The second largest number of maximal independent sets in twinkle graphs

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Abstract

In this paper, we determine the second largest number of maximal independent sets and characterize those extremal graphs achieving these values among all twinkle graphs.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. An *independent set* is a subset S of V such that no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$. For a vertex $x \in V(G)$, let $\mathrm{MI}_{+x}(G) = \{I \in \mathrm{MI}(G) : x \in I\}$ and $\mathrm{MI}_{-x}(G) = \{I \in \mathrm{MI}(G) : x \notin I\}$. The cardinalities of $\mathrm{MI}_{+x}(G)$ and $\mathrm{MI}_{-x}(G)$ are denoted by $mi_{+x}(G)$ and $mi_{-x}(G)$, respectively. Note that $mi(G) = mi_{+x}(G) + mi_{-x}(G)$. The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [12]. The same problem was investigated for certain families of graphs, including trees [5, 13, 14], forests [5], graphs with at most one cycle [5], triangle-free graphs [1, 2].

A *twinkle graph* W is a connected unicyclic graph with the cycle C such that $W - x$ is disconnected for any $x \in V(C)$. Additionally, a connected graph G with vertex set $V(G)$ is called a *quasi-tree graph*, if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree. The concept of quasi-tree graphs was mentioned by H. Liu and M. Lu in [11]. Lin [8, 9] solved the largest and the second largest numbers of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order n. Trivially, the connected graphs with at most one cycle are the union of trees, twinkle graphs and quasi-tree graphs with exactly one cycle. Recently, Lin and Jou [10] investigated the largest cardinality of $mi(G)$ among all twinkle graphs of order n. In this paper, we determine the second largest number of maximal independent sets among all twinkle graphs. We also characterize those extremal graphs achieving these values.

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2 Preliminary

For a graph $G = (V, E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of vertices adjacent to v in G and the *closed neighborhood* $N_G[v]$ is $\{v\} \cup N_G(v)$. Two distinct vertices v_1 and v_2 are called *duplicated vertices* if $N_G(v_1) = N_G(v_2)$. The *degree* of x is the cardinality of $N_G(x)$, denoted by $\deg_G(x)$. A vertex x is a *leaf* if $\deg_G(x) = 1$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. If $A = \{v\}$ is a singleton, we write $G - v$ rather than $G - \{v\}$. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The short notation for the union of n copies of disjoint graphs isomorphic to G is nG. Denote by P_n a *path* with n vertices. The number of edges of a path is its length. Throughout this paper, for simplicity, let $r = \sqrt{2}$.

The batons $B(i, j)$ is the set which are the graphs obtained from a basic path P of $i \geq 1$ vertices by attaching $j \geq 0$ paths of length two to the endpoints of P in all possible ways (see Figure 1).

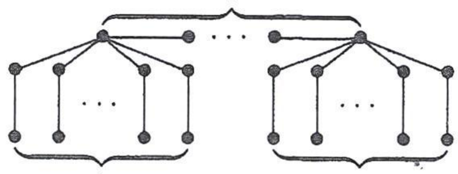


Figure 1: The baton $B(i, j)$ with $j = j_1 + j_2$

Lemma 2.1. ([2, 4]) *For any vertex x in a graph G, $mi(G) \leq mi(G - x) + mi(G - N_G[x])$.*

Lemma 2.2. ([2, 4]) *If u is a leaf adjacent to v in a graph G, $mi(G) = mi(G - N_G[u]) + mi(G - N_G[v])$.*

Lemma 2.3. ([6]) *If a graph G has duplicated leaves u_1 and u_2, then $mi(G) = mi(G - u_2)$.*

Lemma 2.4. ([4]) *If G is the union of two disjoint graphs G_1 and G_2, then $mi(G) = mi(G_1) \cdot mi(G_2)$.*

We observe that $G - x$ and $G - N_G[x]$ will form forests for the vertex x in Lemma 2.1 if we choose x on the cycle. In addition, if the v in Lemma 2.2

is a leaf in a twinkle graph, then $G - N_G[u]$ and $G - N_G[v]$ may be twinkle graphs, trees or forests.

Lemmas 2.1 and 2.2 will be needed in proofs of Theorem 3.1 and Lemma 3.3. The preceding observations tell us that the results listed in Theorems 2.5 through 2.11 on the largest, second largest, and third largest numbers of maximal independent sets of trees and forests lay the foundation of our proof.

The results of the largest numbers of maximal independent sets for trees and forests are illustrated in Theorems 2.5 and 2.6, respectively.

Theorem 2.5. ([5]) *If T is a tree of order $n \geq 1$, then $mi(G) \leq t_1(n)$, where*

$$t_1(n) = \begin{cases} r^{n-1}, & \text{if } n \text{ is odd,} \\ r^{n-2}+1, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, $mi(T) = t_1(n)$ if and only if $T \in T_1(n)$, where

$$T_1(n) = \begin{cases} B(1, \frac{n-1}{2}), & \text{if } n \text{ is odd,} \\ B(2, \frac{n-2}{2}) \text{ or } B(4, \frac{n-4}{2}), & \text{if } n \text{ is even,} \end{cases}$$

Theorem 2.6. ([5]) *If F is a forest of order $n \geq 1$, then $mi(G) \leq f_1(n)$, where*

$$f_1(n) = \begin{cases} r^{n-1}, & \text{if } n \text{ is odd,} \\ r^{n}, & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, $mi(F) = f_1(n)$ if and only if $F \in F_1(n)$, where

$$F_1(n) = \begin{cases} B(1, \frac{n-1-2s}{2}) \cup sP_2, & \text{if } n \text{ is odd,} \\ \frac{n}{2}P_2, & \text{if } n \text{ is even,} \end{cases}$$

where $0 \leq s \leq \frac{n-1}{2}$.

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.7 and 2.8, respectively.

Theorem 2.7. ([6]) *If T is a tree of order $n \geq 4$ with $T \notin T_1(n)$, then $mi(T) \leq t_2(n)$, where*

$$t_2(n) = \begin{cases} r^{n-2}, & \text{if } n \geq 4 \text{ is even,} \\ 3, & \text{if } n = 5, \\ 3r^{n-5}+1, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t_2(n)$ if and only if $T \in \{T_2'(8), T_2''(8), P_{10}, T_2(n)\}$, where $T_2(n)$ and $T_2'(8)$, $T_2''(8)$ are shown in Figures 2.

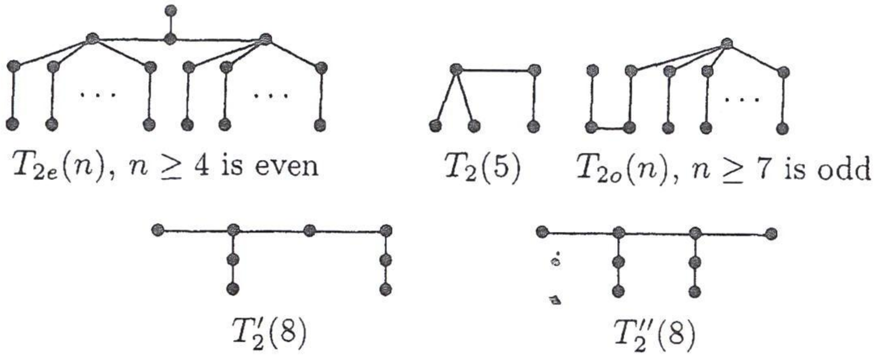


Figure 2: The trees $T_2(n)$, $T_2'(8)$ and $T_2''(8)$

Theorem 2.8. ([6]) *If F is a forest of order $n \geq 4$ with $F \notin F_1(n)$, then $mi(F) \leq f_2(n)$, where*

$$f_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \geq 4 \text{ is even,} \\ 3, & \text{if } n = 5, \\ 7r^{n-7}, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_2(n)$ if and only if $F \in F_2(n)$, where

$$F_2(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even,} \\ T_2(5) \text{ or } P_1 \cup P_4, & \text{if } n = 5, \\ P_7 \cup \frac{n-7}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

The results of the third largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.9 and 2.10, respectively.

Theorem 2.9. ([3]) *If T is a tree with $n \geq 7$ vertices having $T \notin T_i(n)$, $i = 1, 2$, then $mi(T) \leq t_3(n)$, where*

$$t_3(n) = \begin{cases} 3r^{n-5}, & \text{if } n \geq 7 \text{ is odd;} \\ 7, & \text{if } n = 8; \\ 15, & \text{if } n = 10; \\ 7r^{n-8} + 2, & \text{if } n \geq 12 \text{ is even.} \end{cases}$$

Furthermore, $mi(T) = t_3(n)$ if and only if $T \in \{T_3(8), T_3'(10), T_3''(10), T_3(n)\}$, where $T_3(n)$ and $T_3(8), T_3'(10), T_3''(10)$ are shown in Figures 3 and 4, respectively.

$T_{3e}(n)$, $n \geq 12$ is even $\qquad$ $T_{3o}(n)$, $n \geq 7$ is odd

Figure 3: The trees $T_3(n)$

$T_3(8)$ $\qquad$ $T_3'(10)$ $\qquad$ $T_3''(10)$

Figure 4: The trees $T_3(8), T_3'(10)$ and $T_3''(10)$

Theorem 2.10. ([7]) *If F is a forest with $n \geq 8$ vertices having $F \notin F_i(n)$, $i = 1, 2$, then $mi(F) \leq f_3(n)$, where*

$$f_3(n) = \begin{cases} 5r^{n-6}, & \text{if } n \geq 8 \text{ is even;} \\ 13r^{n-9}, & \text{if } n \geq 9 \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f_3(n)$ if and only if $F \in F_3(n)$, where

$$F_3(n) = \begin{cases} T_1(6) \cup \frac{n-6}{2} P_2, & \text{if } n \geq 8 \text{ is even;} \\ T_2(9) \cup \frac{n-9}{2} P_2, & \text{if } n \geq 9 \text{ is odd.} \end{cases}$$

Theorem 2.11. ([10]) *If W is a twinkle graph of order $n \geq 6$, then $mi(W) \leq w_1(n)$, where*

$$w_1(n) = \begin{cases} 4, & \text{if } n = 6, \\ 13, & \text{if } n = 9, \\ 3r^{n-5}, & \text{if } n = 7,\ n \geq 11 \text{ is odd,} \\ r^{n-2} + 1, & \text{if } n \geq 8 \text{ is even.} \end{cases}$$

Furthermore, $mi(W) = w_1(n)$ if and only if $W \in W_1(n)$, where

$$W_1(n) = \begin{cases} W_1(6), & \text{if } n = 6, \\ W_1(9), & \text{if } n = 9, \\ W_{1o}(n), & \text{if } n = 7,\ n \geq 11 \text{ is odd,} \\ W_{1e}(n) \text{ or } W_{1e}^*(n), & \text{if } n \geq 8 \text{ is even,} \end{cases}$$

where $W_1(6)$, $W_1(9)$, $W_{1o}(n)$, $W_{1e}(n)$ and $W_{1e}^*(n)$ are shown in Figure 5.

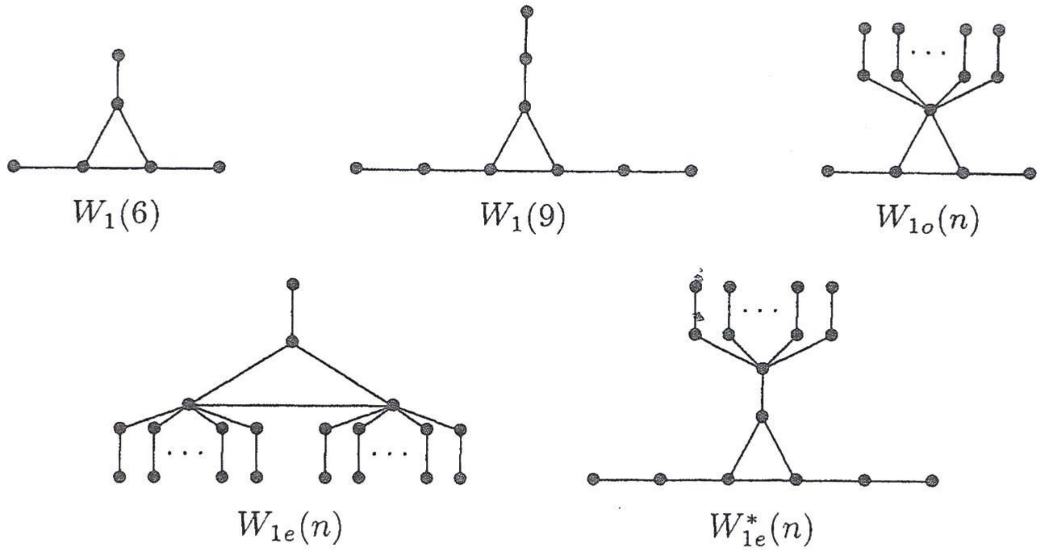


Figure 5: The graphs $W_1(6)$, $W_1(9)$, $W_{1o}(n)$, $W_{1e}(n)$ and $W^*_{1e}(n)$

3 Main results

Define the graph $W_2(n)$ of order $n \geq 10$ as follows.

$$W_2(n) = \begin{cases} W_{2e}(n) \text{ or } W^*_{2e}(n), & \text{if } n \geq 10 \text{ is even,} \\ W_{2o}(n), & \text{if } n \geq 11 \text{ is odd,} \end{cases}$$

where $W_{2e}(n)$, $W^*_{2e}(n)$ and $W_{2o}(n)$ are shown in Figure 6.

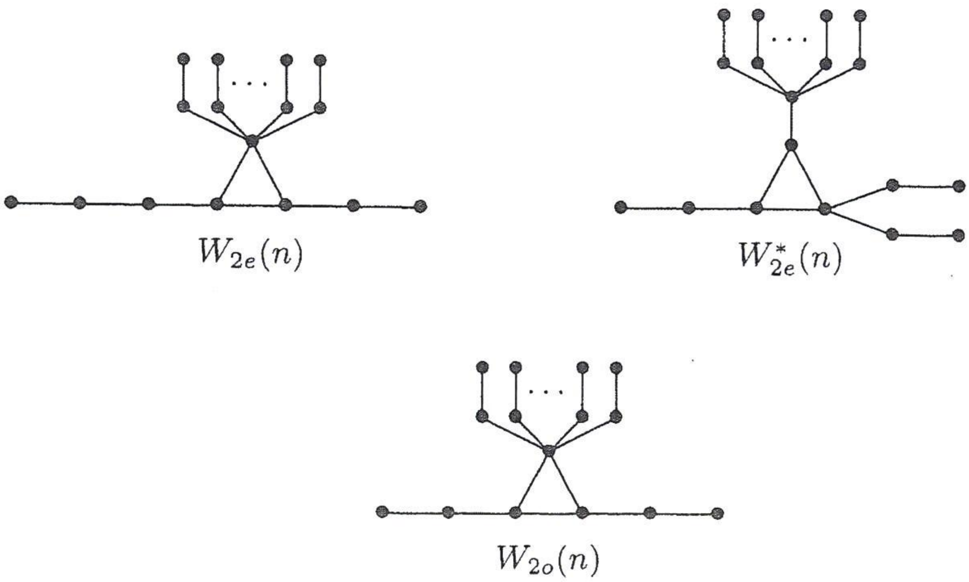


Figure 6: The graphs $W_{2e}(n)$, $W^*_{2e}(n)$ and $W_{2o}(n)$

To calculate $mi(W_2(n))$, we illustrate with $mi(W_{2e}(n))$ as follows. See Figure 7.

$$\begin{aligned} mi(W_{2e}(n)) &= mi_{+x}(W_{2e}(n)) + mi_{-x}(W_{2e}(n)) \\ &= mi(P_3 \cup P_2 \cup \frac{n-8}{2} P_1) \\ &\qquad\qquad + mi(P_7 \cup \frac{n-8}{2} P_2) - mi(\emptyset) \\ &= 4 + 7r^{n-8} - 1 \\ &= 7r^{n-8} + 3. \end{aligned}$$

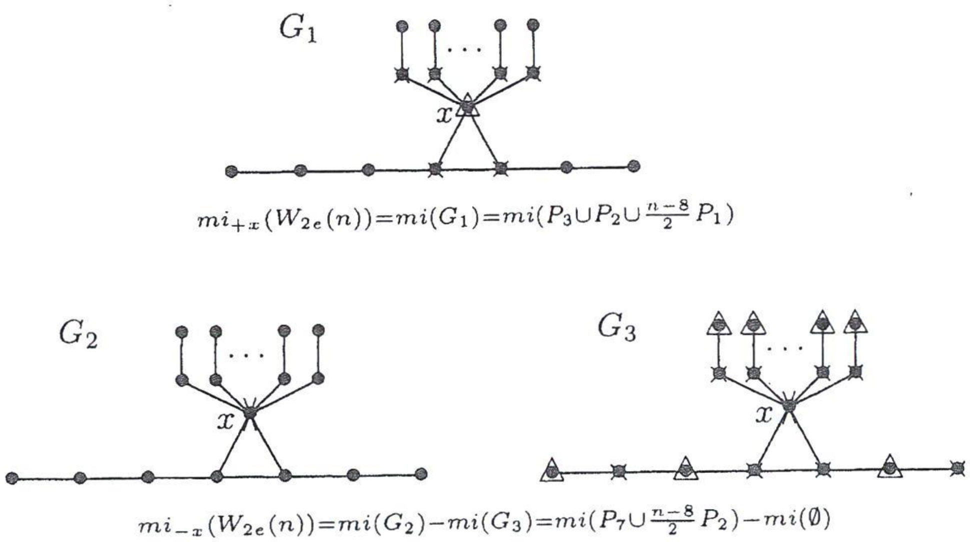


Figure 7: The $mi(W_{2e}(n))$

Let $w_2(n) = mi(W_2(n))$. By similar calculation, we obtain:

$$w_2(n) = \begin{cases} 7r^{n-8} + 3, & \text{if } n \geq 10 \text{ is even}, \\ 5r^{n-7} + 3, & \text{if } n \geq 11 \text{ is odd}. \end{cases}$$

In this paper, we will prove the following result.

Theorem 3.1. *If W is a twinkle graph of order $n \geq 10$ with $W \notin W_1(n)$, then $mi(W) \leq w_2(n)$. Furthermore, the equality holds if and only if $W \in W_2(n)$.*

Lemmas 3.2 and 3.3 are needed for our discussions.

Lemma 3.2. *Suppose that W is a twinkle graph of order $n \geq 10$ having duplicated leaves, then $mi(W) < w_2(n)$.*

Proof. Suppose that W has duplicated leaves u_1 and u_2, then $W' = W - u_2$ is a twinkle graph of order $n - 1$. By Lemma 2.3 and Theorem 2.11, we have that

$$\begin{aligned} mi(W) &= mi(W - u_2) \\ &\leq \begin{cases} 3r^{(n-1)-5}, & \text{if } n \text{ is even,} \\ r^{(n-1)-2} + 1, & \text{if } n \text{ is odd,} \end{cases} \\ &< w_2(n). \end{aligned}$$

This completes the proof. □

By Lemma 3.2, we assume that W is always a twinkle graph of order $n \geq 10$ without duplicated leaves in the remainder of the paper. As an illustration, a path P_n is *attached* to a vertex v on the cycle C in a twinkle graph is exhibited in Figure 8.

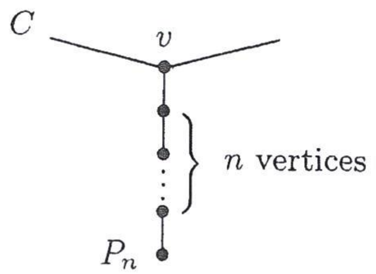
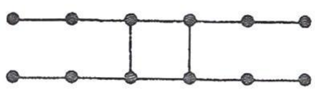


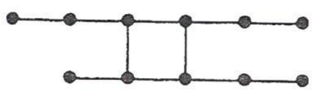
Figure 8: The path P_n is attached to $v \in V(C)$

Lemma 3.3. *Suppose that W is a twinkle graph of order $n \geq 10$ obtained by attaching a P_2 or a P_3 to each vertex in the unique cycle C. Then $mi(W) < w_2(n)$.*

Proof. By the assumption of W and $|W| \geq 10$, we obtain that $|C| \geq 4$. If $|C| = 4$, then there are four possibilities for W. See Figure 9. By simple calculation, we have $\overline{W}^{(i)} < w_2(n)$ for $1 \leq i \leq 4$.



$\overline{W}^{(1)}$



$\overline{W}^{(2)}$

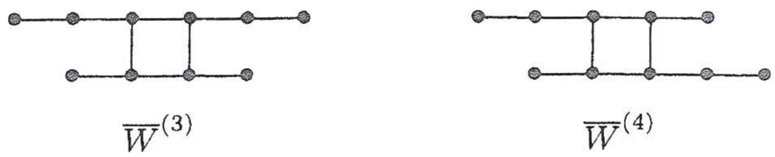


Figure 9: The possible graph W with $|C| = 4$

If $|C| = 5$, then there are eight possibilities for W. See Figure 10. By simple calculation, we have $\widehat{W}^{(i)} < w_2(n)$ for $1 \leq i \leq 8$.

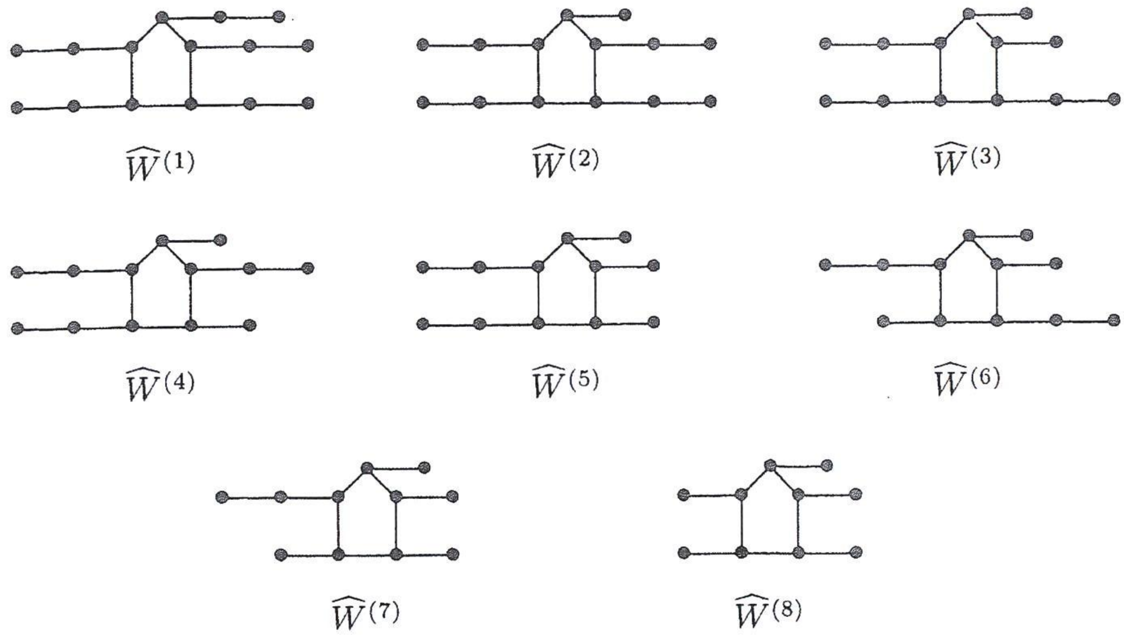


Figure 10: The possible graph W with $|C| = 5$

Hence we can assume that $|C| \geq 6$, that is, $|\{v \in V(C) : \deg_W(v) = 3\}| \geq 6$. We distinguish two cases to consider.

Case 1. There exists a vertex $y \in V(C)$ is attached by a $P_2 : xy$, see Figure 11. Note that $W - N_W[x]$ is a tree of order $n-2$ and $W - N_W[y]$ is a forest of order $n-4$.

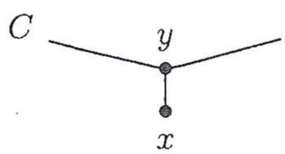


Figure 11: The graph for Case 1

Claim 1. $W - N_W[x] \notin T_1(n-2)$.

Check. Note that

$$\begin{aligned}&|\{v \in V(C) : \deg_{W-N_W[x]}(v) \geq 3\}|\\ =& |\{v \in V(C) : \deg_{W-N_W[x]}(v) = 3\}|\\ =& |\{v \in V(C) : \deg_W(v) = 3\}| - |\{v \in V(C) : v \in N_W[y]\}|\\ =& |C| - 3\\ \geq& 6 - 3\\ =& 3,\end{aligned}$$

which is contradiction to $|\{v \in V(T_1(n-2)) : \deg_{T_1(n-2)}(v) \geq 3\}| \leq 2$. Hence we have that $W - N_W[x] \notin T_1(n-2)$.

Claim 2. $W - N_W[y] \notin F_1(n-4)$.

Check. Note that

$$\begin{aligned}&|\{v \in V(C) : \deg_{W-N_W[y]}(v) \geq 3\}|\\ =& |\{v \in V(C) : \deg_{W-N_W[y]}(v) = 3\}|\\ =& |\{v \in V(C) : \deg_W(v) = 3\}| - |\{v \in V(C) : v \in N_W[y]\}|\\ \geq& 6 - 3\\ =& 3,\end{aligned}$$

which is contradiction to $|\{v \in V(F_1(n-4)) : \deg_{F_1(n-4)}(v) \geq 3\}| \leq 1$. Hence we have that $W - N_W[y] \notin F_1(n-4)$.

By Lemma 2.2, Theorems 2.7 and 2.8, we have that

$$\begin{aligned}mi(W) &= mi(W - N_W[x]) + mi(W - N_W[y])\\ &\leq \begin{cases} r^{(n-2)-2} + 3r^{(n-4)-4}, & n \text{ is even},\\ (3r^{(n-2)-5} + 1) + 7r^{(n-4)-7}, & n \text{ is odd},\end{cases}\\ &= \begin{cases} 7r^{n-8}, & n \text{ is even},\\ 19r^{n-11} + 1, & n \text{ is odd},\end{cases}\\ &< w_2(n).\end{aligned}$$

Case 2. Every vertex in C is attached by a P_3. Choose two P_3's, say $P^{(1)} : x_1y_1z_1$ and $P^{(2)} : x_2y_2z_2$, such that $z_1, z_2 \in V(C)$ and z_1 is adjacent to z_2, see Figure 12.

Claim 3. $W - N_W[x_1] - N_W[x_2] - z_1 \notin T_1(n-5), T_2(n-5)$.

Check. Since $N_W[x_1] = \{x_1, y_1\}$ and $N_W[x_2] = \{x_2, y_2\}$, it follows that $W - N_W[x_1] - N_W[x_2] - z_1$ is a tree of order $n-5$. Note that $|\{v \in V(C) : \deg_{W-N_W[x_1]-N_W[x_2]-z_1}(v) \geq 3\}| = |\{v \in V(C) : \deg_W(v) \geq 3\}| - 3 \geq 6 - 3 = 3$, which is contradiction to $|\{v \in V(T_1(n-5)) : \deg_{T_1(n-5)}(v) \geq 3\}| \leq 2$. On the other hand, every vertex in C is attached by a P_3, it

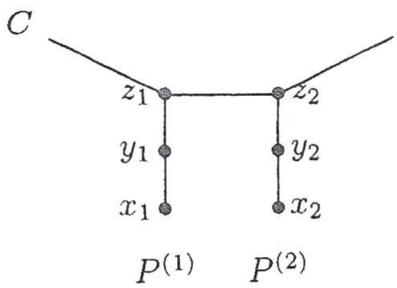


Figure 12: The graph for Case 2

follows that $W - N_W[x_1] - N_W[x_2] - z_1 \notin T_2(n-5)$. Hence we have that $W - N_W[x_1] - N_W[x_2] - z_1 \notin T_1(n-5), T_2(n-5)$. By similar arguments, we have the following.

- $W - N_W[x_1] - N_W[x_2] - z_1$ is a tree of order $n-5$ and $W - N_W[x_1] - N_W[x_2] - z_1 \notin T_1(n-5), T_2(n-5)$.
- $W - N_W[x_1] - N_W[y_2]$ is a tree of order $n-5$ and $W - N_W[x_1] - N_W[y_2] \notin T_1(n-5), T_2(n-5)$.
- $W - N_W[y_1] - N_W[x_2]$ is a tree of order $n-5$ and $W - N_W[y_1] - N_W[x_2] \notin T_1(n-5), T_2(n-5)$.
- $W - N_W[x_1] - N_W[x_2] - N_W[z_1]$ is a forest of order $n-7$.
- $W - N_W[y_1] - N_W[y_2]$ is a tree of order $n-6$.

Hence, by Lemmas 2.1, 2.2, Theorems 2.5–2.9, we have that

$$\begin{aligned}
mi(W) &\overset{\text{Lma 2.2}}{=} mi(W - N_W[x_1]) + mi(W - N_W[y_1]) \\
&\overset{\text{Lma 2.2}}{=} mi(W - N_W[x_1] - N_W[x_2]) \\
&\qquad + mi(W - N_W[x_1] - N_W[y_2]) \\
&\qquad\quad + mi(W - N_W[y_1] - N_W[x_2]) \\
&\qquad\qquad + mi(W - N_W[y_1] - N_W[y_2]) \\
&\overset{\text{Lma 2.1}}{\le} mi(W - N_W[x_1] - N_W[x_2] - z_1) \\
&\qquad + mi(W - N_W[x_1] - N_W[x_2] - N_W[z_1]) \\
&\qquad\quad + mi(W - N_W[x_1] - N_W[y_2]) \\
&\qquad\qquad + mi(W - N_W[y_1] - N_W[x_2]) \\
&\qquad\qquad\quad + mi(W - N_W[y_1] - N_W[y_2]) \\
&\overset{\text{Thms 2.5–2.9}}{\le} \begin{cases} 3 \cdot 3r^{(n-5)-5} + r^{(n-7)-1} + r^{(n-6)-2} + 1, & n \text{ is even}, \\ 3 \cdot r^{(n-5)-2} + r^{n-7} + r^{(n-6)-1}, & n \text{ is odd}, \end{cases} \\
&= \begin{cases} 13r^{n-10} + 1, & n \text{ is even}, \\ 5r^{n-7}, & n \text{ is odd}, \end{cases} \\
&< w_2(n).
\end{aligned}$$

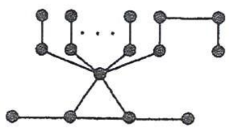
This completes the proof. □

Now we are ready for the main result.

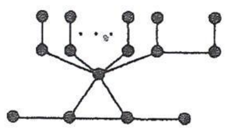
Theorem 3.1. *If W is a twinkle graph of order $n \geq 10$ with $W \notin W_1(n)$, then $mi(W) \leq w_2(n)$. Furthermore, the equality holds if and only if $W \in W_2(n)$.*

Proof. Let W be a twinkle graph of order $n \geq 10$ having $W \notin W_1(n)$ such that $mi(W)$ is as large as possible. We shall prove the result by induction on n. It is true for $n = 10, 11$. Assume that it is true for all $n' < n$. For a leaf u, let $\ell(u, C)$ be the length of the unique shortest path, P, from u to C; as a convention, we will take $v := v_u$, where $\{v_u\} = P \cap N_W(u)$. By Lemma 3.3, we have that there exists a leaf u with $\ell(u, C) \geq 3$ or a vertex $x \in V(C)$ with $\deg_W(x) \geq 4$. Trivially, $W - N_W[u]$ is a graph of order $n-2$. Moreover, if there exists a leaf u with $\ell(u, C) \geq 3$, then the length of $P' = P - \{u, v\}$ is not less than 1. On the other hand, if there exists a vertex $x \in V(C)$ with $\deg_W(x) \geq 4$, we can find a leaf u such that the length of the path joining u and x is not less than 2 by Lemma 3.2. Then $\deg_{W-N_W[u]}(x) \geq \deg_W(x) - 1 \geq 4 - 1 = 3$. Hence $W - N_W[u]$ is a twinkle graph of order $n-2$. Consider the cases of $W - N_W[u] \in W_1(n-2)$ and $W - N_W[u] \notin W_1(n-2)$.

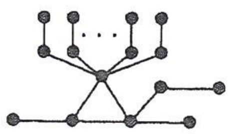
Case 1. $W - N_W[u] \in W_1(n-2)$. Then $W - N_W[u]$ is one of $W_{1o}(n-2)$, $W_{1e}(n-2)$, or $W_{1e}^*(n-2)$ (see Theorem 2.11 and Figure 5). When we add $N_W[u]$ back, there are 14 possibilities for graph W (see Figure 13 for a complete list).



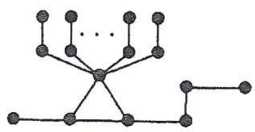
$W^{(1)}$ $[9r^{n-9} + 1]$



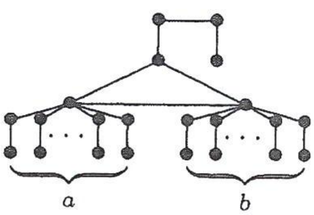
$W^{(2)}$ $[9r^{n-9}]$



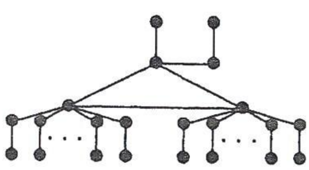
$W^{(3)}$ $[5r^{n-7}]$



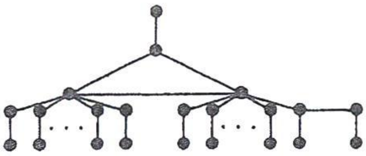
$W^{(4)}$ $[5r^{n-7} + 1]$



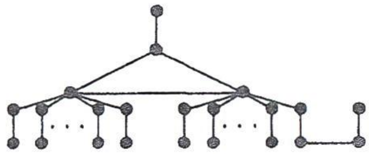
$W^{(5)}$ $[\leq 7r^{n-8} + 3]$



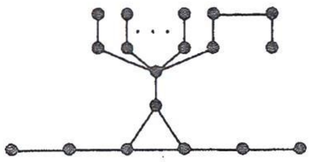
$W^{(6)}$ $[3r^{n-6} + 2]$



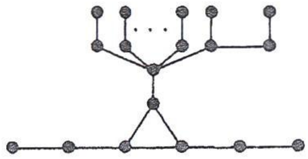
$W^{(7)}$ $[3r^{n-6}+2]$



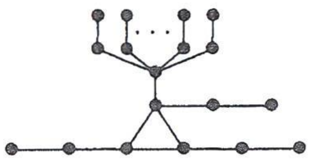
$W^{(8)}$ $[\leq 7r^{n-8}+1]$



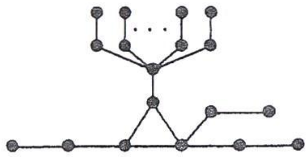
$W^{(9)}$ $[3r^{n-6}+6]$



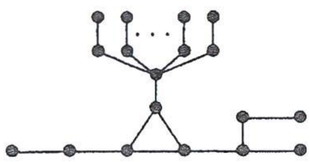
$W^{(10)}$ $[3r^{n-6}+2]$



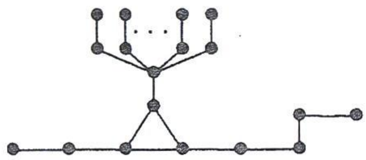
$W^{(11)}$ $[13r^{n-10}+1]$



$W^{(12)}$ $[7r^{n-8}+3]$



$W^{(13)}$ $[13r^{n-10}+1]$



$W^{(14)}$ $[13r^{n-10}+2]$

Figure 13: The possible graphs W

The number inside the brackets in Figure 13 indicates the number of maximal independent sets of the corresponding graphs. Note that $W^{(5)} = W_{2e}(n)$ when $a = \frac{n-8}{2}$, $b = 1$ and $W^{(12)} = W_{2e}^{*}(n)$. By simple calculation, we have $mi(W^{(i)}) < w_2(n)$ for $i \neq 5, 12$.

Case 2. $W - N_W[u] \notin W_1(n-2)$. By the induction hypothesis, $mi(W - N_W[u]) \leq w_2(n-2)$. Now we consider the cases of $W - N_W[v]$.

Subcase 2.1. For each leaf u with $\ell(u, C) = 3$ and for each $x \in V(C)$ with $\deg_W(x) = 3$. Note that n is even. Recall that W has no duplicated leaves. There is only one possibility for graph W. See Figure 14.

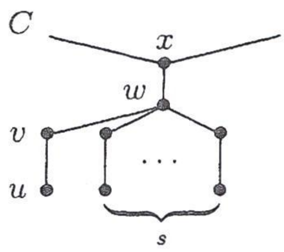


Figure 14: The possible graph W

Suppose that $|C| = 3$, then W is a twinkle graph of order n, where $n = 2a+2b+2c+6$ and $a, b, c \geq 1$, see Figure 15. Note that $1 \leq a \leq \frac{n-10}{2}$.

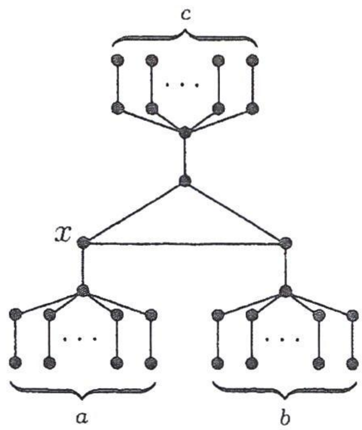


Figure 15: The case of $|C| = 3$

By Lemma 2.1, we have that

$$\begin{aligned}
mi(W) &\leq mi(W-x) + mi(W - N_W[x]) \\
&= r^{2a}(r^{2b+2c+4-2}+1) + r^{2a}r^{(2b+1)-1}r^{(2c+1)-1} \\
&= r^{2a+2b+2c+2} + r^{2a} + r^{2a+2b+2c} \\
&\leq r^{n-4} + r^{n-10} + r^{n-6} \\
&= 13r^{n-10} \\
&< w_2(n).
\end{aligned}$$

Hence we can assume now that $|C| \geq 4$. Note that $W - N_W[v] - x$ is the union of a tree T of order $n-2s-4$ and s copies of P_2, $W - N_W[v] - N_W[x]$ is a forest of order $n-6$. Note that $0 \leq s \leq \frac{n-12}{2}$. Since $|C| \geq 4$, by similar arguments as in the checks of Claims in Lemma 3.3, we have that $T \notin T_i(n-2s-4)$ and $W - N_W[v] - N_W[x] \notin F_i(n-6)$ for $i = 1, 2, 3$. By

Lemma 2.1, Theorems 2.9 and 2.10, we have that

$$\begin{aligned}mi(W-N_W[v]) &\le mi(W-N_W[v]-x)\\ &\qquad\qquad\qquad + mi(W-N_W[v]-N_W[x])\\ &< r^{2s}(7r^{(n-2s-4)-8}+2)+5r^{(n-6)-6}\\ &= 12r^{n-12}+r^{2s+2}\\ &\le 14r^{n-12}\\ &= 7r^{n-10}.\end{aligned}$$

Hence, by Lemma 2.2, we obtain that

$$\begin{aligned}mi(W) &= mi(W-N_W[u])+mi(W-N_W[v])\\ &< w_2(n-2)+7r^{n-10}\\ &= w_2(n).\end{aligned}$$

Subcase 2.2. There exists a leaf u with $\ell(u,C)\neq 3$ or a vertex $x\in V(C)$ with $\deg_W(x)>3$. There are two possibilities for $W-N_W[v]$.

• $W-N_W[v]$ is a forest $\overline{F}$ of order $n-3$. By similar arguments as in the checks of Claims in Lemma 3.3 and W is a twinkle graph with $W\notin W_1(n)$, it follows that $\overline{F}\notin F_1(n-3)$ for n is even and $\overline{F}\notin F_1(n-3), F_2(n-3)$ for n is odd. So, by Theorems 2.8 and 2.10, we have that

$$mi(W-N_W[v])\le\begin{cases}7r^{(n-3)-7}, & \text{if } n \text{ is even},\\ 5r^{(n-3)-6}, & \text{if } n \text{ is odd}.\end{cases}\tag{1}$$

• $W-N_W[v]$ is the union of a forest (may be empty) and a twinkle graph $\overline{W}$ of order t, $6\le t\le n-3$. By the induction hypothesis, Theorems 2.6 and 2.11, we have that

(I): When n is even,

$$mi(W-N_W[v])\le\begin{cases}4r^{(n-3-6)-1}, & \text{if } t=6,\\ r^{(n-3-t)-1}(r^{t-2}+1), & \text{if } t\ge 8 \text{ is even},\\ 13r^{n-3-9}, & \text{if } t=9,\\ 3r^{t-5}r^{n-3-t}, & \text{if } t=7,\ t\ge 11 \text{ is odd}.\end{cases}\tag{2.1}$$

(II): When n is odd,

$$mi(W-N_W[v])\le\begin{cases}4r^{n-3-6}, & \text{if } t=6,\\ r^{n-3-t}(r^{t-2}+1), & \text{if } t\ge 8 \text{ is even},\\ 13r^{(n-3-9)-1}, & \text{if } t=9,\\ 3r^{t-5}r^{(n-3-t)-1}, & \text{if } t=7,\ t\ge 11 \text{ is odd}.\end{cases}\tag{2.2}$$

Thus, by (1), (2.1)and (2.2), we have that

$$mi(W - N_W[v]) \le \begin{cases} \max\{7r^{n-10}, 4r^{n-10}, r^{n-6} + r^{n-4-t}, \\ \qquad 13r^{n-12}, 3r^{n-8}, \} & \text{if } n \text{ is even,} \\ \max\{5r^{n-9}, 4r^{n-9}, r^{n-5} + r^{n-3-t}, \\ \qquad 13r^{n-13}, 3r^{n-9}\}, & \text{if } n \text{ is odd,} \end{cases}$$

$$\le \begin{cases} 7r^{n-10}, & \text{if } n \text{ is even,} \\ 5r^{n-9}, & \text{if } n \text{ is odd.} \end{cases}$$

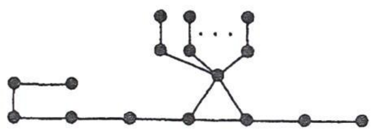
Also, if the equalities hold, then

$$W - N_W[v] = \begin{cases} F_2(n-3), & \text{if } n \text{ is even,} \\ F_3(n-3), & \text{if } n \text{ is odd.} \end{cases}$$

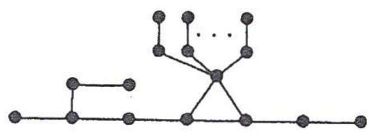
Hence, by Lemma 2.2, we obtain that

$$\begin{aligned} mi(W) &= mi(W - N_W[u]) + mi(W - N_W[v]) \\ &\le \begin{cases} w_2(n-2) + 7r^{n-10}, & \text{if } n \text{ is even,} \\ w_2(n-2) + 5r^{n-9}, & \text{if } n \text{ is odd,} \end{cases} \\ &= \begin{cases} (7r^{n-10} + 3) + 7r^{n-10}, & \text{if } n \text{ is even,} \\ (5r^{n-9} + 3) + 5r^{n-9}, & \text{if } n \text{ is odd,} \end{cases} \\ &= w_2(n). \end{aligned}$$

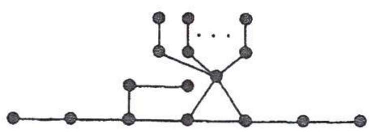
Furthermore, the equality holding imply that $W - N_W[v] = F_2(n-3)$ for even n and $W - N_W[v] = F_3(n-3)$ for odd n and, using the induction hypothesis, $W - N_W[u] \in W_2(n-2)$. Based upon the fact that $W - N_W[u] \in W_2(n-2)$, there are ten possibilities for W for even n and there are six possibilities for W for odd n. See Figure 16. Note that for even n only $W_e^{(8)}$ satisfies $W_e^{(8)} - N_{W_e^{(8)}}[v] = F_2(n-3)$ and for odd n only $W_o^{(4)}$ satisfies $W_o^{(4)} - N_{W_o^{(4)}}[v] = F_3(n-3)$. Since $W_e^{(8)} = W_{2e}(n)$ and $W_o^{(4)} = W_{2o}(n)$, the proof is complete. □



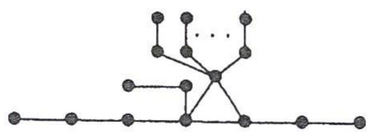
$W_e^{(1)}$



$W_e^{(2)}$



$W_e^{(3)}$



$W_e^{(4)}$

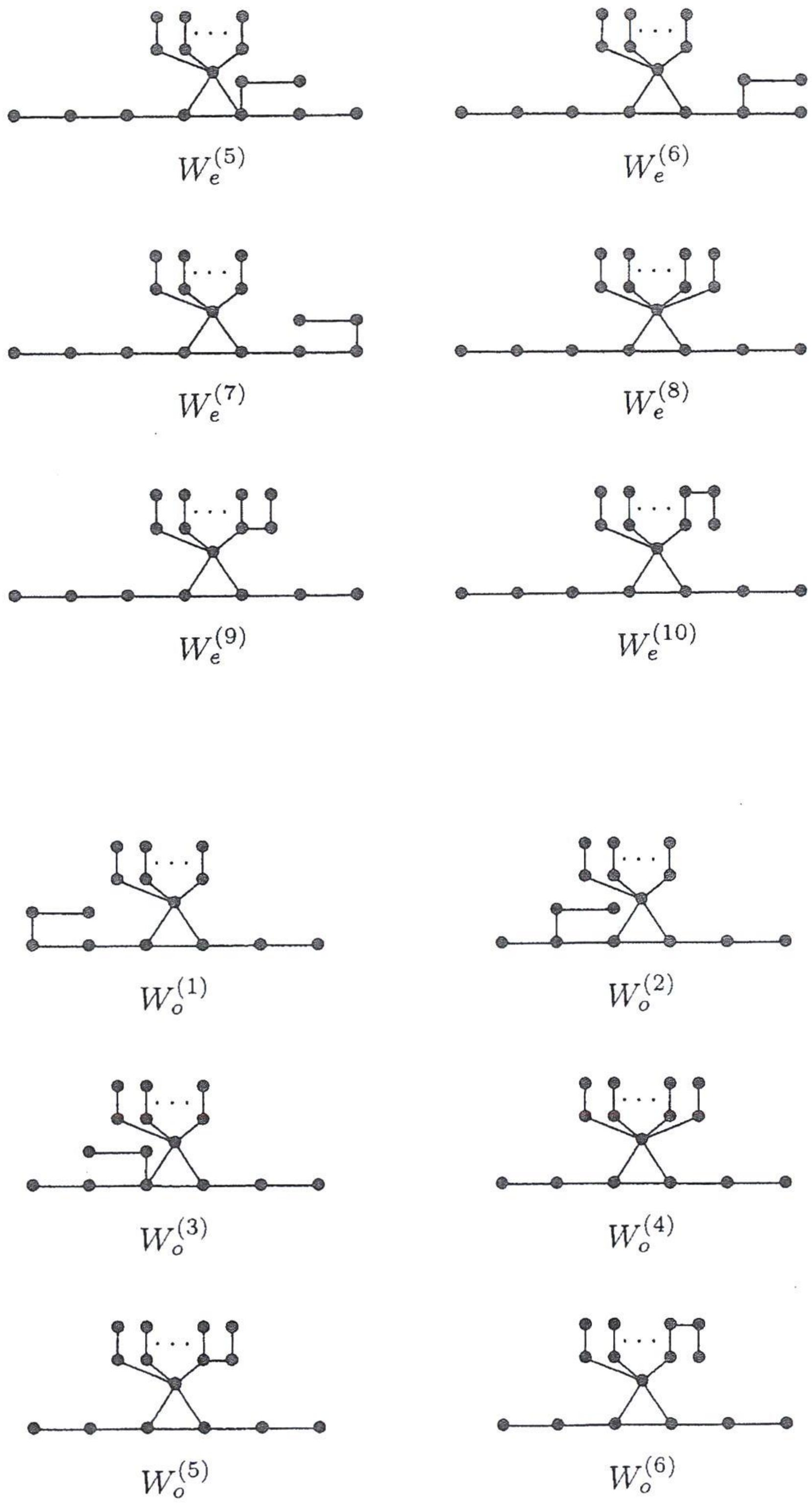


Figure 16: The possible graphs W

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