Vertex-distinguishing I- and VI-total colorings of $S_m \vee F_n$, $S_m \vee W_n$ and $F_n \vee W_n$[1]

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Abstract

We will discuss the vertex-distinguishing I-total colorings and vertex-distinguishing VI-total colorings of three types of graphs: $S_m \vee F_n$, $S_m \vee W_n$ and $F_n \vee W_n$ in this paper. The optimal vertex-distinguishing I (resp. VI)-total colorings of these join graphs are given by the method of constructing colorings according to their structural properties and the vertex-distinguishing I (resp. VI)-total chromatic numbers of them are determined.

Keywords graph, the join of graphs, I-total coloring, vertex-distinguishing I-total coloring, vertex-distinguishing I-total chromatic number

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1 Introduction and Preliminaries

Graph theory is the historical foundation of the science of networks and the basis of information science. The problem in which we are interested is a particular case of the great variety of different ways of labeling a graph. The graph coloring is an important branch in graph theorem and has various applications such as in timetabling of teaching schedule, storage problem, etc. The not necessarily proper edge colorings (i.e. general edge coloring) which is vertex distinguishing had been proposed in [5] and studied in several papers [5]-[10]. The concepts of the vertex-distinguishing I-total coloring and vertex-distinguishing VI-total coloring are introduced in [1]. The vertex-distinguishing I-total chromatic numbers and vertex-distinguishing VI-total chromatic numbers of the complete graph, star, complete bipartite graph, normal double star, wheel, fan, cycle, path and $C_n \vee C_n$ are determined in [1]. Based on these results, Two conjectures (VDITC Conjecture, VDVITC Conjecture) is proposed. The vertex-distinguishing I-total chromatic number of the join of two paths is determined in [2]. The vertex-distinguishing I(resp. VI)-total chromatic numbers of the join of a cycle and a path and join graphs $C_m \vee C_n (m \neq n)$, $C_m \vee W_n$, $C_m \vee F_n$ are determined in [3] and [4] respectively. In this paper we will study the vertex-distinguishing I(resp. VI)-total chromatic numbers of the joins of a star and a fan, a star and a wheel, a fan and a wheel.

We consider the undirected, finite simple graphs only in this paper.

Definition 1 Suppose graphs G and H are disjoint. The join of G and H, denoted by $G \vee H$, is a new graph such that

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$V(G \bigvee H) = V(G) \bigcup V(H)$,
$E(G \bigvee H) = E(G) \bigcup E(H) \bigcup \{uv | u \in V(G), v \in V(H)\}$.

Definition 2 Suppose G is a simple graph of order at least 2, k is a positive integer. A mapping $f : V \bigcup E \to \{1, 2, \ldots, k\}$ is called an I-total coloring of G using k colors, if

(i) for any $uv \in E(G), u \neq v$, we have $f(u) \neq f(v)$;

(ii) for any $uv, vw \in E(G), u \neq w$, we have $f(uv) \neq f(vw)$.

A mapping $f : V \bigcup E \to \{1, 2, \ldots, k\}$ is called an VI-total coloring of G using k colors if for any $uv, vw \in E(G), u \neq w$, we have $f(uv) \neq f(vw)$.

For an I-total coloring (resp. VI-total coloring) f of G and x is a vertex of G, the color set of x under f is the set $\{f(x)\} \cup \{ f(e) \mid e \in E(G)$ and e is incident with $x\}$ (not multiset), which is denoted by $C_f(x)$ or simply $C(x)$.

If an I-total coloring (resp. a VI-total coloring) f of G satisfies $C(u) \neq C(v)$ for any two distinct vertices u and v of G, then f is called a vertex-distinguishing I-total coloring of G (resp. vertex-distinguishing VI-total coloring of G), or a VDIT coloring (resp. a VDVIT coloring) for short. The vertex-distinguishing I-total coloring (resp. vertex-distinguishing VI-total coloring) using k colors is called k-vertex-distinguishing I-total coloring (resp. vertex-distinguishing VI-total coloring), or k-VDIT coloring (resp. k-VDVIT coloring). The number

$$\min\{k : G \text{ has a } k - \text{VDIT coloring}\}$$

is called the vertex-distinguishing I-total chromatic number of graph G and denoted by $\chi^i_{vt}(G)$. The number

$$\min\{k : G \text{ has a } k - \text{VDVIT coloring}\}$$

is called the vertex-distinguishing VI-total chromatic number of graph G and denoted by $\chi^{vi}_{vt}(G)$.

For a graph G, let n_i denote the number of vertices with degree i, $\delta \leq i \leq \Delta$. Suppose

$$\begin{aligned}\zeta(G) = \min\{l | \tbinom{l}{i} + \tbinom{l}{i+1} + \tbinom{l}{i+2} + \cdots + \tbinom{l}{i+s} + \tbinom{l}{i+s+1} \\ \geq n_i + n_{i+1} + \cdots + n_{i+s}, \delta \leq i \leq i+s \leq \Delta, s \geq 0\}.\end{aligned}$$

Obviously we have the following proposition.

Proposition 1 (i) $\zeta(G) \leq \chi^{vi}_{vt}(G) \leq \chi^i_{vt}(G)$; (ii) if $n_\Delta \geq 2$, then $\Delta(G) + 1 \leq \chi^{vi}_{vt}(G) \leq \chi^i_{vt}(G)$.

The following two conjectures can be found in [1].

Conjecture 1 $\chi^i_{vt}(G) = \zeta(G)$ or $\zeta(G) + 1$.

Conjecture 2 $\chi^{vi}_{vt}(G) = \zeta(G)$ or $\zeta(G) + 1$.

In this paper we appointed that the k colors that we will use are $1, 2, \cdots, k$ when a k-VDIT coloring is constructed. If we mention an I(resp. VI)-total coloring using k colors $1, 2, \ldots, k$, then $\overline{C(x)} = \{1, 2, \ldots, k\} \setminus C(x)$ is called the complementary color set of x.

Suppose f is an I(resp. VI)-total coloring of graph G using colors $1, 2, \ldots, k$. Let $\widetilde{C}_k$ denote the cycle such that $V(\widetilde{C}_k) = \{1, 2, \ldots, k\}$; $E(\widetilde{C}_k) = \{i(i+1) \mid i = 1, 2, \ldots, k-1\} \bigcup \{k1\}$.

Note that the elements in $C(x)$ are considered to be ordered and the colors in $C(x)$ are listed from smallest to largest in this paper.

Denote by S_m a star of order $m+1$ with $V(S_m) = \{u_i | i = 0, 1, \ldots, m\}$, $E(S_m) = \{u_0u_i | i = 1, 2, \ldots, m\}$;

Denote by F_n a fan of order $n+1$ with $V(F_n) = \{v_i | i = 0, 1, \ldots, n\}$, $E(F_n) = \{v_0v_i | i = 1, 2, \ldots, n\} \bigcup \{v_iv_{i+1} | i = 1, 2, \ldots, n-1\}$;

Denote by W_n a wheel of order $n+1$ with $V(W_n) = \{w_i | i = 0, 1, \ldots, n\}$, $E(W_n) = \{w_0w_i | i = 1, 2, \ldots, n\} \bigcup \{w_iw_{i+1} | i = 1, 2, \ldots, n-1\} \bigcup \{w_1w_n\}$.

2 Main Results

Theorem 1 If $m, n \geq 2$, then $\chi^i_{vt}(S_m \vee F_n) = m + n + 2$.

Proof. As $\Delta(S_m \vee F_n) = m + n + 1$, $n_\Delta \geq 2$. By proposition 1 we have $\chi^i_{vt}(S_m \vee F_n) \geq m + n + 2$. In the following we need only to give an $(m+n+2)-$VDIT coloring f of $S_m \vee F_n$.

There are three cases to be considered.

Case 1: $m = 2, n \geq 3$.

Obviously, $S_2 \vee F_n \cong P_3 \vee F_n$. Let $V(P_3) = \{u_1, u_2, u_3\}$, $E(P_3) = \{u_1u_2, u_2u_3\}$. Construct an $(n+4)-$VDIT coloring f of $P_3 \vee F_n$ with colors $1, 2, \ldots, n+4$ as follows (Set $v_{n+1} \triangleq v_0$):

$f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 1$, $f(u_1u_2) = 1$, $f(u_2u_3) = n+3$; $f(v_j) = j+2$ for $j = 1, 2, \ldots, n+1$; $f(v_1v_2) = 1$, $f(v_jv_{j+1}) = j+4$ for $j = 2, 3, \ldots, n-1$; $f(v_0v_j) = j-1$ for $j = 3, 4, \ldots, n$; $f(v_0v_1) = n+3$, $f(v_0v_2) = n+4$; $f(u_1v_1) = n+2$, $f(u_1v_j) = j$ for $j = 2, 3, \ldots, n+1$; $f(u_2v_j) = j+1$ for $j = 1, 2, \ldots, n+1$; $f(u_3v_0) = 1$, $f(u_3v_j) = j+2$ for $j = 1, 2, \ldots, n$. We may see Figure 1 about this coloring in the next page.

Then under this coloring, we have the following color sets:

$C(u_1) = \{1, 2, 3, \ldots, n+2\}$; $C(u_2) = \{1, 2, 3, \ldots, n+3\}$;
$C(u_3) = \{1, 3, 4, \ldots, n+3\}$; $C(v_0) = \{1, 2, 3, \ldots, n-1, n+1, \ldots, n+4\}$;
$C(v_1) = \{1, 2, 3, n+2, n+3\}$; $C(v_2) = \{1, 2, 3, 4, 6, n+4\}$;
$C(v_j) = \{j-1, j, j+1, j+2, j+3, j+4\}$, for $j = 3, 4, \ldots, n-1$;
$C(v_n) = \{n-1, n, n+1, n+2, n+3\}$.

It's easy to verify that f is an I-total coloring. Next we have to confirm that f is vertex-distinguishing.

(1) $n = 3$.

There are 3 vertices u_2, v_0 and v_2 whose color sets contain 6 colors. This time, $\overline{C(u_2)} = \{7\}$, $\overline{C(v_0)} = \{3\}$, $\overline{C(v_2)} = \{5\}$. So $C(u_2)$, $C(v_0)$ and $C(v_2)$ are different.

There are 4 vertices u_1, u_3, v_1 and v_3 whose color sets contain 5 colors. This time, $\overline{C(u_1)} = \{6, 7\}$, $\overline{C(u_3)} = \{2, 7\}$, $\overline{C(v_1)} = \{4, 7\}$, $\overline{C(v_3)} = \{1, 7\}$. So $C(u_1)$, $C(u_3)$, $C(v_1)$ and $C(v_3)$ are distinct.

(2) $n = 4$.

There are 2 vertices u_2 and v_0 whose color sets contain 7 colors. This time, $\overline{C(u_2)} = \{8\}$, $\overline{C(v_0)} = \{4\}$. So $C(u_2) \neq C(v_0)$.

There are 4 vertices u_1, u_3, v_2 and v_3 whose color sets contain 6 colors. This time, $\overline{C(u_1)} = \{7, 8\}$, $\overline{C(u_3)} = \{2, 8\}$, $\overline{C(v_2)} = \{5, 7\}$, $\overline{C(v_3)} = \{1, 8\}$. So $C(u_1)$, $C(u_3)$, $C(v_2)$ and $C(v_3)$ are different.

There are 2 vertices v_1 and v_4 whose color sets contain 5 colors. This time, $\overline{C(v_1)} = \{4, 5, 8\}$, $\overline{C(v_4)} = \{1, 2, 8\}$. So $C(v_1) \neq C(v_4)$.

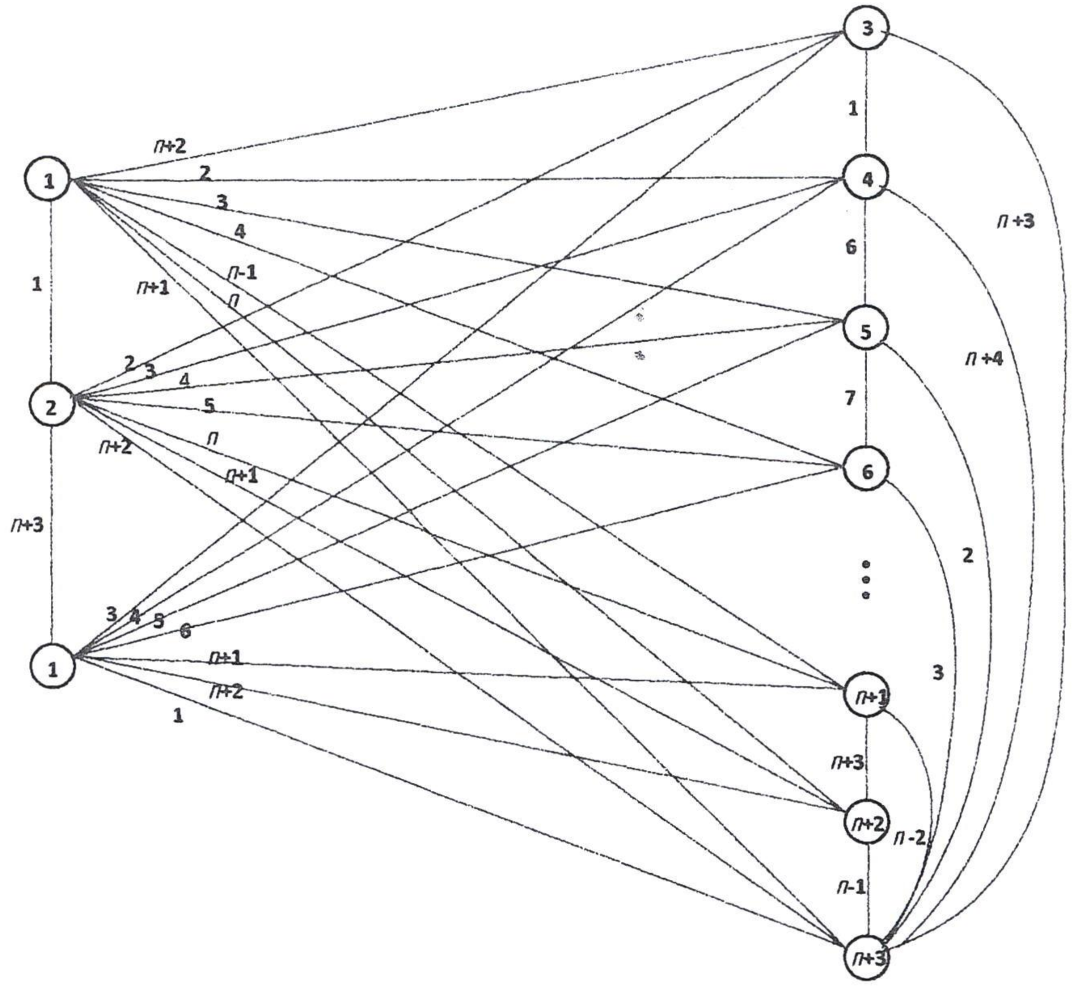

Figure 1: An $(n+4)$-VDIT coloring of $P_3 \vee F_n (n \geq 3)$.

(3) $n \geq 5$.

There are 2 vertices u_2 and v_0 whose color sets contain $n+3$ colors. This time, $\overline{C(u_2)} = \{n+4\}$, $\overline{C(v_0)} = \{n\}$. So $C(u_2) \neq C(v_0)$.

There are 2 vertices u_1 and u_3 whose color sets contain $n+2$ colors. This time, $\overline{C(u_1)} = \{n+3, n+4\}$, $\overline{C(u_3)} = \{2, n+4\}$. So $C(u_1) \neq C(u_3)$.

There are exactly $n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ such that $|C(v_j)| = 6, j = 2, 3, \ldots, n-1$. The subgraph induced by all colors of $C(v_j)(j = 3, 4, \ldots, n-1)$ in $\widetilde{C}_{n+4}$ is a path of order 6. The starting point is $j-1, j = 3, 4, \ldots, n-1$. As the beginning points of the $n-3$ paths are distinct. Thus $C(v_3), C(v_4), \ldots, C(v_{n-1})$ are mutually different. Furthermore, $C(v_2)$ contains color 1 while $C(v_j)(j = 3, 4, \ldots, n-1)$ doesn't contain color 1. So $C(v_2) \neq C(v_j)(j = 3, 4, \ldots, n-1)$.

There are 2 vertices v_1 and v_n whose color sets contain 5 colors. $C(v_1)$ contains color 1 while $C(v_n)$ doesn't contain color 1. So $C(v_1) \neq C(v_n)$.

Thus we get an $(n+4)-$VDIT coloring f of $P_3 \vee F_n$. So $\chi^i_{vt}(S_2 \vee F_n) =$

$n+4 (n \geq 3)$.

Case 2: $m = n = 2$.

Based on the coloring f in Case 1, we change the colors of u_1v_0, u_1v_1 and u_1v_2 such that $f(u_1v_0) = 2$, $f(u_1v_1) = 6$ and $f(u_1v_2) = 5$ with other conditions unchanged. It is easy to verify that the resulting coloring is a 6−VDIT coloring of $P_3 \vee F_2$ by giving the color set of each vertex. So $\chi^i_{vt}(S_2 \vee F_2) = 6$.

Case 3: $m \geq 3, n \geq 2$.

We only need to give an $(m+n+2)$−VDIT coloring of $S_m \vee F_n$. Construct a mapping $f : V(S_m \vee F_n) \cup E(S_m \vee F_n) \to \{1, 2, \ldots, m+n+2\}$ as follows (Note that $v_{n+1} \triangleq v_0$):

$f(u_0) = 2$, $f(u_i) = 1$ for $i = 1, 2, \ldots, m$; $f(u_0u_i) = n+i+1$ for $i = 1, 2, \ldots, m$; $f(v_j) = j+2$ for $j = 1, 2, \ldots, n+1$; $f(v_0v_j) = j-2$ for $j = 3, 4, \ldots, n$; $f(v_0v_1) = n+m+1$, $f(v_0v_2) = n+m+2$; $f(v_jv_{j+1}) = n+m$ if $j \in \{1, 2, \ldots, n-1\}$ and j is an odd number; $f(v_jv_{j+1}) = n+m+1$ if $j \in \{1, 2, \ldots, n-1\}$ and j is an even number; $f(u_iv_j) = i+j-1$ for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n+1$; $f(u_0v_1) = n+m+2$, $f(u_0v_j) = j-1$, for $j = 2, 3, \ldots, n+1$. We may see Figure 2 about this coloring in the next page.

Then under this coloring

(1) $n = 2$

$C(u_i) = \{1\} \bigcup \{i, i+1, i+2, i+3\}$ for $i = 1, 2, \ldots, m$;

$C(u_0) = \{1, 2, 4, 5, \ldots, m+4\}$; $C(v_0) = \{2, 3, 4, \ldots, m+4\}$;

$C(v_1) = \{1, 2, 3, \ldots, m, m+2, m+3, m+4\}$;

$C(v_2) = \{1, 2, 3, \ldots, m+1, m+2, m+4\}$.

(2) $n \geq 3$

$C(u_i) = \{1\} \bigcup \{i, i+1, i+2, \ldots, i+n+1\}$ for $i = 1, 2, \ldots, m$;

$C(u_0) = \{1, 2, 3, \ldots, n, n+2, n+3, \ldots, n+m+2\}$;

$C(v_0) = \{1, 2, 3, \ldots, n-2, n, n+1, \ldots, n+m+2\}$;

$C(v_1) = \{1, 2, 3, \ldots, m, n+m, n+m+1, n+m+2\}$;

$C(v_2) = \{1, 2, 3, \ldots, m+1, n+m, n+m+1, n+m+2\}$;

$C(v_j) = \{j-2, j-1, j, \ldots, j+m-1, n+m, n+m+1\}$ for $j = 3, 4, \ldots, n-1$;

$C(v_n) = \{n-2, n-1, n, \ldots, n+m-1, n+m+1\}$ if n is an odd number; $C(v_n) = \{n-2, n-1, n, \ldots, n+m-1, n+m\}$ if n is an even number.

It isn't difficult to verify that f is an I-total coloring. We can easily summarize up the following facts.

Fact 1: $C(u_2), C(u_3), \ldots, C(u_m)$ are mutually different.

Proof. The subgraph induced by all colors of $C(u_i) \backslash \{1\} (i = 2, 3, \ldots, m)$ in $\widetilde{C}_{m+n+2}$ is a path of order $n+2$. The initial point is i for $i = 2, 3, \ldots, m$. As the initial points of the $m-1$ paths are distinct, Therefore $C(u_2)$, $C(u_3)$, $\cdots$, $C(u_m)$ are mutually different.

Fact 2: $C(v_2), C(v_3), \ldots, C(v_{n-1})$ are mutually different.

Proof. The subgraph induced by all colors of $C(v_j) \setminus \{n+m, n+m+1\} (j = 3, 4, \ldots, n-1)$ in $\widetilde{C}_{m+n+2}$ is a path of order $m+2$. The starting point is $j-2, j = 3, 4, \ldots, n-1$. Because the initial points of the $n-3$ paths are distinct, $C(v_3), C(v_4), \ldots, C(v_{n-1})$ are mutually different. Furthermore,

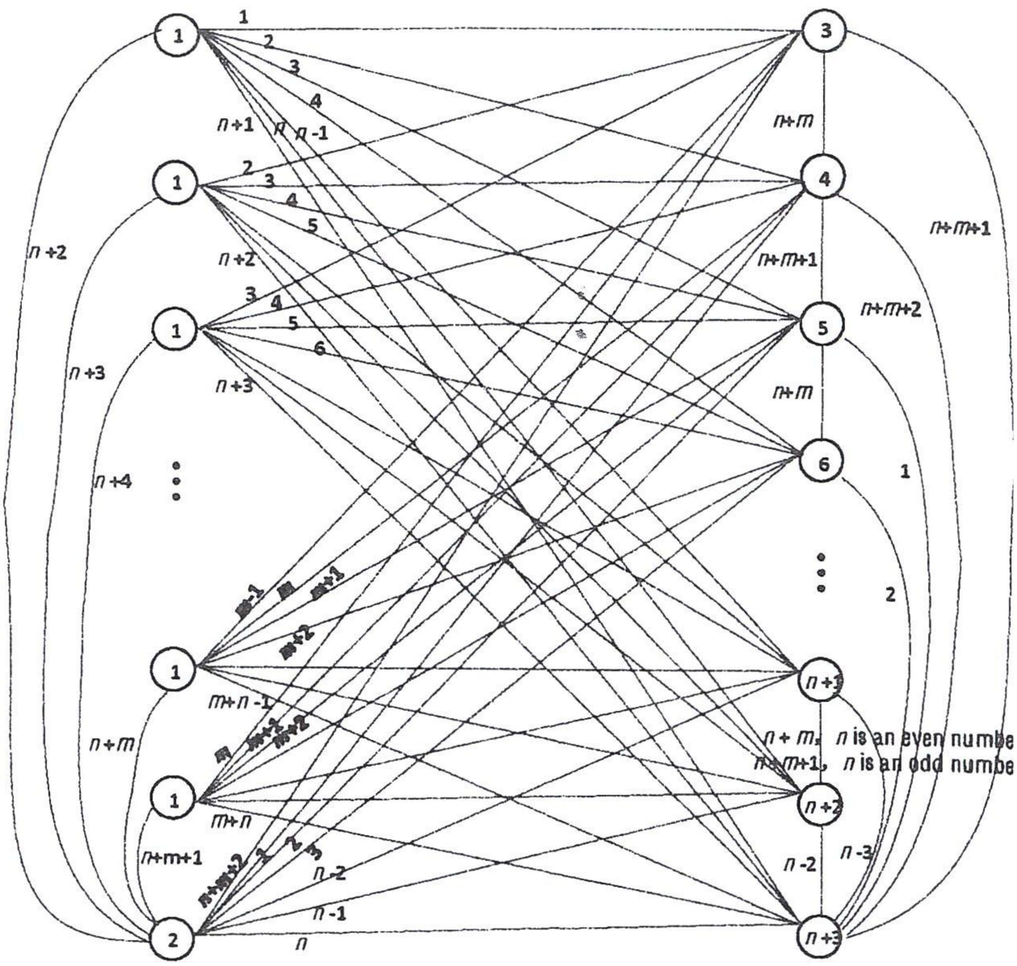

Figure 2: An $(m+n+2)$-VDIT coloring of $S_m \vee F_n (m \geq 3, n \geq 2)$.

$C(v_2)$ contains color $n+m+2$ while $C(v_j)(j = 3, 4, \ldots, n-1)$ doesn't contain color $n+m+2$. So $C(v_2) \neq C(v_j)(j = 3, 4, \ldots, n-1)$.

Now we will confirm that f is vertex-distinguishing.

(1) $n = 2$.

There are 4 vertices u_0, v_0, v_1 and v_2 whose color sets contain $m+3$ colors. This time, $\overline{C(u_0)} = \{3\}$, $\overline{C(v_0)} = \{1\}$, $\overline{C(v_1)} = \{m+1\}$, $\overline{C(v_2)} = \{m+3\}$. So $C(u_0)$, $C(v_0)$, $C(v_1)$ and $C(v_2)$ are different.

There are exactly $m-1$ vertices $u_2, u_3, \ldots, u_m$ such that $|C(u_i)| = 5$, $i = 2, 3, \ldots, m$. According to fact 1, $C(u_2), C(u_3), \ldots, C(u_m)$ are mutually different.

There is only one vertex u_1 whose color set contains 4 colors.

(2) $n = 3$.

There are 3 vertices u_0, v_0 and v_2 whose color sets contain $m+4$ colors. This time, $\overline{C(u_0)} = \{4\}$, $\overline{C(v_0)} = \{2\}$, $\overline{C(v_2)} = \{m+2\}$. So $C(u_0)$, $C(v_0)$

and $C(v_2)$ are different.

i) For $m = 3$, there are 4 vertices u_2, u_3, v_1 and v_3 whose color sets contain 6 colors. This time, $\overline{C(u_2)} = \{7,8\}$, $\overline{C(u_3)} = \{2,8\}$, $\overline{C(v_1)} = \{4,5\}$, $\overline{C(v_3)} = \{6,8\}$. So $C(u_2)$, $C(u_3)$, $C(v_1)$ and $C(v_3)$ are distinct.

ii) For $m \geq 4$, there are 2 vertices v_1 and v_3 whose color sets contain $m+3$ colors. This time, $\overline{C(v_1)} = \{m+1, m+2\}$, $\overline{C(v_3)} = \{m+3, m+5\}$. So $C(v_1) \neq C(v_3)$. Meanwhile, there are exactly $m-1$ vertices $u_2, u_3, \ldots, u_m$ such that $|C(u_i)| = 6, i = 2, 3, \ldots, m$. According to fact 1, $C(u_2), C(u_3), \ldots, C(u_m)$ are mutually different.

There is only one vertex u_1 whose color set contains 5 colors.

(3) $n \geq 4$.

a) $n = m$.

There is only one vertex u_1 whose color set contains $m+2$ colors.

There are $m+1$ vertices $u_2, u_3, \ldots, u_m, v_1$ and v_n whose color sets contain $m+3$ colors. According to fact 1, $C(u_2), C(u_3), \ldots, C(u_m)$ are mutually different. Furthermore, $C(v_n)$ and $C(u_i)(i = 2, 3, \ldots, m)$ don't contain color $n+m+2$ while $C(v_1)$ contains color $n+m+2$. So $C(v_n) \neq C(v_1)$, and $C(u_i) \neq C(v_1)$ for $i = 2, 3, \ldots, m$. Meanwhile $C(u_i)$ $(i = 2, 3, \ldots, m)$ contains color 1, but $C(v_n)$ doesn't contains color 1. So $C(u_i) \neq C(v_n)$ for $i = 2, 3, \ldots, m$. Above all, $C(u_2)$,$C(u_3)$,...,$C(u_m)$,$C(v_1)$ and $C(v_n)$ are mutually distinct.

There are exactly $n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ such that $|C(v_j)| = m+4, j = 2, 3, \ldots, n-1$. According to fact 2, $C(v_2), C(v_3), \ldots, C(v_{n-1})$ are mutually different.

There are 2 vertices u_0 and v_0 whose color sets contain $2m+1$ colors. This time, $\overline{C(u_0)} = \{n+1\}$, $\overline{C(v_0)} = \{n-1\}$. So $C(u_0) \neq C(v_0)$.

b) $n = m+1$.

There are 3 vertices u_1, v_1 and v_n whose color sets contain $m+3$ colors. $C(u_1)$ and $C(v_n)$ don't contain color $n+m+2$ while $C(v_1)$ contains. Therefore, $C(u_1) \neq C(v_1)$, and $C(v_n) \neq C(v_1)$. Moreover, $C(u_1)$ contains color 1, but $C(v_n)$ doesn't contain color 1. So $C(u_1) \neq C(v_n)$. In summary, $C(u_1) \neq C(v_1) \neq C(v_n)$.

There are $m+n-3$ vertices $u_2, u_3, \ldots, u_m$ and $v_2, v_3, \ldots, v_{n-1}$ such that $|C(u_i)| = m+4, i = 2, 3, \ldots, m$ and $|C(v_j)| = m+4, j = 2, 3, \ldots, n-1$. According to fact 1, $C(u_2), C(u_3), \ldots, C(u_m)$ are mutually different. On the basis of fact 2, $C(v_2), C(v_3), \ldots, C(v_{n-1})$ are mutually distinct. $C(u_i)$ $(i = 2, 3, \ldots, m-1)$ doesn't contain color $n+m+1$ while $C(v_j)$ $(j = 2, 3, \ldots, n-1)$ contains $n+m+1$. So $C(u_i) \neq C(v_j)$ for $i = 2, 3, \ldots, m-1$ and $j = 2, 3, \ldots, n-1$. Now we need only to prove $C(u_m) \neq C(v_j)$ for $j = 2, 3, \ldots, n-1$. Obviously, $C(v_2)$ contains color $n+m+2$, but $C(u_m)$ doesn't contain color $n+m+2$. So $C(u_m) \neq C(v_2)$. It's easy to know $C(u_m) \neq C(v_j)$ for $j = 3, 4, \ldots, n-1$. Above all, $C(u_i) \neq C(v_j)$ for $i = 2, 3, \ldots, m$ and $j = 2, 3, \ldots, n-1$.

There are 2 vertices (u_0 and v_0) whose color sets contain $2m+2$ colors. $C(u_0) \neq C(v_0)$.

c) $n = m+2$.

There are 2 vertices v_1 and v_n whose color sets contain $m+3$ colors $C(v_1) \neq C(v_n)$.

There are $n-1$ vertices $u_1, v_2, v_3, \ldots, v_{n-1}$ whose color sets contain $m+4$ colors. On the basis of fact 2, $C(v_2), C(v_3), \ldots, C(v_{n-1})$ are mutually distinct. $C(v_j)(j=2,3,\ldots,n-1)$ contains color $n+m+1$, but $C(u_1)$ doesn't contain color $n+m+1$. So $C(u_1) \neq C(v_j)$ for $j=2,3,\ldots,n-1$.

There are exactly $m-1$ vertices $u_2, u_3, \ldots, u_m$ such that $|C(u_i)| = m+5, i=2,3,\ldots,m$. According to fact 1, $C(u_2), C(u_3), \ldots, C(u_m)$ are mutually different.

There are 2 vertices u_0 and v_0 whose color sets contain $2m+3$ colors. $C(u_0) \neq C(v_0)$.

d) The remainder cases.

There is only one vertex (u_1) whose color set contains $n+2$ colors. There are exactly $m-1$ vertices $u_2, u_3, \ldots, u_m$ such that $|C(u_i)| = n+3, i=2,3,\ldots,m$. There are 2 vertices v_1 and v_n whose color sets contain $m+3$ colors. There are exactly $n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ such that $|C(v_j)| = m+4$ with $j=2,3,\ldots,n-1$. There are 2 vertices (u_0 and v_0) whose color sets contain $n+m+1$ colors. The color sets of all vertices are different.

So we get an $(m+n+2)-$VDIT coloring f of $S_m \vee F_n (m \geq 3, n \geq 2)$.

The proof is completed.

Theorem 2 If $m \geq 2, n \geq 3$, then $\chi^i_{vt}(S_m \vee W_n) = m+n+2$.

Proof. Obviously, we know that $\Delta(S_m \vee W_n) = m+n+1$, $n_\Delta \geq 2$. According to proposition 1, we know $\chi^i_{vt}(S_m \vee W_n) \geq m+n+2$. Now we need only to give an $(m+n+2)-$VDIT coloring f of $S_m \vee W_n$.

There are three cases to be considered.

Case 1: $m=2, n \geq 3$.

Now based on the $(n+4)-$VDIT coloring f of $P_3 \vee F_n$ appeared in the proof of Theorem 1, we add the edge $v_1 v_n$ and color it $n+4$. Let $w_j = v_j$ $(j=0,1,\ldots,n)$. Under this coloring, the color sets of other vertices stay the same except for w_1 and w_n. $C(w_1) = C(v_1) \bigcup \{n+4\}$, $C(w_n) = C(v_n) \bigcup \{n+4\}$. It's easy to verify that f is an I-total coloring. Next we have to confirm that f is vertex-distinguishing.

(1) $n=3$.

There are 2 vertices u_1 and u_3 whose color sets contain 5 colors. This time, $\overline{C(u_1)} = \{6,7\}$, $\overline{C(u_3)} = \{2,7\}$. So $C(u_1) \neq C(u_3)$.

There are 5 vertices u_2, w_0, w_1, w_2 and w_3 whose color sets contain 6 colors. This time, $\overline{C(u_2)} = \{7\}$, $\overline{C(w_0)} = \{3\}$, $\overline{C(w_1)} = \{4\}$, $\overline{C(w_2)} = \{5\}$, $\overline{C(w_3)} = \{1\}$. So $C(u_2)$, $C(w_0)$, $C(w_1)$, $C(w_2)$ and $C(w_3)$ are different.

(2) $n=4$.

There are 6 vertices u_1, u_3, w_1, w_2, w_3 and w_4 whose color sets contain 6 colors. This time, $\overline{C(u_1)} = \{7,8\}$, $\overline{C(u_3)} = \{2,8\}$, $\overline{C(w_1)} = \{4,5\}$, $\overline{C(w_2)} = \{5,7\}$, $\overline{C(w_3)} = \{1,8\}$, $\overline{C(w_4)} = \{1,2\}$. Therefore, $C(u_1)$, $C(u_3)$, $C(w_1)$, $C(w_2)$, $C(w_3)$ and $C(w_4)$ are distinct.

There are 2 vertices u_2 and w_0 whose color sets contain 7 colors. This time, $\overline{C(u_2)} = \{8\}$, $\overline{C(w_0)} = \{4\}$. So $C(u_2) \neq C(w_0)$.

(3) $n \geq 5$.

There are n vertices $w_1, w_2, \ldots, w_n$ such that $|C(w_j)| = 6$ with $j = 1, 2, \ldots, n$. The subgraph induced by $C(w_j)$ $(j = 3, 4, \ldots, n)$ in $\widetilde{C}_{n+4}$ is a path of order 6. The initial point is $j-1$ with $j = 3, 4, \ldots, n$. As the initial points of the $n-2$ paths are distinct, therefore $C(w_3), C(w_4), \ldots, C(w_n)$ are mutually different. Obviously, $C(w_1) \neq C(w_2)$, $C(w_1) \neq C(w_j)$ $(j = 3, 4, \ldots, n)$, $C(w_2) \neq C(w_j)$ $(j = 3, 4, \ldots, n)$. Thus, $C(w_1), C(w_2), \cdots, C(w_n)$ are mutually different.

There are 2 vertices u_1 and u_3 whose color sets contain $n+2$ colors. We have proved $C(u_1) \neq C(u_3)$ in the proof of Theorem 1.

There are 2 vertices u_2 and w_0 whose color sets contain $n+3$ colors. We have confirmed $C(u_2) \neq C(w_0)$ in the proof of Theorem 1.

Thus we construct an $(n+4)-$VDIT coloring f of $P_3 \vee W_n$. So $\chi^i_{vt}(S_2 \vee W_n) = n + 4 (n \geq 3)$.

Case 2: $m = n = 3$.

Based on the following $(m+n+2)-$VDIT coloring f of $S_m \vee W_n$ appeared in Case 3, we exchange the colors of u_0 and w_2. Namely, let $f(u_0) = 4$, $f(w_2) = 2$. By listing the color sets of all vertices, we can prove that it is a 8–VDIT coloring of $S_3 \vee W_3$ easily. So $\chi^i_{vt}(S_3 \vee W_3) = 8$.

Case 3: $m = 3, n \geq 4$ or $m \geq 4, n \geq 3$.

By the $(m+n+2)-$VDIT coloring f of $S_m \vee F_n (m \geq 3, n \geq 2)$ appeared in the proof of Theorem 1, we add the edge $v_1 v_n$ and color it $n+m$. We change the color of edge $v_1 v_2$ and let $f(v_1 v_2) = n + m - 1$. Meanwhile, $f(v_j v_{j+1}) = n + m + 2$ if $j \in \{1, 2, \ldots, n-1\}$ and j is an odd number; $f(v_j v_{j+1}) = n + m + 1$, if $j \in \{1, 2, \ldots, n-1\}$ and j is an even number; Let $w_j = v_j (j = 0, 1, \ldots, n)$. We may see Figure 3 about this coloring in the next page.

Under this coloring, the color sets of u_i $(i = 0, 1, \ldots, m)$ and w_0 keep the same. Under this coloring,

$C(w_1) = \{1, 2, 3, \ldots, m, n+m-1, n+m, n+m+1, n+m+2\}$;

$C(w_2) = \{1, 2, 3, \ldots, m+1, n+m-1, n+m+1, n+m+2\}$;

$C(w_j) = \{j-2, j-1, j, \ldots, j+m-1, n+m+1, n+m+2\}$, for $j = 3, 4, \ldots, n-1$;

$C(w_n) = \{n-2, n-1, n, \ldots, n+m, n+m+1\}$ if n is an odd number; $C(w_n) = \{n-2, n-1, n, \ldots, n+m, n+m+2\}$ if n is an even number.

It isn't difficult to verify that f is an I-total coloring. Similarly, we have the following facts.

Fact 1: $C(u_2), C(u_3), \ldots, C(u_m)$ are mutually different.

The proof is the same as the proof of Fact 1 appeared in the proof of Theorem 1.

Fact 2: $C(w_1), C(w_2), \ldots, C(w_n)$ are mutually different.

Proof. The subgraph induced by $C(w_j) \setminus \{n+m+1, n+m+2\} (j = 3, 4, \ldots, n-1)$ in $\widetilde{C}_{m+n+2}$ is a path of order $m+2$. The initial point is $j-2, j = 3, 4, \ldots, n-1$. Since the initial points of the $n-3$ paths are distinct, $C(w_3), C(w_4), \ldots, C(w_{n-1})$ are mutually different. Furthermore, $C(w_1)$ and $C(w_2)$ contain color $n+m-1$ while $C(w_j) (j = 3, 4, \ldots, n-1)$ doesn't contain color $n+m-1$. So $C(w_1) \neq C(w_j), C(w_2) \neq C(w_j) (j =$

for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, n+1$; $f(w_0v_1) = 2n+2$, $f(w_0v_j) = j -$ for $j = 2, 3, \ldots, n+1$. We may see Figure 4 about this coloring.

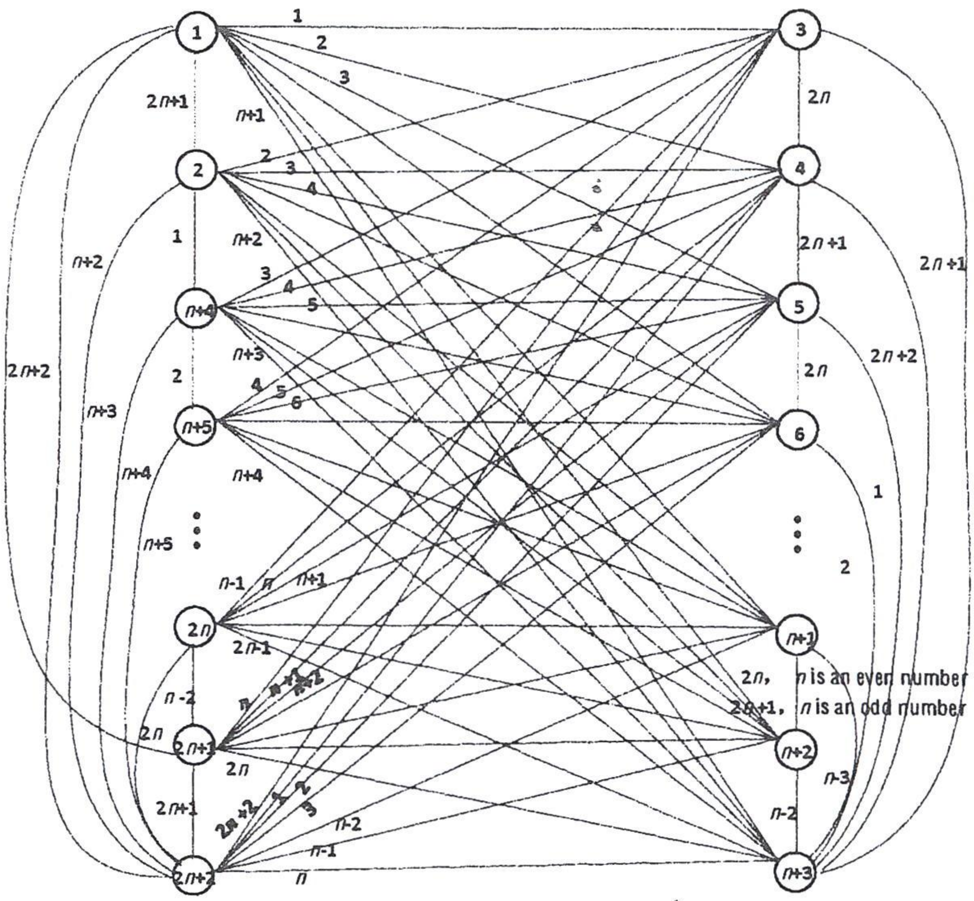

Figure 4: An $(2n+2)$-VDIT coloring of $F_n \vee W_n (n \geq 4)$.

Then under this coloring
$C(w_0) = \{1, 2, 3, \ldots, n, n+2, n+3, \ldots, 2n+2\}$;
$C(w_1) = \{1, 2, 3, \ldots, n+2, 2n+1, 2n+2\}$;
$C(w_2) = \{1, 2, 3, \ldots, n+3, 2n+1\}$;
$C(w_i) = \{i-2, i-1, i, \ldots, i+n+1\}$ for $i = 3, 4, \ldots, n-1$;
$C(w_n) = \{n-2, n, n+1, \ldots, 2n+2\}$;
$C(v_0) = \{1, 2, 3, \ldots, n-2, n, n+1, \ldots, 2n+2\}$;
$C(v_1) = \{1, 2, 3, \ldots, n, 2n, 2n+1, 2n+2\}$;
$C(v_2) = \{1, 2, 3, \ldots, n+1, 2n, 2n+1, 2n+2\}$;
$C(v_j) = \{j-2, j-1, j, \ldots, j+n-1, 2n, 2n+1\}$ for $j = 3, 4, \ldots, n-1$;
$C(v_n) = \{n-2, n-1, n, \ldots, 2n-1, 2n+1\}$ if n is an odd number; $C(v_n) = \{n-2, n-1, n, \ldots, 2n-1, 2n\}$ if n is an even number.

It is easy to verify that f is an I-total coloring. Now we need only to confirm that f is vertex-distinguishing.

There are 2 vertices v_1 and v_n whose color sets contain $n+3$ colors. This time, $C(v_1)$ contains color 1, but $C(v_n)$ doesn't contain color 1. So $C(v_1) \neq C(v_n)$.

There are $2n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ and $w_1, w_2, \ldots, w_n$ such that $|C(v_j)| = n+4$ for $j = 2, 3, \ldots, n-1$ and $|C(w_i)| = n+4$ with $i = 1, 2, \ldots, n$. The subgraph induced by $C(w_i)$ $(i = 3, 4, \ldots, n-1)$ in $\widetilde{C}_{2n+2}$ is a path of order $n+4$. The initial point is $i-2, i = 3, 4, \ldots, n-1$. Because the initial points of the $n-3$ paths are distinct, $C(w_3), C(w_4), \ldots, C(w_{n-1})$ are mutually different. By analyzing the color set, we can easily know that $C(w_1), C(w_2), \ldots, C(w_n)$ are mutually different. Meanwhile, the subgraph induced by $C(v_j) \setminus \{2n, 2n+1\}$ $(j = 3, 4, \ldots, n-1)$ in $\widetilde{C}_{2n+2}$ is a path of order $n+2$. The initial point is $j-2, j = 3, 4, \ldots, n-1$. As the initial points of the $n-3$ paths are distinct, $C(v_3), C(v_4), \ldots, C(v_{n-1})$ are mutually different. Obviously, $C(v_2) \neq C(v_j)(j = 3, 4, \ldots, n-1)$. By analyzing the color set, we can easily know that $C(v_j) \neq C(w_i)$ for $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n-1$.

There are 2 vertices v_0 and w_0 whose color sets contain $2n+1$ colors. This time, $\overline{C(v_0)} = \{n-1\}$, $\overline{C(w_0)} = \{n+1\}$. So $C(v_0) \neq C(w_0)$.

Above all, we get a $(2n+2)$-VDIT coloring f of $F_n \vee W_n (n \geq 4)$. So $\chi^i_{vt}(F_n \vee W_n) = 2n+2 (n \geq 4)$.

Case 2: $n = 3$.

Based on the coloring f in Case 1, we change the colors of w_2 and w_3 such that $f(w_2) = 7$ and $f(w_3) = 2$ with other conditions unchanged. According to this specific coloring, we have $\chi^i_{vt}(F_3 \vee W_3) = 8$.

The proof is over.

According to Proposition 1, we can get the following three theorems.

Theorem 4 If $m, n \geq 2$, then $\chi^{vi}_{vt}(S_m \vee F_n) = m+n+2$.

Theorem 5 If $m \geq 2, n \geq 3$, then $\chi^{vi}_{vt}(S_m \vee W_n) = m+n+2$.

Theorem 6 If $n \geq 3$, then $\chi^{vi}_{vt}(F_n \vee W_n) = 2n+2$.

For the above join graphs, we can calculate the value of ζ according to its expression. $\zeta(S_m \vee F_n) = m+n+2$, $\zeta(S_m \vee W_n) = m+n+2$, $\zeta(F_n \vee W_n) = 2n+2$. We find that the conclusions obtained in this paper are consistent with Conjecture 1 and Conjecture 2. We substituted using ζ by proposition 1(ii) to give a lower bound of χ^i_{vt} and χ^{vi}_{vt} in this paper. That's to say, we still can determine the lower bounds of the vertex-distinguishing I[or VI]-total chromatic number by computing ζ. Therefore, proposing conjectures is an important milestone in solving mathematical problems.

Acknowledgement The authors would like to thank the referees for their valuable comments and helpful suggestions.

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for $i = 1, 2, \ldots, n; j = 1, 2, \ldots, n+1$; $f(w_0v_1) = 2n+2$, $f(w_0v_j) = j -$ for $j = 2, 3, \ldots, n+1$. We may see Figure 4 about this coloring.

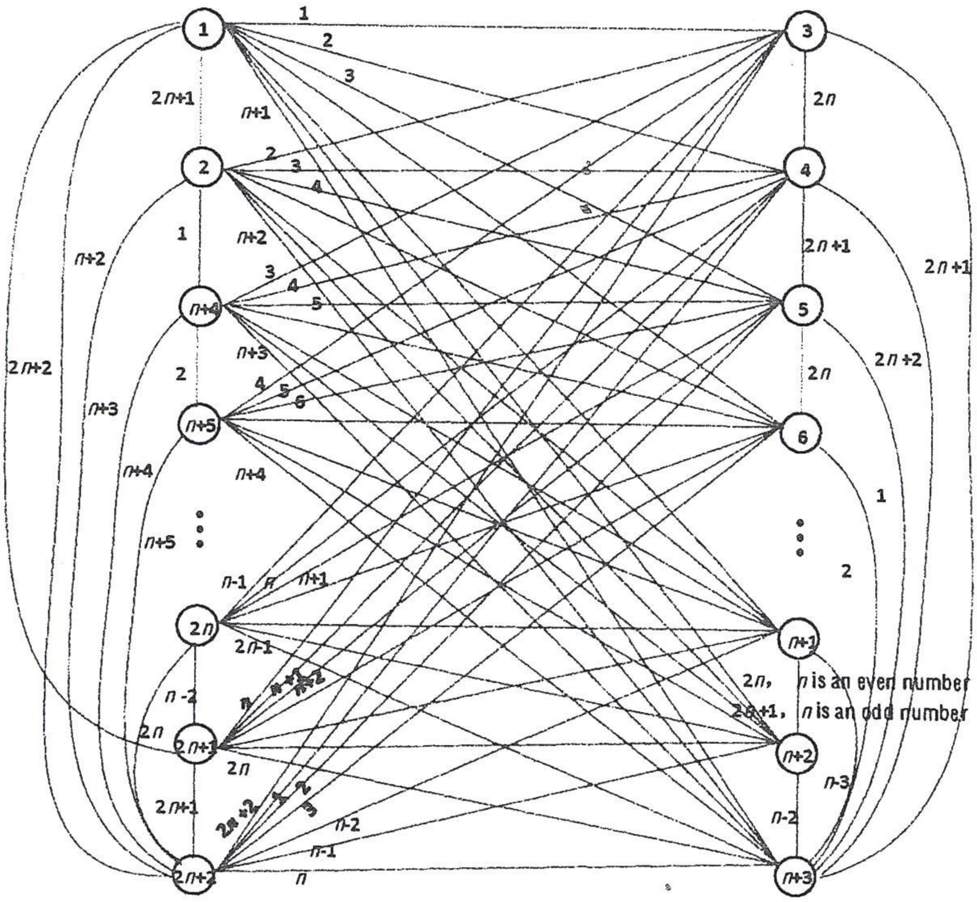

Figure 4: An $(2n+2)$-VDIT coloring of $F_n \vee W_n (n \geq 4)$.

Then under this coloring

$C(w_0) = \{1, 2, 3, \ldots, n, n+2, n+3, \ldots, 2n+2\}$;
$C(w_1) = \{1, 2, 3, \ldots, n+2, 2n+1, 2n+2\}$;
$C(w_2) = \{1, 2, 3, \ldots, n+3, 2n+1\}$;
$C(w_i) = \{i-2, i-1, i, \ldots, i+n+1\}$ for $i = 3, 4, \ldots, n-1$;
$C(w_n) = \{n-2, n, n+1, \ldots, 2n+2\}$;
$C(v_0) = \{1, 2, 3, \ldots, n-2, n, n+1, \ldots, 2n+2\}$;
$C(v_1) = \{1, 2, 3, \ldots, n, 2n, 2n+1, 2n+2\}$;
$C(v_2) = \{1, 2, 3, \ldots, n+1, 2n, 2n+1, 2n+2\}$;
$C(v_j) = \{j-2, j-1, j, \ldots, j+n-1, 2n, 2n+1\}$ for $j = 3, 4, \ldots, n-1$
$C(v_n) = \{n-2, n-1, n, \ldots, 2n-1, 2n+1\}$ if n is an odd number
$C(v_n) = \{n-2, n-1, n, \ldots, 2n-1, 2n\}$ if n is an even number.

It is easy to verify that f is an I-total coloring. Now we need only to confirm that f is vertex-distinguishing.

There are 2 vertices v_1 and v_n whose color sets contain $n+3$ colors. This time, $C(v_1)$ contains color 1, but $C(v_n)$ doesn't contain color 1. So $C(v_1) \neq C(v_n)$.

There are $2n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$ and $w_1, w_2, \ldots, w_n$ such that $|C(v_j)| = n+4$ for $j = 2, 3, \ldots, n-1$ and $|C(w_i)| = n+4$ with $i = 1, 2, \ldots, n$. The subgraph induced by $C(w_i)$ $(i = 3, 4, \ldots, n-1)$ in $\widetilde{C}_{2n+2}$ is a path of order $n+4$. The initial point is $i-2, i = 3, 4, \ldots, n-1$. Because the initial points of the $n-3$ paths are distinct, $C(w_3), C(w_4), \ldots, C(w_{n-1})$ are mutually different. By analyzing the color set, we can easily know that $C(w_1), C(w_2), \ldots, C(w_n)$ are mutually different. Meanwhile, the subgraph induced by $C(v_j) \setminus \{2n, 2n+1\}$ $(j = 3, 4, \ldots, n-1)$ in $\widetilde{C}_{2n+2}$ is a path of order $n+2$. The initial point is $j-2, j = 3, 4, \ldots, n-1$. As the initial points of the $n-3$ paths are distinct, $C(v_3), C(v_4), \ldots, C(v_{n-1})$ are mutually different. Obviously, $C(v_2) \neq C(v_j)(j = 3, 4, \ldots, n-1)$. By analyzing the color set, we can easily know that $C(v_j) \neq C(w_i)$ for $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n-1$.

There are 2 vertices v_0 and w_0 whose color sets contain $2n+1$ colors. This time, $\overline{C(v_0)} = \{n-1\}$, $\overline{C(w_0)} = \{n+1\}$. So $C(v_0) \neq C(w_0)$.

Above all, we get a $(2n+2)$-VDIT coloring f of $F_n \vee W_n (n \geq 4)$. So $\chi_{vt}^{i}(F_n \vee W_n) = 2n+2 (n \geq 4)$.

Case 2: $n = 3$.

Based on the coloring f in Case 1, we change the colors of w_2 and w_3 such that $f(w_2) = 7$ and $f(w_3) = 2$ with other conditions unchanged. According to this specific coloring, we have $\chi_{vt}^{i}(F_3 \vee W_3) = 8$.

The proof is over.

According to Proposition 1, we can get the following three theorems.

Theorem 4 If $m, n \geq 2$, then $\chi_{vt}^{vi}(S_m \vee F_n) = m+n+2$.

Theorem 5 If $m \geq 2, n \geq 3$, then $\chi_{vt}^{vi}(S_m \vee W_n) = m+n+2$.

Theorem 6 If $n \geq 3$, then $\chi_{vt}^{vi}(F_n \vee W_n) = 2n+2$.

For the above join graphs, we can calculate the value of ζ according to its expression. $\zeta(S_m \vee F_n) = m+n+2$, $\zeta(S_m \vee W_n) = m+n+2$, $\zeta(F_n \vee W_n) = 2n+2$. We find that the conclusions obtained in this paper are consistent with Conjecture 1 and Conjecture 2. We substituted using ζ by proposition 1(ii) to give a lower bound of χ_{vt}^{i} and χ_{vt}^{vi} in this paper. That's to say, we still can determine the lower bounds of the vertex-distinguishing I[or VI]-total chromatic number by computing ζ. Therefore, proposing conjectures is an important milestone in solving mathematical problems.

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