism between the cycle matroids of G_1 and G_2. That is, β satisfies the following equivalent conditions.*

1. A subset $S \subseteq E(G_1)$ is the edge set of a maximal forest of G_1 if and only if $\beta(S)$ is the edge set of a maximal forest of G_2.

2. There are oriented versions $\vec{G}_1$ and $\vec{G}_2$ such that vectors in $\mathbb{Z}^{E(G_1)}$ corresponding to circuits of G_1 are matched by β to vectors in $\mathbb{Z}^{E(G_2)}$ corresponding to circuits of G_2.

Recall that a circuit in a graph is a minimal closed path. The vector corresponding to a circuit is obtained by following the circuit according to one of the two orientations, and placing ± 1 in the e coordinate of the vector for each edge e that appears on the circuit, with $+1$ (resp. -1) representing agreement (resp. disagreement) between the $\vec{G}$ direction of e and the direction of e on the circuit. The subgroup of $\mathbb{Z}^{E(G)}$ generated by these vectors is called the *cycle group* of G, or the *lattice of integral flows* of G.

The last property we discuss concerns the relationship between Laplacian matrices and maximal forests.

If M is a maximal forest of G and $\vec{M}$ inherits edge directions from $\vec{G}$ then the matrix-tree theorem tells us that $N_{V_0}(\vec{M})$ is a unimodular submatrix of $N_{V_0}(\vec{G})$, which includes the columns corresponding to edges of M. For convenience we adopt a notational shorthand: if $\vec{G} - E(M)$ is the directed graph obtained from $\vec{G}$ by removing all the edges of M, then we define

$$C(M) := N_{V_0}(\vec{M})^{-1} \cdot N_{V_0}(\vec{G} - E(M)).$$

This useful matrix appears in several references [1, 4, 9, 10], but it does not seem to have a standard name. We use the letter C because the rows represent the fundamental cuts of G with respect to M. More information about $C(M)$ is given in Sections 4 and 5.

If $\ell = m$ or G is a forest then $C(M)$ is the empty 0×0 matrix; otherwise, $C(M)$ is an $(n - c(G)) \times (m - n + c(G))$ matrix. Of course the $C(M)$ notation is incomplete, as it does not mention $\vec{G}$ or V_0. Notice also that

$$N_{V_0}(\vec{G}) = N_{V_0}(\vec{M}) \cdot \begin{pmatrix} I & C(M) \end{pmatrix} \cdot P_M,$$

where I is an identity matrix of order $n - c(G)$ and P_M is a permutation matrix that permutes the columns of $\begin{pmatrix} I & C(M) \end{pmatrix}$ into the order of $E(G)$ used for the columns of $N_{V_0}(\vec{G})$. Permutation matrices satisfy $P_M P_M^T = I$, so

$$L_{V_0}(G) = N_{V_0}(\vec{G}) \cdot N_{V_0}(\vec{G})^T = N_{V_0}(\vec{M}) \cdot \begin{pmatrix} I + C(M)C(M)^T \end{pmatrix} \cdot N_{V_0}(\vec{M})^T.$$

If $G = M$ is a forest then $C(M)$ is empty, and the equation holds with $I + C(M)C(M)^T = I$. If $m = \ell$, on the other hand, then the equation holds vacuously – all the matrices are empty.

We deduce the following.

Property XI The congruence class of $I + C(M)C(M)^T$ over $\mathbb{Z}$ is the same as that of $L_{V_0}(G)$.

Property XI tells us that we may think of the reduced forms of properties I – X as applying to $I + C(M)C(M)^T$ matrices rather than $L_{V_0}(G)$ matrices, up to congruence over $\mathbb{Z}$. For instance the equivalence between conditions 4 and 1 of property X may be rephrased like this: G_1 and G_2 are 2-isomorphic if and only if there are oriented versions $\vec{G}_i$, maximal forests M_i, and subsets $V_{0i} \subseteq V(G_i)$ containing one vertex from each connected component of each graph, such that $I + C(M_1)C(M_1)^T$ and $I + C(M_2)C(M_2)^T$ are congruent over $\mathbb{Z}$.

3 An example

Before discussing dual Laplacian matrices, we consider a small example.

Suppose G has two vertices and three parallel non-loop edges. A maximal forest M of G consists of one edge. Depending on the edge directions, $C(M)$ is $\begin{pmatrix} 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -1 \end{pmatrix}$. In every case, $I + C(M)C(M)^T$ is the 1×1 matrix whose only entry is 3; this is the same as $L_{V_0}(G)$.

Foreshadowing the results of the next sections, notice that if I' is the 2×2 identity matrix then $I' + C(M)^T C(M)$ is one of these two matrices.

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The first matrix is the reduced Laplacian of K_3, the dual graph of G. The trace of the unreduced Laplacian $L(K_3)$ is 6. In contrast, the second matrix is not a reduced Laplacian of any graph. Moreover, if we enlarge this matrix to a 3×3 matrix whose rows and columns sum to $\mathbf{0}$ we get a matrix of trace 10,

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ -3 & -3 & 6 \end{pmatrix}.$$

4 Dual Laplacian matrices

Here is another famous definition of Whitney [13]; again, we refer to Oxley [9] for a thorough discussion.

Definition 13 *Let G_1 and G_2 be graphs. Then G_1 and G_2 are* abstract duals *if and only if there is a bijection $\beta : E(G_1) \to E(G_2)$ that defines an isomorphism between the cycle matroid of G_1 and the bond matroid of G_2. That is, β satisfies the following equivalent conditions.*

1. *A subset $S \subseteq E(G_1)$ is the edge set of a maximal forest of G_1 if and only if $\beta(S)$ is the complement of the edge set of a maximal forest of G_2.*

2. *There are oriented versions $\vec{G}_1$ and $\vec{G}_2$ such that vectors in $\mathbb{Z}^{E(G_1)}$ corresponding to circuits of G_1 are matched by β to vectors in $\mathbb{Z}^{E(G_2)}$ corresponding to edge cuts of G_2.*

If W is a proper subset of $V(G)$ then the vector corresponding to the edge cut determined by W is obtained by placing ± 1 in the e coordinate of the vector for each non-loop edge e that is incident on just one vertex of W, with $+1$ (resp. -1) representing an edge directed toward W (resp. away from W) in $\vec{G}$. The subgroup of $\mathbb{Z}^{E(G)}$ generated by these vectors is called the *cut group* of G, or the *lattice of integral cuts* of G.

It is easy to see from Definitions 12 and 13 that there is a strong connection between 2-isomorphism and abstract duality: if G_1 and G_2 are abstract duals, then every graph 2-isomorphic to G_1 is an abstract dual of every graph 2-isomorphic to G_2. It is not so easy to see another famous theorem of Whitney [13]: G has an abstract dual if and only if G is planar.

It turns out that if I' is an identity matrix of order $m - n + c(G)$, then almost all of the fundamental properties of Laplacians listed in Section 2 have analogues for matrices of the form $I' + C(M)^T C(M)$.

Definition 14 *If G is a graph with a maximal forest M then any matrix congruent over $\mathbb{Z}$ to $I' + C(M)^T C(M)$ is a* reduced dual Laplacian *matrix of G.*

When $C(M)$ is the empty matrix – i.e., when $m = \ell$ or G is a forest – the matrix $I' + C(M)^T C(M)$ of Definition 5 should be interpreted as being equal to I'. It is empty if G is a forest.

Definition 15 *If G is a graph then an* unreduced *dual Laplacian matrix of G is obtained by adjoining a row and column to a reduced dual Laplacian matrix of G, so that the rows and columns of the resulting matrix sum to* **0**.

Notice that compared to Definitions 5 and 6, Definitions 14 and 15 are "backward": we start with reduced dual Laplacian matrices, and construct unreduced dual Laplacian matrices by enlarging the reduced ones. To make sure there is no misunderstanding we should emphasize that dual Laplacians do not require dual graphs: every graph has reduced and unreduced dual Laplacian matrices, whether the graph is planar or nonplanar. If $G = M$ is a forest, the only reduced dual Laplacian matrix of G is the empty 0×0 matrix, and the only unreduced dual Laplacian matrix of G is the

1×1 matrix $\mathbf{0}$. Otherwise the reduced dual Laplacian matrices of G are symmetric $(m - n + c(G)) \times (m - n + c(G))$ matrices.

In general we use $*$ to indicate dual Laplacian matrices and their properties. For instance $L_{V_0}^*(G)$ denotes a reduced dual Laplacian of G obtained using V_0, and $L^*(G)$ denotes an unreduced dual Laplacian matrix of G. It is important to keep in mind that unlike $L_{V_0}(G)$ and $L(G)$, the notations $L_{V_0}^*(G), L^*(G)$ are not well defined. In consequence there is no property III*. However, we will see in Section 5 that these matrices satisfy the following property.

Property IX* $L_{V_0}^*(G)$ and $L^*(G)$ are well defined up to congruence over $\mathbb{Z}$.

That is, the reduced dual Laplacian matrices of G are all congruent over $\mathbb{Z}$, and the unreduced dual Laplacian matrices of G are all congruent over $\mathbb{Z}$.

Here are some other properties of dual Laplacian matrices.

Property I* $L^*(G)$ and $L_{V_0}^*(G)$ are symmetric matrices with integer entries.

Property II* $L^*(G)$ and $L_{V_0}^*(G)$ are not changed if bridges are added to G or removed from G.

Property IV* The sum of the columns of $L^*(G)$ is $\mathbf{0}$; and the same for the rows. The reduced version of property IV* is that reduced dual Laplacian matrices satisfy the matrix-tree theorem, just as reduced Laplacian matrices do. That is, $\det(I' + C(M)^T C(M))$ is the number of maximal forests of G [4, Theorem 14.7.3].

Property II* implies that the dual version of property VI is rather different from the original:

Property VI* If G is not connected then any connected graph obtained by adding bridges to G has the same L^* and $L_{V_0}^*$ matrices as G.

Before stating a property VII*, it is helpful to discuss $C(M)$ a little more. Recall that $C(M) = N_{V_0}(\vec{M})^{-1} \cdot N_{V_0}(\vec{G} - E(M))$. The rows of $C(M)$ correspond to the rows of $N_{V_0}(\vec{M})^{-1}$, which are indexed by the same set that indexes the columns of $N_{V_0}(\vec{M})$, i.e., $E(M)$. The columns of $C(M)$ correspond to the columns of $N_{V_0}(\vec{G} - E(M))$, which are indexed by the edges of $G - E(M)$.

Now, consider the matrix $\begin{pmatrix} C(M)^T & -I' \end{pmatrix}$. The columns of I' inherit an indexing from the rows of $C(M)^T$, which are the columns of $C(M)$; so the columns of I' are indexed by the edges of $G - E(M)$. Of course the columns of $C(M)^T$ are the rows of $C(M)$, and as was just discussed they are indexed by $E(M)$. All in all, then, the columns of $\begin{pmatrix} C(M)^T & -I' \end{pmatrix}$ are indexed by the edges of G. We define $F(M)$ to be the matrix

$$F(M) := \begin{pmatrix} C(M)^T & -I' \end{pmatrix} \cdot P_M,$$

where P_M is a permutation matrix that permutes the columns of

$$\begin{pmatrix} C(M)^T & -I' \end{pmatrix}$$

into the order of $E(G)$ used for the columns of $N_{V_0}(\vec{G})$, as before. N.b. Like $C(M)$, the notation $F(M)$ is incomplete; it does not mention either $\vec{G}$ or V_0.

It turns out that $F(M)$ plays a role dual to that of $N_{V_0}(\vec{G})$. We give more details in Section 5, but we can certainly see the following.

Property VII* If $\widehat{F}(M)$ is the matrix obtained from $F(M)$ by adjoining a new row equal to the negative of the sum of the rows of $F(M)$, then

$$F(M)F(M)^T = I' + C(M)^T C(M) = L_{V_0}^*(G) \ \text{ and } \ \widehat{F}(M)\widehat{F}(M)^T = L^*(G).$$

Recall that properties VIII and IX differ in that the former involves $\vec{G}$ and the latter involves G. In Section 5 we see that there is an analogous difference between properties VIII* and IX*.

Property VIII* For a fixed choice of edge directions in $\vec{G}$, the $F(M)$ matrices that arise from different choices of M and V_0 are all strictly row equivalent to each other over $\mathbb{Z}$.

Before proceeding we take a moment to describe the effect on $F(M)$ of changing the direction of an edge e, while holding M and V_0 fixed. (a) If $e \notin E(M)$, then reversing the direction of e has the effect of multiplying the e column of $N_{V_0}(\vec{G} - E(M))$ by -1. This in turn has the effect of multiplying the e column of $C(M) = N_{V_0}(\vec{M})^{-1} \cdot N_{V_0}(\vec{G} - E(M))$ by -1. The effect on $F(M) = \begin{pmatrix} C(M)^T & -I' \end{pmatrix} \cdot P_M$ is to multiply the e row of $C(M)^T$ by -1 while leaving the $-I'$ block of $F(M)$ unchanged. Of course multiplying *part* of a row by -1 is not an elementary row operation. (b) If $e \in E(M)$, then reversing the direction of e has the effect of multiplying the e column of $N_{V_0}(\vec{M})$ by -1. This in turn has the effect of multiplying the e row of $C(M) = N_{V_0}(\vec{M})^{-1} \cdot N_{V_0}(\vec{G} - E(M))$ by -1. The effect on $F(M)$ is to multiply the e column of $C(M)^T$ by -1. Again, this effect is not an elementary row operation. These observations explain why property VIII* requires a fixed choice of edge directions.

On the other hand, Property IX* does not require a fixed choice of edge directions. The preceding paragraph gives two reasons for this. (a) If $e \notin E(M)$ then multiplying the e column of $C(M)$ by -1 has the effect of replacing $C(M)$ with $C(M)U$, where U is the elementary matrix corresponding to the column multiplication. As $U^T U = I'$, the effect on $I' + C(M)^T C(M)$ is to replace it with

$$I' + U^T C(M)^T C(M)U = U^T \cdot (I' + C(M)^T C(M)) \cdot U,$$

which is congruent to $I' + C(M)^T C(M)$ over $\mathbb{Z}$. (b) If $e \in E(M)$ then multiplying the e row of $C(M)$ by -1 has no effect on $I' + C(M)^T C(M)$.

So far, we have stated properties I*, II*, IV*, VI*, VII*, VIII* and IX*. There is no property III*, and property XI* is Definition 14. The two remaining dual properties, V* and X*, are Theorems 3 and 4, stated in the introduction.

5   Properties II*, VIII* and IX*

Let $\vec{G}$ be a directed version of a graph G, and M a maximal forest of G. Let I' be the identity matrix of order $m - n + c(G)$, and let $F(M)$ be the matrix

$$F(M) = \begin{pmatrix} C(M)^T & -I' \end{pmatrix} \cdot P_M$$

mentioned above. Then $F(M)F(M)^T = I' + C(M)^T C(M)$ is a reduced dual Laplacian matrix of G. Also

$$F(M) \cdot N_{V_0}(\vec{G})^T = \begin{pmatrix} C(M)^T & -I' \end{pmatrix} \cdot P_M P_M^T \cdot \begin{pmatrix} I \\ C(M)^T \end{pmatrix} \cdot N_{V_0}(\vec{M})^T$$

$$= \begin{pmatrix} C(M)^T - C(M)^T \end{pmatrix} \cdot N_{V_0}(\vec{M})^T = \mathbf{0},$$

so each row of $F(M)$ is orthogonal to all the rows of $N_{V_0}(\vec{G})$.

Notice that $F(M)$ is an $(m - n + c(G)) \times m$ matrix and $N_{V_0}(\vec{G})$ is an $(n - c(G)) \times m$ matrix. Both matrices have linearly independent rows, so it follows that the row spaces of these two matrices are orthogonal complements in the vector space $\mathbb{Q}^m$. Because of the I and $-I'$ blocks of

$$N_{V_0}(\vec{M})^{-1} \cdot N_{V_0}(\vec{G}) \cdot P_M^{-1} = \begin{pmatrix} I & C(M) \end{pmatrix} \text{ and } F(M) \cdot P_M^{-1} = \begin{pmatrix} C(M)^T & -I' \end{pmatrix},$$

it is easy to deduce that the groups generated by the rows of $F(M)$ and $N_{V_0}(\vec{G})$ are orthogonal complements in the free abelian group $\mathbb{Z}^{E(G)}$. That is, the rows of $F(M)$ generate the cycle group (also called the lattice of integral flows) of G. The fact that the rows of $F(M)$ represent a basis of the cycle group implies directly that $F(M)F(M)^T$ is a Gram matrix for the lattice of integral flows of G, as mentioned by Godsil and Royle [4, Chapter 14].

Each row of $F(M)$ has precisely one nonzero entry from I', so each row of $F(M)$ corresponds to a circuit of G that includes precisely one edge outside M. That is, the rows of $F(M)$ represent the *fundamental circuits* of G with respect to M. The observation of the preceding paragraph – that the cycle group of G is generated by the fundamental circuits with respect to M, for every maximal forest M – is a well-known elementary property of the fundamental circuits. In textbooks of graph theory or matroid theory

like [2] or [9], this elementary property of fundamental circuits is often stated only for cycle spaces defined over fields; but as noted above it is easy to deduce the integral version, because of the I and $-I'$ blocks in the matrices. The statement over $\mathbb{Z}$ is more common in textbooks of algebraic topology, like [8]; it is also discussed by Bacher, de la Harpe and Nagnibeda [1, Lemma 2].

The same cycle group is generated by the rows of $F(M)$, independent of the choices of M and V_0. We deduce property VIII*: All of the $F(M)$ matrices associated with $\vec{G}$ are strictly row equivalent over $\mathbb{Z}$.

That is, if M and M' are maximal forests of G then $UF(M) = F(M')$ for some unimodular matrix U. It follows that

$$U\left(I' + C(M)^T C(M)\right) U^T = UF(M)F(M)^T U^T \tag{2}$$
$$= F(M')F(M')^T = I' + C(M')^T C(M'),$$

so $I' + C(M)^T C(M)$ and $I' + C(M')^T C(M')$ are congruent over $\mathbb{Z}$. We conclude that all the reduced dual Laplacian matrices of $\vec{G}$ provided by Definition 14 are congruent to each other over $\mathbb{Z}$. As discussed at the end of Section 4, changing edge directions does not affect reduced dual Laplacian matrices, up to congruence; it follows that $\vec{G}$ may be replaced by G in the preceding sentence. This is the reduced form of property IX*.

For the unreduced form of property IX*, notice that (2) implies that if $L^*(G)$ and $L'^*(G)$ are the matrices obtained from $I' + C(M)^T C(M)$ and $I' + C(M')^T C(M')$ (respectively) by adjoining a new row and column so that the row and column sums are $\mathbf{0}$, then

$$L'^*(G) = \begin{pmatrix} 1 & -\mathbf{1} \\ \mathbf{0} & I' \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I' + C(M')^T C(M') \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ -\mathbf{1} & I' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\mathbf{1} \\ \mathbf{0} & I' \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I' + C(M)^T C(M) \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U^T \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ -\mathbf{1} & I' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\mathbf{1} \\ \mathbf{0} & I' \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{pmatrix} \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{0} & I' \end{pmatrix} L^*(G) \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{1} & I' \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U^T \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ -\mathbf{1} & I' \end{pmatrix}.$$

If M is a maximal forest of G and e is a bridge of G then $e \in E(M)$ and e does not appear in any circuit of G, so every entry of the e column of $F(M)$ is 0. It follows that $F(M)F(M)^T$ is exactly the same as the reduced dual Laplacian matrix $F(M - e)F(M - e)^T$ of $G - e$. This is property II*.

As mentioned above, the rows of $F(M)$ correspond to fundamental circuits with respect to M. Circuit-cutset duality is reflected in the fact that the rows of $\begin{pmatrix} I & C(M) \end{pmatrix}$ correspond to fundamental cuts with respect to M, and this fact implies that $\begin{pmatrix} I & C(M) \end{pmatrix} \cdot \begin{pmatrix} I & C(M) \end{pmatrix}^T = I + C(M)C(M)^T$ is a Gram matrix for the lattice of integral cuts of G. See [1, Lemma 2] or [4, Theorem 14.2.4] for a detailed discussion.

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6   Theorems 1, 3 and 4

The following matrix result will be useful. The argument is adapted from [11].

Lemma 16 *Let A be an $n \times m$ integer matrix, and let a be the number of nonzero columns in A. Then either of these two conditions implies the other.*

1. *There are a directed graph $\vec{G}$ and a unimodular matrix C such that $CA = N(\vec{G})$.*

2. *There are a symmetric integer matrix B and a unimodular matrix C such that $B = CAA^T C^T$, the row sum of B is $\mathbf{0}$, and $Tr(B) \leq 2a$.*

If C satisfies one condition then C also satisfies the other condition. Moreover, every matrix B in condition 2 has $Tr(B) = 2a$.

Proof. For the implication $1 \implies 2$, suppose C is unimodular and $N(\vec{G}) = CA$. Then the number a of nonzero columns of A is the same as the number $m - \ell$ of nonzero columns of $N(\vec{G})$. The matrix $B = CAA^T C^T = L(G)$ has row sum $\mathbf{0}$ and trace $Tr(B) = 2a = 2(m - \ell)$ by properties IV and V of unreduced Laplacian matrices.

For $2 \implies 1$, suppose C is unimodular and $B = CAA^T C^T$ has row sum $\mathbf{0}$ and trace $Tr(B) \leq 2a$. If $\mathbf{1}$ denotes a vector whose entries are all 1 then $\mathbf{1} \cdot B = \mathbf{0}$, because the rows of B sum to $\mathbf{0}$. Hence $0 = \mathbf{1} \cdot B \cdot \mathbf{1} = \mathbf{1} \cdot CAA^T C^T \cdot \mathbf{1} = (\mathbf{1} \cdot CA) \cdot (\mathbf{1} \cdot CA)^T = \|\mathbf{1} \cdot CA\|^2$, so $\mathbf{1} \cdot CA = \mathbf{0}$. That is, the rows of CA sum to $\mathbf{0}$. It follows that each nonzero column of CA has at least one positive entry and at least one negative entry.

As C is nonsingular, A and CA both have a nonzero columns. The trace $Tr(B) = Tr(CA \cdot (CA)^T)$ is the sum of the squares of the entries of CA, so since every nonzero column of CA has at least two nonzero entries,

$$Tr(B) = \sum_{i,j} (CA)_{ij}^2 \geq 2a,$$

with equality only if every nonzero column of CA has exactly two nonzero entries, both of absolute value 1.

The hypothesis $Tr(B) \leq 2a$ implies that the equality $Tr(B) = 2a$ holds. The rows of CA sum to $\mathbf{0}$, so it follows that every nonzero column of CA has exactly two nonzero entries, $+1$ and -1. That is, CA is the incidence matrix of a directed graph. ∎

6.1 Proof of Theorem 1

Recall that property X includes Theorem 1. As discussed in Section 2, the equivalence of conditions 2, 3 and 4 of property X is well known. The implication $2 \implies 1$ follows immediately from property VII. For $1 \implies 2$, suppose G_1 and G_2 are graphs each of which has ℓ loops, and suppose U is a unimodular matrix with $U L_{V_{01}}(G_1) U^T = L_{V_{02}}(G_2)$. We may assume without loss of generality that $|E(G_2)| = m_2 \leq m_1 = |E(G_1)|$.

Let G_1' be the connected graph obtained from G_1 by identifying all the vertices of V_{01} to a single vertex v_1, and let G_2' be the connected graph obtained from G_2 by identifying all the vertices of V_{02} to a single vertex w_1. Let $V_{01}' = \{v_1\}$ and $V_{02}' = \{w_1\}$. Then $L_{V_{01}}(G_1) = L_{V_{01}'}(G_1')$ and $L_{V_{02}}(G_2) = L_{V_{02}'}(G_2')$, so $U L_{V_{01}'}(G_1') U^T = L_{V_{02}'}(G_2')$. Let

$$W = \begin{pmatrix} 1 & -\mathbf{1} \\ \mathbf{0} & I \end{pmatrix}, \quad X = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{0} & I \end{pmatrix},$$

where I is an identity matrix. Let $Z = WXY$, and order the vertices of $V(G_1')$ and $V(G_2')$ with v_1 and w_1 first (respectively). Then

$$ZL(G_1')Z^T = WX \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & L_{V_{01}'}(G_1) \end{pmatrix} X^T W^T = W \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & L_{V_{02}'}(G_2) \end{pmatrix} W^T,$$

which is $L(G_2')$. Let $B = L(G_2') = ZN(\vec{G}_1')N(\vec{G}_1')^T Z^T$. Properties IV and V of Laplacian matrices tell us that the row sum of B is $\mathbf{0}$ and $Tr(B) = 2(m_2 - \ell) \leq 2(m_1 - \ell)$, which is twice the number of nonzero columns of $N(\vec{G}_1')$. Applying Lemma 16 with $A = N(\vec{G}_1')$ and $C = Z$, we conclude that there is a directed graph $\vec{G}_3$ such that $ZN(\vec{G}_1') = N(\vec{G}_3)$. Moreover,

$$L(G_3) = N(\vec{G}_3)N(\vec{G}_3)^T = ZN(\vec{G}_1')\left(ZN(\vec{G}_1')\right)^T = ZL(G_1')Z^T = L(G_2'),$$

so property III tells us that G_3 is isomorphic to G_2', except possibly for the placement of loops. Loop placement does not affect incidence matrices, so we conclude that $ZN(\vec{G}_1') = N(\vec{G}_2')$, i.e., the unreduced incidence matrices of $\vec{G}_1'$ and $\vec{G}_2'$ are strictly row equivalent over $\mathbb{Z}$. It follows that $N_{V_{01}'}(\vec{G}_1')$ and $N_{V_{02}'}(\vec{G}_2')$ are also strictly row equivalent over $\mathbb{Z}$; these are the same matrices as $N_{V_{01}}(\vec{G}_1)$ and $N_{V_{02}}(\vec{G}_2)$.

6.2 The unreduced version of Theorem 1

The unreduced version of Theorem 1 is not so different from the reduced version, but we provide details for the sake of completeness.

Lemma 17 *The rank of $L(G)$ over $\mathbb{Q}$ is $n - c(G)$.*

Proof. Properties IV and VI tell us that the rank of $L(G)$ over $\mathbb{Q}$ is no more than $n - c(G)$. The matrix-tree theorem tells us that a reduced Laplacian of G is a nonsingular submatrix of $L(G)$, of order $n - c(G)$. ∎

Lemma 18 *If V_0 includes one vertex from each connected component of G then $L(G)$ is congruent over $\mathbb{Z}$ to the matrix*

$$\begin{pmatrix} L_{V_0}(G) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Proof. Properties IV and VI tell us that the displayed matrix is $U L(G) U^T$, where U is obtained from an identity matrix by changing the vw entry to 1 whenever $v \in V_0$, $v \neq w$ and v, w lie in the same connected component of G. ∎

Corollary 19 *Suppose the unreduced Laplacian matrices of G_1 and G_2 are congruent over $\mathbb{Z}$. Then the reduced Laplacian matrices of G_1 and G_2 are congruent over $\mathbb{Z}$.*

Proof. As $L(G_1)$ and $L(G_2)$ are congruent over $\mathbb{Z}$, Lemma 18 tells us that

$$A_1 = \begin{pmatrix} L_{V_{01}}(G_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} L_{V_{02}}(G_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

are also congruent over $\mathbb{Z}$. Hence there is a unimodular matrix U with $U A_1 U^T = A_2$. Also, the fact that $L(G_1)$ and $L(G_2)$ are congruent implies that they have the same rank; so according to Lemma 17, $L_{V_{01}}(G_1)$ and $L_{V_{02}}(G_2)$ have the same size. It follows that

$$U A_1 U^T = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} L_{V_{01}}(G_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} U_1^T & U_3^T \\ U_2^T & U_4^T \end{pmatrix} = A_2$$

requires $U_1 L_{V_{01}}(G_1) U_1^T = L_{V_{02}}(G_2)$ and $U_1 L_{V_{01}}(G_1) U_3^T = \mathbf{0}$.

As $L_{V_{01}}(G_1)$ and $L_{V_{02}}(G_2)$ are both nonsingular, $U_1 L_{V_{01}}(G_1) U_1^T = L_{V_{02}}(G_2)$ implies that U_1 is nonsingular too. Then $U_1 L_{V_{01}}(G_1) U_3^T = \mathbf{0}$ implies that $U_3 = \mathbf{0}$, so $\det(U) = \det(U_1) \det(U_4)$. Necessarily then U_1 is unimodular, so $U_1 L_{V_{01}}(G_1) U_1^T = L_{V_{02}}(G_2)$ implies that $L_{V_{01}}(G_1)$ and $L_{V_{02}}(G_2)$ are congruent over $\mathbb{Z}$. ∎

Here is the unreduced version of Theorem 1.

Proposition 20 *Let G_1 and G_2 be graphs with the same number of loops. Then $L(G_1)$ and $L(G_2)$ are congruent over $\mathbb{Z}$ if and only if G_1 and G_2 are 2-isomorphic graphs with the same number of vertices and the same number of connected components.*

Proof. If $L(G_1)$ and $L(G_2)$ are congruent over $\mathbb{Z}$, then they certainly have the same rank and size. It follows that G_1 and G_2 have the same values for $n - c(G)$ and n, so G_1 and G_2 have the same number of vertices and the same number of connected components. Corollary 19 and Theorem 1 tell us that G_1 and G_2 are 2-isomorphic.

For the converse, suppose G_1 and G_2 are 2-isomorphic graphs with the same number of vertices and the same number of connected components. Then the matrices

$$A_1 = \begin{pmatrix} L_{V_{01}}(G_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} L_{V_{02}}(G_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

have the same size. Theorem 1 tells us that $L_{V_{01}}(G_1)$ and $L_{V_{02}}(G_2)$ are congruent over $\mathbb{Z}$, so A_1 and A_2 are congruent over $\mathbb{Z}$. According to Lemma 18, it follows that $L(G_1)$ and $L(G_2)$ are congruent over $\mathbb{Z}$. $\blacksquare$

6.3 Proof of Theorem 3

Let G be a graph with m edges and b bridges. Recall that Theorem 3 has two parts. 1. If $L^*(G)$ is an unreduced dual Laplacian matrix of G then $Tr(L^*(G))$ is an even integer $\geq 2(m - b)$. 2. G is planar if and only if G has an unreduced dual Laplacian matrix with $Tr(L^*(G)) = 2(m - b)$.

It is easy to verify that $Tr(L^*(G))$ is an even integer. The row sum of $L^*(G)$ is $\mathbf{0}$, so the sum of the entries of $L^*(G)$ is 0. It follows that $-Tr(L^*(G))$ is the sum of the off-diagonal entries of $L^*(G)$; this sum is even because $L^*(G)$ is symmetric.

It is also easy to verify one direction of part 2. If G is planar then G has an abstract dual G^*, and Theorem 4 tells us that $L(G^*)$ is an unreduced dual Laplacian matrix of G. (Theorem 4 is proven below; there is no circularity because the proof does not involve Theorem 3.) As G^* has m edges and b loops, property V guarantees that $Tr(L(G^*)) = 2(m - b)$.

We verify part 1 and the other direction of part 2 simultaneously, by proving that if $L^*(G)$ is an unreduced dual Laplacian matrix of G with $Tr(L^*(G)) \leq 2(m - b)$ then $Tr(L^*(G)) = 2(m - b)$ and G is planar.

Suppose that G is a graph with an unreduced dual Laplacian matrix $L^*(G)$ such that $Tr(L^*(G)) \leq 2(m - b)$. According to Definitions 14 and 15, G has a maximal forest M such that $L^*(G)$ is obtained from a matrix congruent to $I' + C(M)^T C(M)$ by adjoining a row and column to make the row and column sums equal to $\mathbf{0}$. Let

$$F(M) = \begin{pmatrix} C(M)^T & -I' \end{pmatrix} \cdot P_M$$

as in Section 4, and let A be the matrix obtained from $F(M)$ by adjoining a new first row with all entries equal to 0. Then the number of nonzero

columns of A is the same as the number of nonzero columns of $F(M)$, and according to the discussion in Section 5 this is the number of edges of G that appear in circuits of G. That is, A has $a = m - b$ nonzero columns.

Suppose U is unimodular and $L^*(G)$ is obtained by adjoining a row and column to $U(I' + C(M)^T C(M))U^T$, in such a way that the row and column sums equal $\mathbf{0}$. Then we have

$$L^*(G) = \begin{pmatrix} 1 & -\mathbf{1} \\ \mathbf{0} & I' \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & U(I' + C(M)^T C(M))U^T \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ -\mathbf{1} & I' \end{pmatrix}$$

$$= ZAA^T Z^T, \text{ where } Z = \begin{pmatrix} 1 & -\mathbf{1} \\ \mathbf{0} & I' \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{pmatrix} = \begin{pmatrix} 1 & -\mathbf{1} \cdot U \\ \mathbf{0} & U \end{pmatrix}.$$

Then A, $B = L^*(G)$ and $C = Z$ satisfy part 2 of Lemma 16, so the lemma guarantees that $Tr(L^*(G)) = 2a = 2(m - b)$ and there is a directed graph $\vec{G}^*$ such that $ZA = N(\vec{G}^*)$. The group generated by the rows of $N(\vec{G}^*)$ is the group of cuts of G^*, and as noted at the beginning of Section 5, the group generated by the rows of $F(M)$ is the group of cycles of G. The equation $ZA = N(\vec{G}^*)$ implies that these two groups are the same, so if $\beta : E(G) \to E(G^*)$ is the bijection that matches edges according to the correspondence between columns of A and $N(\vec{G}^*)$, then cuts of G^* correspond to cycles of G under β. That is, G and G^* are abstract duals; hence both are planar.

6.4   Theorem 3 and MacLane's criterion

The planarity criterion of MacLane [7] is this: G is planar if and only if there is a $GF(2)$ basis for its cycle space, in which each edge appears no more than twice. If we augment such a basis with one more element, equal (modulo 2) to the sum of the basis elements, then the resulting set has the property that every non-bridge edge appears precisely twice.

In one direction, the relationship with Theorem 3 is simple. If $L^*(G)$ is an unreduced dual Laplacian matrix of G, then there is a $\mathbb{Z}$ basis B of the group of cycles of G, such that $L^*(G)$ records the dot products among the vectors in the set B' obtained by augmenting B with one more element, equal to the negative of the sum of the elements of B. Notice that every non-bridge edge of G is represented at least once among the elements of B, and at least twice among the elements of B'. Each diagonal entry of $L^*(G)$ is a positive integer, at least as large as the number of edges represented in the corresponding element of B'. (A diagonal entry will be larger than the number of edges represented in the corresponding element of B' if the absolute value of some coordinate of that element is more than 1.) It follows that $Tr(L^*(G)) \geq 2(m - b)$, with equality only if each non-bridge edge is represented in precisely two elements of B'. Clearly then

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$Tr(L^*(G)) = 2(m - b)$ implies that B satisfies MacLane's criterion.

The opposite direction is not so immediate, as MacLane's criterion provides only a $GF(2)$ basis, not a $\mathbb{Z}$ basis. Of course if G satisfies MacLane's criterion then G is planar, and it is easy to verify Theorem 3 for planar graphs, as indicated in Subsection 6.3.

6.5 Proof of Theorem 4

Let G be a planar graph. Theorem 4 asserts that these two statements about a graph G^* are equivalent. 1. G and G^* are abstract duals. 2. The number of loops in G^* is the same as the number of bridges in G, and a reduced Laplacian matrix of G^* is a reduced dual Laplacian matrix of G.

If G and G^* are abstract duals then there are oriented versions $\vec{G}, \vec{G}^*$ and a bijection $\beta : E(G) \to E(G^*)$ under which the cycle vectors of G correspond to the cut vectors of G^*. If we match the columns of $F(M)$ and $N_{V_0^*}(\vec{G}^*)$ according to β, then the rows of $F(M)$ and $N_{V_0^*}(\vec{G}^*)$ generate the same group. Recall that $F(M)$ has $m - n + c(G)$ rows by definition, and the number of rows in $N_{V_0^*}(\vec{G}^*)$ is the same as the number of edges in a maximal forest of G^*. As G and G^* are abstract duals, the number of edges in a maximal forest of G^* is $m - |E(M)| = m - (n - c(G))$, the same as the number of rows in $F(M)$. It follows that $F(M)$ and $N_{V_0^*}(\vec{G}^*)$ are strictly row equivalent over $\mathbb{Z}$, so there is a unimodular U with $N_{V_0^*}(\vec{G}^*) = UF(M)$. Then $L_{V_0^*}(G^*) = N_{V_0^*}(\vec{G}^*)N_{V_0^*}(\vec{G}^*)^T = UF(M)F(M)^T U^T$ is a reduced dual Laplacian matrix of G. Also, the number of loops in G^* is the number of **0** columns of $N_{V_0^*}(\vec{G}^*)$, and the number of bridges in G is the number of **0** columns of $F(M)$; if the matrices are row equivalent these numbers must be equal. This verifies the implication $1 \implies 2$.

Suppose condition 2 holds. As G is planar, it has an abstract dual D. Applying the implication $1 \implies 2$ to D in place of G^*, we conclude that the number of loops in G^* is the same as the number of loops in D, and both $L_{V_0^*}(G^*)$ and a reduced Laplacian of D are reduced dual Laplacians of G. But then $L_{V_0^*}(G^*)$ and a reduced Laplacian of D are congruent to each other over $\mathbb{Z}$, so Theorem 1 tells us that G^* and D are 2-isomorphic. As D is an abstract dual of G, so is G^*.

7   Two examples

Example 1 Suppose G is the graph pictured in Figure 1, with bold edges indicating the spanning tree M with $E(M) = \{e_1, e_2, e_4, e_5, e_6, e_8\}$. Using
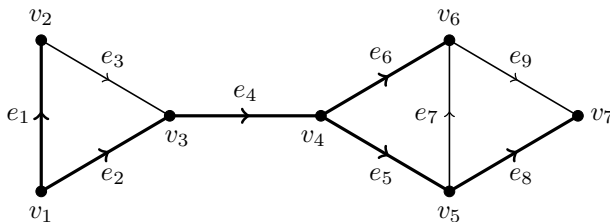
Figure 1: The graph G in Example 1.

the indicated edge directions and $V_0 = \{v_3\}$, we obtain

$$
\begin{array}{c}
\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array}
\left(
\begin{array}{ccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}
\right) = N(\vec{G}),
\end{array}
$$

so

$$
\left(
\begin{array}{ccccccccc}
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1
\end{array}
\right) = F(M)
$$

and

$$
\left(
\begin{array}{cccc}
6 & -3 & -1 & -2 \\
-3 & 3 & 0 & 0 \\
-1 & 0 & 3 & -2 \\
-2 & 0 & -2 & 4
\end{array}
\right) = \widehat{F}(M)\widehat{F}(M)^T,
$$

in the notation of Section 4.

Notice that the trace of $\widehat{F}(M)\widehat{F}(M)^T$ is $16 = 2(m-b)$, so in the notation of Subsection 6.3, U can be taken to be an identity matrix. As predicted by the argument of Subsection 6.3, it turns out that $ZA = \widehat{F}(M)$ is the incidence matrix of a graph G^*. This graph is pictured in Figure 2, with bold edges indicating the spanning tree M^* with $E(M^*) = \{e_3, e_7, e_9\}$. It is not difficult to verify that G^* is an abstract dual of G, but it happens that the two graphs are not geometric duals, i.e., they cannot be drawn together in the plane in such a way that each graph has one vertex in each complementary region of the other graph. One way to see this is to observe that there is no vertex of G^* incident only on e_7, e_8 and e_9, but every drawing of G has a complementary region with boundary $\{e_7, e_8, e_9\}$. This example illustrates the fact that Theorem 4 involves abstract rather than geometric duality.
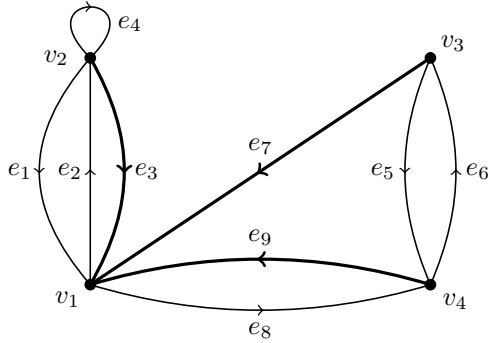
21

Figure 2: The graph G^* in Example 1.

Example 2 In Example 1 it happens that G has a maximal forest M such that $Tr(\widehat{F}(M)\widehat{F}(M)^T) = 2(m - b)$. That is, the planarity criterion of Theorem 3 is satisfied by an unreduced dual Laplacian matrix obtained directly from a matrix of the form $I' + C(M)^T C(M)$. It is not always the case that Theorem 3 is satisfied so readily. For instance, consider the graph G of Figure 3. Then G has $m = 9$ edges, none of which is a bridge. As G has 5 vertices, an unreduced dual Laplacian matrix $L^*(G)$ is a 6×6 matrix. The diagonal entries of $L^*(G)$ are the dot products with themselves of certain nonzero elements of the cycle group of G, and the smallest cycles of G are of length 3, so if $Tr(L^*(G)) = 18$ then each diagonal entry of $L^*(G)$ must correspond to a 3-cycle of G. This is not possible for an $\widehat{F}(M)\widehat{F}(M)^T$ matrix, because the $-I'$ block of $F(M)$ guarantees that the row adjoined to $F(M)$ in constructing $\widehat{F}(M)$ has more than $m-n+c(G) = 5$ nonzero entries. We leave it as an exercise for the reader to verify that nevertheless, G does have an unreduced dual Laplacian matrix $L^*(G)$ with $Tr(L^*(G)) = 18$.
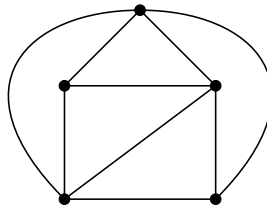

Figure 3: Example 2.

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