$(0,1)$-Matrices, Discrepancy, and Preservers II*†

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Abstract

Let m and n be positive integers, and let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be non-negative integral vectors. Let $A(R, S)$ be the set of all $m \times n$ $(0,1)$-matrices with row sum vector R and column vector S. Let R and S be non increasing, and let $F(R, S)$ be the $m \times n$ $(0,1)$-matrix where for each i, the i^{th} row of $F(R, S)$ consists of r_i 1's followed by $n - r_i$ 0's, called Ferrers matrices. The discrepancy of an $m \times n$ $(0,1)$-matrix A, $disc(A)$, is the number of positions in which $F(R, S)$ has a 1 and A has a 0. In this paper we investigate linear operators mapping $m \times n$ matrices over the binary Boolean semiring to itself that preserve sets related to the discrepancy. In particular we characterize linear operators that preserve both the set of Ferrers matrices and the set of matrices of discrepancy one.

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1 Introduction

In the area of graph theory or discrete matrix theory, dete mining if an object belongs to a specified set may be a dif cult problem. In the current topic, finding the discrepancy a (0, 1)-matrix is an NP-complete problem, [4]. One method identifying objects in a set is to identify a known set of matric with that property and apply transformations that preserve th property to expand the known set.

In the study of plant species versus biological pollinators, bipartite graph is an obvious tool for analysis. To study bipa tite graphs we often use the reduced adjacency matrix (a (0, 1 matrix). Thus, we will study (0, 1)-matrices whose arithmet is Boolean, that is the same arithmetic as for the real numbe except that 1+1=1. With this arithmetic there is a isomorphis between the set of bipartite graphs (using union and interse tion as the arithmetic) and the set of $m \times n$, Boolean, (0, 1 matrices. We let $\mathcal{M}_{m,n}(\mathbb{B})$ denote this set of all $m \times n$ Boolea (0, 1)-matrices. Let $E_{i,j}$ be the matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ which h exactly one nonzero entry, a one in the (i, j) position. L $J_{m,n} \in \mathcal{M}_{m,n}(\mathbb{B})$ denote the matrix of all ones, $O_{m.n} \in \mathcal{M}_{m,n}($ denote the zero matrix and I_n denote the $n \times n$ identity matrix. no confusion arises, we suppress the subscripts and write $J, O,$ If A and B are matrices in $\mathcal{M}_{m,n}(\mathbb{B})$, and $b_{i,j} = 0$ implies $a_{i,j} =$ we say that B *dominates* A and write $B \sqsupseteq A$ or $A \sqsubseteq B$.

A linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ is a mapping T which is a ditive, that is $T(A + B) = T(A) + T(B)$, and homogeneou $T(\alpha A) = \alpha T(A)$. It is easily seen that a linear operator $\mathcal{M}_{m,n}(\mathbb{B})$ is any additive map such that $T(O) = O$. Let T $\mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. We say that T *p serves* a set $\mathcal{X} \subseteq \mathcal{M}_{m,n}(\mathbb{B})$ if $A \in \mathcal{X}$ implies that $T(A) \in$ The operator *strongly preserves* the set $\mathcal{X}$ if $A \in \mathcal{X}$ if and onl $T(A) \in \mathcal{X}$. Thus, "T strongly preserves the set $\mathcal{X}$" is equivale to saying "T preserves the set $\mathcal{X}$ and T preserves its comp ment, $\mathcal{M}_{m,n}(\mathbb{B})\backslash\mathcal{X}$".

Let f be a function on $\mathcal{M}_{m,n}(\mathbb{B})$. We say that T *preserves* f if T preserves the set $\{X \in \mathcal{M}_{m,n}(\mathbb{B}) | f(X) = r\}$ for each r in the image of f. That is,

T preserves f if and only if, for each r in the image of f,

$$T \text{ (strongly) preserves } f^{-1}(r). \tag{1}$$

In [1] there were four equivalent definitions for Ferrers matrices given. Here we shall prefer:

Definition 1.1 *An $m \times n$ matrix A of zeros and ones is called a* Ferrers matrix *if $a_{i,j} = 1$ implies that for all $k \leq i$ and $\ell \leq j$ $a_{k,\ell} = 1$,*

Note that every $1 \times n$ and $m \times 1$ matrix of zeros and ones which has non increasing row and column sums is a Ferrers matrix, and the transpose of a Ferrers matrix is a Ferrers matrix. So, henceforth we assume that $2 \leq m \leq n$.

Let $\mathbb{Z}_+$ denote the set of all nonnegative integers so that $\mathbb{Z}_+^k$ is the set of all k tuples of nonnegative integers. Let $Q(m,n) = \{(R,S) | R \in \mathbb{Z}_+^n, S \in \mathbb{Z}_+^m, n \geq r_1 \geq r_2 \geq \ldots \geq r_m, m \geq s_1 \geq s_2 \geq \ldots \geq s_n\}$. That is, $Q(m,n)$ is the set of ordered tuples of non increasing sequences of length m and n from $\{0,1,2,\ldots,n\}$ and $\{0,1,2,\ldots,m\}$ respectively.

Let $(R,S) \in Q(m,n)$ and define $A(R,S)$ to be the subset of $\mathcal{M}_{m,n}(\mathbb{B})$ consisting of matrices with r_i nonzero entries in row i and s_j nonzero entries in column j where r_i is the i^{th} component of R and s_j is the j^{th} component of S. Note that in order that $A(R,S) \neq \emptyset$, we must have that $r_1 + r_2 + \ldots + r_m = s_1 + s_2 + \ldots + s_n$.

Definition 1.2 *Let $B \in A(R,S)$ for some $(R,S) \in Q(m,n)$. The* discrepancy *of B, $disc(B)$, is the minimum number of ones* *exchanged with zeros in the same row of B that yields a Ferre matrix.*

It was shown in [1] that the discrepancy of a zero one matr is not independent of permutation of columns which mainta the non increasing nature of the columns.

The discrepancy is only defined for matrices in $A(R,S)$ f $(R,S) \in Q(m,n)$. For matrices not in $A(R,S)$ for some (R,S) $Q(m,n)$, let the discrepancy be ∞.

2 Preservers.

Note that for transformations $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{K}$ for $\mathcal{K}$ a monoi T is nonsingular means only that $T(X) = O$ only if $X = O$. U like transformations on vector spaces (over a field), being no singular does not imply invertibility. If Z is a matrix in $\mathcal{M}_{m,n}($ and $Y \sqsubseteq Z$, let $Z \setminus Y = Q$ where $q_{i,j} = 1$ if and only if $z_{i,j} =$ and $y_{i,j} = 0$.

In [1] the following theorem was established:

Theorem 2.1 [1, Theorem 4.2] *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}($ be a bijective linear operator that maps the set of Ferrers mat ces to itself. Then either:*

1. *T is the identity; or*

2. *$m = n$ and T is the transpose operator.*

We now continue that investigation into the preservers sets of $(0,1)$-matrices defined by discrepancy.

Lemma 2.1 *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operat If T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferr matrices ($disc(A) = 0$), then T is nonsingular.*

Proof. Let $J_{(k)} = \begin{bmatrix} J_{k,n} \\ O_{m-k,n} \end{bmatrix}$. Suppose $T(X) = O$ for some $X \neq O$. Then there exists (k, ℓ) such that $T(E_{k,\ell}) = O$.

Case 1. $\ell < n$. In this case, $T(J_{(k)}) = T(J_{(k)} \setminus E_{k,\ell})$, while $disc(J_{(k)}) = 0$ and $disc(J_{(k)} \setminus E_{k,\ell}) = 1$, which contradicts that T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferrers matrices.

Case 2. $\ell = n$. In this case, $T(J_{(k)} \setminus (E_{k,n-1} \cup E_{k,n})) = T(J_{(k)} \setminus E_{k,n-1})$ since $T(E_{k,n}) = O$. But $disc(J_{(k)} \setminus (E_{k,n-1} \cup E_{k,n})) = 0$ while $disc(J_{(k)} \setminus E_{k,n-1}) = 1$ contradicting that T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferrers matrices.

Having arrived at a contradiction in both cases we have that T is nonsingular. ■

Suppose that A is a matrix of 0's and 1's of discrepancy 1. Then, there is exactly one entry equal to 1 that can be exchanged with a zero entry in the same row to give a Ferrers matrix. Call that entry an *exchangeable 1*.

Example 2.1 *Let $F_2, F_3, \ldots, F_m$ be Ferrers matrices in $\mathcal{M}_{m,n}(\mathbb{B})$ such that for $i = 2, \ldots, m$, F_i has only zero entries in rows $i, i+1, \ldots, m$. Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be defined by $T(E_{i,j}) = E_{i,j}$ for all (i,j) such that $j < n$, $T(E_{1,n}) = E_{1,n}$, and $T(E_{i,n}) = E_{i,n} + F_i$ for $i = 2, \ldots, m$. Then T preserves the set of matrices of discrepancy 0 (Ferrers matrices) and the set of matrices of discrepancy 1. To verify this observe that if the last column of a matrix is zero its image is itself. If A is a Ferrers matrix with nonzero last column, then by definition 1.1, the last column of A consists of j ones followed by $m-j$ zeros for some j, and the first $n-1$ columns is a Ferrers matrix. So $T(A)$ has last column the same as the last column of A and the first $n-1$ columns is the sum of the first $n-1$ columns of A plus $j-1$ Ferrers matrices whose only nonzero entries are in the first $j-1$*

rows. That is $A = \begin{bmatrix} J_{j,n} \\ F \end{bmatrix}$ *where* F *is an* $(m-j) \times n$ *Ferre matrix.. Thus* $T(A) = A$ *and hence* $T(A)$ *is a Ferrers matrix.*

If A *is a matrix of discrepancy 1, Suppose that the exchang able 1 is in the first* $n-1$ *columns. Then* A *has all nonze entries in any row that has an entry in the last column, so th* $T(A) = A$ *has discrepancy 1. Suppose that the exchangeable is in the last column. say the exchangeable 1 is the* (i,n) *entr Then* $T(A \setminus E_{i,n}) = A \setminus E_{i.n}$. *Thus,* $T(A) = T((A \setminus E_{i,n}) + E_{i,n})$ $T(A \setminus E_{i,n}) + T(E_{i,n}) = A + F_i$. *Further,* $disc(A + F_i) = 1$ *sin* F_i *has only nonzero entries in the first* $i-1$ *rows and is a Ferre matrix. Thus* $T(A)$ *has discrepancy 1.*

If $U \in \mathcal{M}_{m,n}(\mathbb{B})$ let $<U>$ denote the semimodule consisti of all matrices dominated by U, and let $|U|$ denote the numb of nonzero entries in U.

Lemma 2.2 *Let* $U, V \in \mathcal{M}_{m,n}(\mathbb{B})$ *with* $|U| \geq |V|$. *If* $T : <$ $> \to <V>$ *is nonsingular and for any cell* $E \sqsubseteq U$, $T(U \setminus E)$ $T(U)$, *then* $|U| = |V|$ *and* T *is bijective.*

Proof. Let $q = |U|$. Suppose that $|T(E)| > 1$ for some c $E \sqsubseteq U$. Order the cells H_i of $<U>$ by $H_1 = E$ and H_i is chos arbitrarily from the remaining $|U| - i$ cells. If for any $i >$ $T(H_1 + \ldots + H_i) = T(H_1 + \ldots + H_{i-1})$ then $T(U \setminus H_i) = T(U$ a contradiction. But then, we must have $|T(H_1 + \ldots + H_q)|$ $q \geq |V|$, a contradiction. Thus $|T(H_i)| = 1$ for all i. That is maps cells to cells since T is nonsingular.

If $T(E) = T(F)$ for two distinct cells E and F dominat by U, then $T(U) = T((U \setminus (F) + (F)) = T(U \setminus F) + T(F)$ $T(U \setminus F) + T(E) = T(U \setminus F)$ since $E \sqsubseteq U \setminus F$, a contradicti Thus, $|U| = |V|$ and T is a bijection.

Lemma 2.3 *Let* $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ *be a linear ope tor. If* T *preserves the set* $\{A \mid disc(A) = 1\}$ *and the set*

Ferrers matrices ($disc(A) = 0$), then the restriction of the map T to the semimodule $<[J_{m,n-2}\ O_{m,2}]>$ maps $<[J_{m,n-2}\ O_{m,2}]>$ bijectively onto $<T([J_{m,n-2}\ O_{m,2}])>$.

Proof. We first show that T is bijective. Let $U = [J_{m,n-2}\ O_{m,2}]$. Suppose that $|T(U)| > m(n-2)$.

Let $X_{0,0} = [J_{m,n-2}\ O_{m,2}]$. We now define matrices lexicographically starting with $X_{1,1} = [J_{m,n-2}\ O_{m,2}] + E_{1,n} = X_{0,0} + E_{1,n}$, and $X_{1,2} = [J_{m,n-2}\ O_{m,2}] + E_{1,n} + E_{1,n-1} = X_{1,1} + E_{1,n-1}$, etc. Continuing we get:

$$\begin{array}{rcll}
X_{1,1} &=& X_{0,0} + E_{1,n} & \text{and} \quad X_{1,2} = X_{1,1} + E_{1,n-1} \\
X_{2,1} &=& X_{1,2} + E_{2,n} & \text{and} \quad X_{2,2} = X_{2,1} + E_{2,n-1} \\
&\vdots&& \qquad\qquad\quad \vdots \\
X_{j,1} &=& X_{j-1,2} + E_{j,n} & \text{and} \quad X_{j,2} = X_{j,1} + E_{j,n-1} \\
&\vdots&& \qquad\qquad\quad \vdots \\
X_{m,1} &=& X_{m-1,2} + E_{m,n} & \text{and} \quad X_{m,2} = X_{m,1} + E_{m,n-1}
\end{array}$$

so that $X_{m,2} = J_{m,n}$.

Since $|T(U)| > m(n-2)$ and $|T(X_{m,2})| \leq mn$, there is some j such that either $|T(X_{j,1})| = |T(X_{j-1,2})|$ or $|T(X_{j,2})| = |T(X_{j,1})|$. But since $X_{j,1} \sqsupseteq X_{j-1,2}$ and $X_{j,2} \sqsupseteq X_{j,1}$ we have that either $T(X_{j,1}) = T(X_{j-1,2})$ or $T(X_{j,2}) = T(X_{j,1})$. Now, since for any $k \geq 1$, $disc(X_{k,1}) = 1$ and $disc(X_{k,2}) = 0$, we have contradicted that T preserves both the set of matrices of discrepancy 0 and the set of matrices of discrepancy 1. Thus, $|T(U)| \leq m(n-2)$.

By Lemma 2.2, the restriction of operator T to the semimodule $<[J_{m,n-2}\ O_{m,2}]>$ maps $<[J_{m,n-2}\ O_{m,2}]>$ bijectively onto $<T([J_{m,n-2}\ O_{m,2}])>$. ■

Let X be any matrix in $\mathcal{M}_{m,n}(\mathbb{B})$. Let $R_i(X)$ denote the $1 \times n$ matrix consisting of the i^{th} row of X, and $C_j(X)$ denote the $m \times 1$ matrix consisting of the j^{th} column of X.

Lemma 2.4 *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operato If T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferre matrices ($disc(A) = 0$), then $T([J_{m,n-2}\ \ O_{m,2}]) = [J_{m,n-2}\ \ O_{m,2}$*

Proof. Let $U = T([J_{m,n-2}\ \ O_{m,2}])$ and let L be the restri tion of T to the semimodule $<[J_{m,n-2}\ \ O_{m,2}]>$, That is, L $<[J_{m,n-2}\ \ O_{m,2}]> \to <U>$ is a linear operator that preserv the set $\{A \mid disc(A) = 1\}$ and the set of Ferrers matric $(disc(A) = 0)$. By Lemma 2.3 L is bijective.

Since L is bijective, $L(E_{1,1}) = E_{r,s}$ for some r, s. But the onl Ferrers matrix that has only one nonzero entry is $E_{1,1}$. Thu $L(E_{1,1}) = E_{1,1}$.

Consider $L(E_{1,2})$. The only two Ferrers matrices with exact two nonzero entries are $E_{1,1} + E_{1,2}$ and $E_{1,1} + E_{2,1}$.

Case 1: $L(E_{1,2}) = E_{1.2}$. Then, $L(E_{1,3})$ must be $E_{1,3}$ becau L is bijective and, hence, $L(E_{1,1} + E_{1,2} + E_{1,3})$ must be a Ferre matrix and the only possible other choice would be $E_{2,1}$ which impossible because then $L(E_{1,1} + E_{1,3})$ which has discrepancy would be a Ferrers matrix, contracting that L preserves the s $\{A \mid disc(A) = 1\}$. Following this type of argument, we arri at $T(R_1([J_{m,n-2}\ \ O_{m,2}]) \sqsubseteq R_1(U)$.

Continuing in like manner we get that $L(C_1) = C_1$, an since $L(E_{j,1}) = E_{j,1}$ we get that $L(R_i([J_{m,n-2}\ \ O_{m,2}]) \sqsubseteq R_i(U$ But since $|U| = |[J_{m,n-2}\ \ O_{m,2}]|$ we must have that each row of has the same number of nonzero entries as does $[J_{m,n-2}\ \ O_{m,}$ That is $U = [J_{m,n-2}\ \ O_{m,2}]$.

Case 2: $L(E_{1,2}) = E_{2,1}$. Then, $L(E_{1,3})$ must be $E_{3,1}$ becau L is bijective and, hence, $L(E_{1,1} + E_{1,2} + E_{1,3})$ must be a Ferre matrix and the only possible other choice would be $E_{1,2}$ whi is impossible because then $L(E_{1,1} + E_{1,3})$ which has discrepan 1 would be a Ferrers matrix, contracting that L preserves t set $\{A \mid disc(A) = 1\}$.

Following this pattern, we arrive at $L(R_1([J_{m,n-2}\ \ O_{m,2}])$

$C_1(U)$. But for any $j > 2$, $E_{1,1}+E_{1,j}$ has discrepancy 1 while the sum of any two cells in column one has either discrepancy zero or infinity. Thus, case 2 is impossible and the lemma follows. ■

In the following lemma, for α and β sequences of positive integers, let $A[\alpha|\beta]$ denote the submatrix of A on rows indexed by α and columns indexed by β.

Lemma 2.5 *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. If T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferrers matrices ($disc(A) = 0$), then the restriction of the map T to the semimodule $<[J_{m,n-2}\ O_{m,2}]>$ is the identity map from $<[J_{m,n-2}\ O_{m,2}]>$ onto itself.*

Proof. Let $L : \mathcal{M}_{m,n-2}(\mathbb{B}) \to \mathcal{M}_{m,n-2}(\mathbb{B})$ be defined for $X \in \mathcal{M}_{m,n-2}(\mathbb{B})$, $L(X) = T([X\ O_{m,2}])[1,2,\ldots,m|1,2,\ldots,n-2]$. Then, by the above lemmas, L is bijective and preserves the set of Ferrers matrices and the set of matrices of discrepancy 1. By Theorem 2.1, L is either the identity, or $m = n-2$ and L is the transpose operator. However, considering the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, whose discrepancy is 1 but whose transpose has discrepancy 2, one sees that the transpose operator does not preserve discrepancy 1, hence L is the identity operator. The lemma now follows by the definition of L. ■

In the following lemma let $J^{(\ell)} = [J_{m,\ell}\ O_{m,n-\ell}]$ and let $X^{(\ell)} = J^{(n-2)} + E_{1,n-1} + E_{2,n-1} + \ldots + E_{\ell,n-1}$.

Lemma 2.6 *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. If T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferrers matrices ($disc(A) = 0$), then the restriction of the map T to the semimodule $<[J_{m,n-1}\ O_{m,1}]>$ is the identity map from $<[J_{m,n-1}\ O_{m,1}]>$ onto itself.*

Proof. By Corollary 2.5, for any $j \leq n-2$ and any i, $T(E_{i,j})$ $E_{i,j}$. Consider $T(E_{i,n-1})$. Say for some $(r,s) \neq (i,n-1)$ th $E_{r,s} \sqsubseteq T(E_{i,n-1})$. Let k be the smallest k such that some su cell is dominated by $T(E_{k,n-1})$, that is $E_{r,s} \sqsubseteq T(E_{k,n-1})$ for sor $(r,s) \neq (k,n-1)$.

Case 1: $s \leq n-2$, or $r < k$ and $s = n-1$. In this ca $X^{(k)} \sqsupseteq E_{r,s}$ and hence $T(X^{(k)}) = T(X^{(k)} \setminus E_{r,s})$. Let Y $(X^{(k)} + E_{r,n}) \setminus E_{r,s}$ and $Z = X^{(k)} + E_{r,n}$. Then, $T(Y) = T($ but $disc(Y) = 1$ while $disc(Z) = 0$, a contradiction.

Case 2: $r > k$ and $s = n-1$. Let $Y = X^{(r)} + E_{1,n} + E_{2,n} + \ldots + E$ and $Z = Y \setminus E_{r,n-1}$. Then, $T(Y) = T(Z)$, but $disc(Y) = 0$ wh $disc(Z) = 1$, a contradiction.

Case 3: $s = n$. Note that by the above cases and Corollary 2 if $E_{r,n-1} \sqsubseteq T(X^{(i)})$ then $r = i$.

Subcase 1: $r > k$. In this case, $T(X^{(k)})$ must have discrepar 1 while $disc(X^{(k)}) = 0$, a contradiction.

Subcase 2: $r < k$. Let $Y = J^{(n-3)} + (E_{1,n-2} + E_{2,n-2} + \cdots$ $E_{k-1,n-2}) + E_{k,n-1}$. Then, by the above cases, $disc(Y) = 1$ wl $disc(T(Y)) \geq 2$ since $T(Y)$ has zeros in entries $(k, n-2)$ a $(r, n-1)$ and ones in entries $(k, n-1)$ and (r, n). This contradi that T preserves discrepancy 1.

Subcase 3: $r = k$. Let $Y = J^{(n-4)} + (E_{1,n-3} + E_{2,n-3} + \cdots$ $E_{k-1,n-3}) + (E_{1,n-2} + E_{2,n-2} + \ldots + E_{k-1,n-2}) + E_{k,n-1}$. Tl $disc(Y) = 1$ and $disc(T(Y))$ is at least two, a contradiction.

Since a contradiction has arisen in all the above cases a since T is nonsingular by Lemma 2.1 we have that $T(E_{i,n-1}$ $E_{i,n-1}$ for all i. That is, the restriction of the map T to semimodule $<[J_{m,n-1} \ O_{m,1}]>$ is the identity map.

Lemma 2.7 *Let* $T : \mathcal{M}_{m,n}(\mathbb{B}) \rightarrow \mathcal{M}_{m,n}(\mathbb{B})$ *be a linear erator. If* T *preserves the set* $\{A \mid disc(A) = 1\}$ *and set of Ferrers matrices* ($disc(A) = 0$)*, then the image of I dominates* $E_{r,s}$ *only for* $r < k$ *unless* $(r,s) = (k,n)$*, tha*

$$T(E_{k,n}) \sqsubseteq \begin{bmatrix} J_{k-1,n} \\ O \end{bmatrix} + E_{k,n}.$$

Proof. Throughout this proof, we shall use Lemma 2.6 without specific mention.

Suppose that T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferrers matrices ($disc(A) = 0$), and that for some $(r, s) \neq (k, n)$, $T(E_{k,n}) \sqsupseteq E_{r,s}$. We consider two cases.

Case 1. $r \geq k$ and $s \leq n - 1$. Here, let $Y = \begin{bmatrix} J_{r-1,n} \\ O \end{bmatrix} + E_{r,1} + \ldots + E_{r,s-1} + E_{r,s+1} + \ldots + E_{r,n}$. Then $T(Y) \sqsupseteq E_{r,s}$ and $T(E_{r,s}) = E_{r,s}$, so that $T(Y) = T(Y + E_{r,s})$. But, $disc(Y) = 1$ while $disc(Y + E_{r,s}) = 0$, a contradiction.

Case 2. $r > k$ and $s = n$, Then, since $\begin{bmatrix} J_{r-1,n} \\ O \end{bmatrix}$ is a Ferrers matrix, so is $T\left(\begin{bmatrix} J_{r-1,n} \\ O \end{bmatrix}\right)$. But $T\left(\begin{bmatrix} J_{r-1,n} \\ O \end{bmatrix}\right) \sqsupseteq E_{r,n}$, so by the second equivalent definition of a Ferrers matrix we have that $T\left(\begin{bmatrix} J_{r-1,n} \\ O \end{bmatrix}\right) \sqsupseteq E_{r,n-1}$, a contradiction by Case 1 above.

Thus, if $T(E_{n,k}) \sqsupseteq E_{r.s}$ we have that either $r < k$ or $(r, s) = (k, n)$. That is $T(E_{k,n}) \sqsubseteq \begin{bmatrix} J_{k-1,n} \\ O \end{bmatrix} + E_{k,n}$. ■

Lemma 2.8 *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. If T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferrers matrices ($disc(A) = 0$), then $T(E_{k,n}) \sqsupseteq E_{k,n}$.*

Proof. Let k be the first positive integer such that $T(E_{k,n}) \not\sqsupseteq E_{k,n}$. By Lemma 2.7, $T(E_{k,n}) \sqsubseteq \begin{bmatrix} J_{(k-1),n} \\ O \end{bmatrix} + E_{k,n}$. Thus if $T(E_{k,n}) \not\sqsupseteq E_{k,n}$ then $T(E_{k,n}) \sqsubseteq \begin{bmatrix} J_{k-1,n} \\ O \end{bmatrix}$. But then, the discrepancy of $\begin{bmatrix} J_{(k-1),n} \\ O \end{bmatrix} + E_{k,n}$ is 1 while, the discrepanc of $T\left(\begin{bmatrix} J_{k-1,n} \\ O \end{bmatrix} + E_{k,n}\right)$ is 0 since $T\left(\begin{bmatrix} J_{(k-1),n} \\ O \end{bmatrix} + E_{k,n}\right) =$ $\begin{bmatrix} J_{(k-1),n} \\ O \end{bmatrix}$, a contradiction.

Lemma 2.9 *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operato If T preserves the set $\{A \mid disc(A) = 1\}$ and the set of Ferre matrices ($disc(A) = 0$), then*

$$\left[\left(T(E_{k,n}) \circ \begin{bmatrix} J_{k-1,n} \\ O \end{bmatrix}\right) + (E_{1,1} + E_{2,1} + \ldots + E_{k-1,1})\right]$$

is a Ferrers matrix.

Proof. Consider the matrix $Z = (E_{1,1}+E_{2,1}+\ldots+E_{k-1,1})+E_{k,}$ Then $T(Z)$ must be a matrix of discrepancy 1 and $T(Z)$ has a zeros in rows $k, \ldots, m$ except the $(k,n)^{th}$ entry.

Theorem 2.2 *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear oper tor. Then, T preserves the set $\{A \mid disc(A) = 1\}$ and the set Ferrers matrices ($disc(A) = 0$), if and only if there are matrice $F_2, F_3, \ldots, F_m$ in $\mathcal{M}_{m,n}(\mathbb{B})$ such that $F_i \circ \begin{bmatrix} O_{i-1,n} \\ J_{m-i+1,n} \end{bmatrix} = O$ ar $F_i + (E_{1,1} + E_{2,1} + \ldots + E_{i-1,1})$ is a Ferrers matrix, $i = 2, \ldots r$ and such that $T(E_{i,j}) = E_{i,j}$ for $j < n$, $T(E_{1,n}) = E_{1,n}$, ar $T(E_{i,n}) = E_{i,n} + F_i$ for $i = 2, \ldots, m$.*

Proof. Suppose that T preserves the set $\{A \mid disc(A) =$ and the set of Ferrers matrices ($disc(A) = 0$). By Lemma 2 $T(E_{i,j}) = E_{i,j}$ for $j < n$. Let $F_i = T(E_{i,n}) \circ \begin{bmatrix} J_{i-1,n} \\ O \end{bmatrix}$ f $i = 2, \ldots, m$. By Lemmas 2.7 and 2.8 $T(E_{i,n}) = F_i + E_{i,n}$ f $i = 2, \ldots, m$ and $T(E_{1,n}) = E_{1,n}$. By Lemma 2.9, $F_i + (E_{1,1}$ $E_{2,1} + \ldots + E_{i-1,1})$ is a Ferrers matrix, $i = 2, \ldots m$.

Now suppose there are matrices, $F_2, F_3, \ldots, F_m$ in $\mathcal{M}_{m,n}(\mathbb{B})$ such that $F_i \circ \begin{bmatrix} O_{i-1,n} \\ J_{m-i+1,n} \end{bmatrix} = O$ and $F_i + (E_{1,1} + E_{2,1} + \ldots + E_{i-1,1})$ is a Ferrers matrix, $i = 2, \ldots m$, and such that $T(E_{i,j}) = E_{i,j}$ for $j < n$, $T(E_{1,n}) = E_{1,n}$, and $T(E_{i,n}) = E_{i,n} + F_i$ for $i = 2, \ldots, m$. By the proof in Example 2.1 T preserves the set of Ferrers matrices and the set of matrices of discrepancy 1. ■

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