A Short Note on 3-GDDs with 3 Groups and Block Size 4

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Abstract

Necessary conditions for the existence of a 3-GDD$(n, 3, k; \Lambda_1, \Lambda_2)$ are obtained along with some non-existence results. We also prove that these necessary conditions are sufficient for the existence of a 3-GDD$(n, 3, 4; \Lambda_1, \Lambda_2)$ for n even.

1 Preliminaries

Definition 1.1. *A group divisible design GDD$(n, m, k; \lambda_1, \lambda_2)$ is a collection of k-subsets, called blocks, of an nm-set X, where the elements of X are partitioned into m subsets (called groups) of size n each; pairs of distinct elements within the same group are called first associates and appear together in λ_1 blocks while any two elements not in the same group are called second associates and appear together in λ_2 blocks.*

Definition 1.2. *A t-(v, k, λ) design, or a t-design, is a pair (X, B) where X is a v-set of points and B is a collection of k-subsets (blocks) of X with the property that every t-subset of X is contained in exactly λ blocks. The parameter λ is called the index of the design.*

Hanani ([1], pp 706-707) proved that the necessary conditions are sufficient for the existence of 3-$(v, 4, \lambda)$, i.e., 3-$(v, 4, \lambda)$ exists if and only if :

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$\lambda \equiv 1, 5, 7$ or $11 \pmod{12}$ and $v \equiv 2$ or $4 \pmod 6$;
$\lambda \equiv 2$ or $10 \pmod{12}$ and $v \equiv 1, 2, 4, 5, 8,$ or $10 \pmod{12}$;
$\lambda \equiv 3$ or $9 \pmod{12}$ and $v \equiv 0 \pmod 2$;
$\lambda \equiv 4$ or $8 \pmod{12}$ and $v \equiv 1$ or $2 \pmod 3$;
$\lambda \equiv 6 \pmod{12}$ and $v \equiv 0, 1,$ or $2 \pmod 4$;
$\lambda \equiv 0 \pmod{12}$.

Sarvate and Bezire defined a 3-GDD$(n, 2, k; \Lambda_1, \Lambda_2)$ in [2] and obtained some necessary conditions for the case $k = 4$.

Definition 1.3. *A 3-GDD$(n, 2, k; \Lambda_1, \Lambda_2)$ is a set X of $2n$ elements partitioned into two parts of size n called groups together with a collection of k-subsets of X called blocks, such that*
(i) every 3-subset of each group occur in Λ_1 blocks and
(ii) every 3-subset where two elements are from one group and one element from the other group occurs in Λ_2 blocks.

They [2] also gave the following fundamental construction.

Theorem 1. *A 3-GDD$(n, 2, 4; 0, 1)$ exists for even n and a 3-GDD$(n, 2, 4; 0, 2)$ exists for all positive integers n.*

Above Definition 1.3 is generalized below:

Definition 1.4. *A 3-GDD$(n, m, k; \Lambda_1, \Lambda_2)$ is a set X of mn elements partitioned into m parts of size n called groups together with a collection of k-subsets of X called blocks, such that*
(i) every 3-subset of configuration $(3, 0)$, i.e. where all 3 elements are from the same group occur in Λ_1 blocks,
(ii) every 3-subset of configuration $(2, 1)$ where two elements are from one group and one element from the other group, or of configuration $(1, 1, 1)$, i.e. three elements from different groups, occurs in Λ_2 blocks.

Example 1.1. *A 3-GDD$(3, 3, 4; 0, 4)$ with $X = \{1, 2, 3, a, b, c, x, y, z\}$; $G_1 = \{1, 2, 3\}$, $G_2 = \{a, b, c\}$, $G_3 = \{x, y, z\}$. Blocks are written as columns:*

$$\begin{array}{ccccccccc} 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 & 3 \\ a & b & c & c & a & b & b & c & a \\ b & c & a & a & b & c & c & a & b \end{array} \qquad \begin{array}{ccccccccc} 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 & 3 \\ x & x & y & x & y & x & y & x & x \\ y & z & z & z & z & y & z & y & z \end{array}$$

$$\begin{array}{ccccccccc} a & b & c & a & b & c & a & b & c \\ b & c & a & b & c & a & b & c & a \\ x & x & x & x & x & x & y & y & y \\ y & y & y & z & z & z & z & z & z \end{array}$$

1	1	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1	2
2	3	3	2	3	3	2	3	3	2	3	3	2	3	3	2	3	3
a	a	a	b	b	b	c	c	c	a	a	a	b	b	b	c	c	c
x	y	z	y	z	x	z	x	y	x	y	z	y	z	x	z	x	y

a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b
b	c	c	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c
z	z	z	y	y	y	x	x	x	x	x	x	z	z	z	y	y	y
1	3	2	2	1	3	3	2	1	1	3	2	2	1	3	3	2	1

x	x	y	x	x	y	x	x	y	x	x	y	x	x	y	x	x	y
y	z	z	y	z	z	y	z	z	y	z	z	y	z	z	y	z	z
2	2	2	1	1	1	3	3	3	2	2	2	1	1	1	3	3	3
a	c	b	c	b	a	b	a	c	c	b	a	b	a	c	a	c	b

Remark 1. *Suppose a Resolvable t-$(n, k, 1)$ exists, where $n = sk$ then by deleting a parallel class, we get t-GDD$(k, s, k; 0, 1)$.*

In addition, we can relax condition (ii) and require that every 3-subset where each element is from a different group occurs in Λ_3 blocks, where Λ_3 may not be equal to Λ_2 and we will denote such a design by a 3-PBIBD$(n, m, k; \Lambda_1, \Lambda_2, \Lambda_3)$ or a 3-GDD$(n, m, k; \Lambda_1, \Lambda_2, \Lambda_3)$.

Example 1.2. *A 3-PBIBD$(3, 3, 4; 0, 1, 2)$ with $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5, 6\}$, $G_3 = \{7, 8, 9\}$, and the blocks written below in columns:*

1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
2	3	1	2	3	1	2	3	1	5	6	4	6	4	5	4	5	6
4	5	6	5	6	4	6	4	5	6	4	5	4	5	6	5	6	4
7	8	9	9	7	8	8	9	7	7	8	9	9	7	8	8	9	7

1	2	3	1	2	3	1	2	3
6	4	5	4	5	6	5	6	4
7	8	9	9	7	8	8	9	7
8	9	7	7	8	9	9	7	8

In fact, Hanani has used the concept of 3-GDDs with $\Lambda_1 = 0$ to construct -$(v, 4, \lambda)$'s. He used the notation $P_n^{''}[k, \lambda, nt]$ to represent 3-PBIBD$(n, t, k;$ $, 0, \lambda)$ and the notation $Q_n^{''}[k, \lambda, nt]$ to represent 3-PBIBD$(n, t, k; 0, \lambda, \lambda)$ e. 3-GDD$(n, t, k; 0, \lambda)$. His following result and existence of designs such s $Q_6^{''}(4, 1)$ (i.e. 3-GDD$(6, 3, 4; 0, 1)$) will be used in this note.

Theorem 2. *([1], Proposition 5) If $n'|n$ and a 3-PBIBD$(n', t, 4; 0, 0, \lambda)$ exists then a 3-PBIBD$(n, t, 4; 0, 0, \lambda)$ exists.*

In the next section, we obtain some necessary conditions for the existence of a 3-GDD$(n, 3, k; \Lambda_1, \Lambda_2)$. Towards this aim, assuming a 3-GDD exists, we count the number of blocks, λ_1, containing a first associate pair, λ_2, the number of blocks containing a second associate pair and the required number of blocks, say b, and the number of blocks containing a given element x (called the replication number r for x), for a 3-GDD.

2 Necessary Conditions

Let λ_1 denote the number of blocks containing $\{x_1, x_2\}$ where x_1 and x_2 are from the same block and let λ_2 denote the number of blocks containing $\{x, y\}$ where x and y are from different groups. For each pair $\{x_1, x_2\}$ belonging to one group, there are $n-2$ $(3,0)$ triples which appear Λ_1 times in the 3-GDD$(n, m, k; \Lambda_1, \Lambda_2)$. Also the pair $\{x_1, x_2\}$ can form $n(m-1)$ triples with an element y belonging to other groups. Such triples appear Λ_2 times in the 3-GDD$(n, m, k; \Lambda_1, \Lambda_2)$, if it exists. If a block contains $\{x_1, x_2\}$ then it has exactly $k-2$ triples which contains both x_1 and x_2. Thus we have

$$\lambda_1 = \frac{\Lambda_1(n-2) + \Lambda_2 n(m-1)}{k-2}. \quad (1)$$

For a pair $\{x, y\}$, where x and y are elements from different groups; following three types of triples can be formed:

i) $\{x, y, x_i\}$, where x and x_i are from the same group.

ii) $\{x, y, y_i\}$, where y and y_i are from the same group.

iii) $\{x, y, z_i\}$, where x, y and z_i are all from different groups.

This gives

$$\lambda_2 = \frac{2(n-1) + n(m-2)}{k-2}\Lambda_2. \quad (2)$$

Hence every 3-GDD$(n, m, k; \Lambda_1, \Lambda_2)$ is also a 2-GDD$(n, m, k; \lambda_1, \lambda_2)$.

Assuming a 3-GDD exists, the replication number r of an element x is the number of blocks containing x. Suppose a 3-GDD$(n, m, k; \Lambda_1, \Lambda_2)$ exists with groups $G_1, G_2, \ldots, G_m$. Without loss of generality, let $x \in G_1$ and let r be the replication number for x. There are $\binom{n-1}{2}$ 3-subsets containing x, where all elements are from the same group G_1. A triple of configuration

$(2,1)$ may be of type $\{x, x_i, y_j\}$, where x_i is an element from G_1 and y_j from any of the remaining $m-1$ groups or of the type $\{x, y_i, y_j\}$, where y_i and y_j are from a group other than G_1. Also, for triples with $(1,1,1)$ configuration, we have $\{x, y_i, z_j\}$, where y_i and z_j are elements from two different groups other than G_1. Moreover x appears in $\binom{k-1}{2}$ 3-subsets of every block. Thus we have

$\binom{k-1}{2}r = \binom{n-1}{2}\Lambda_1 + ((n-1)(m-1)n + (m-1)\binom{n}{2})\Lambda_2 + \binom{m-1}{2}n \cdot n\Lambda_2.$
Hence we have

$$r = \frac{(n-1)(n-2)\Lambda_1 + n(m-1)(mn+n-3)\Lambda_2}{(k-1)(k-2)} \tag{3}$$

The number of blocks in a 3-GDD, if exists, is denoted by b. Using $mnr = bk$, we get

$$b = mn\frac{\Lambda_1(n-1)(n-2) + \Lambda_2(m-1)(n^2-3n+mn^2)}{k(k-1)(k-2)}. \tag{4}$$

Substituting $m = 3$ and $k = 4$ in Equations 1, 2, 3, and 4, we get the following parameters for a 3-GDD$(n, 3, 4; \Lambda_1, \Lambda_2)$.

$$\lambda_1 = \frac{\Lambda_1(n-2) + 2\Lambda_2 n}{2} = \frac{\Lambda_1(n-2)}{2} + \Lambda_2 n, \tag{5}$$

$$\lambda_2 = \frac{(3n-2)\Lambda_2}{2}, \tag{6}$$

$$r = \frac{(n-1)(n-2)\Lambda_1 + (8n^2-6n)\Lambda_2}{6} \tag{7}$$

and

$$b = \frac{n}{8}(\Lambda_1(n-1)(n-2) + 2n\Lambda_2(4n-3)) \tag{8}$$

As these parameters must be integers, one of the necessary conditions for the existence of a 3-GDD$(n, 3, 4; \Lambda_1, \Lambda_2)$ is

$$(n-1)(n-2)\Lambda_1 + 2n^2\Lambda_2 \equiv 0 \pmod 6. \tag{9}$$

from Equation 7. From the above equation, when $\Lambda_1 \equiv 0 \pmod 3$, $2n^2\Lambda_2 \equiv 0 \pmod 6$ and hence either n or Λ_2 must be congruent to 0 modulo 3. Further, when $m = 3$, if a 3-GDD$(n, 3, 4; 0, \Lambda_2)$ exists, there can be only 2 types of blocks $(2,1,1)$ or $(2,2)$. Note, there are $3n^2(n-1)\Lambda_2$ triples of the form $(2,1)$ and $n^3\Lambda_2$ triples of the type $(1,1,1)$ in the blocks of the 3-GDD. Blocks of the type $(2,1,1)$ cover two triples of type $(1,1,1)$ per block and also covers two $(2,1)$ triples. The remaining $(2,1)$ triples must occur in the blocks of the type $(2,2)$. Each such block contributes 4 triples. Hence a necessary condition for the case when $\Lambda_1 = 0$ is

$$3n^2(n-1)\Lambda_2 - n^3\Lambda_2 \equiv 0 \pmod 4. \tag{10}$$

Which implies

$$n^2(2n-3)\Lambda_2 \equiv 0 \pmod 4. \tag{11}$$

Hence if n is even, there is no restriction on Λ_2 but if n is odd then $\Lambda_2 \equiv 0 \pmod 4$. What is interesting is this condition is not sufficient for the existence, as we will see, (Remark 2), that a 3-GDD$(5,3,4;0,4)$ does not exist.

Theorem 3. *If there exists a 3-GDD$(n,m,k;0,\Lambda)$ and a 3-(n,k,Λ) then a 3-(nm,k,Λ) exists.*

Proof. Let X be a 3-GDD$(n,m,k;0,\Lambda)$ and Y be a 3-(n,k,Λ). The blocks of X and the blocks of Y on each group together give a 3-(nm,k,Λ). □

Corollary 3.1. *If a 3-(n,k,Λ) exists and a 3-(nm,k,Λ) does not exist then a 3-GDD$(n,m,k;0,\Lambda)$ does not exist.*

Remark 2. *As an application of the corollary above a 3-GDD$(4,3,4;0,1)$ and a 3-GDD$(5,3,4;0,4)$ can not exist.*

In fact, we can get several such examples, for instance, using Corollary 3.1, we get

Theorem 4. *A 3-GDD$(n,3,4;0,\Lambda_2)$ does not exist for $n \equiv 2,4 \pmod 6$ and $\Lambda_2 \equiv 1,5 \pmod 6$.*

Proof. A 3-$(n,4,1)$ exists when $n \equiv 2,4 \pmod 6$. But a 3-$(3n,4,\Lambda_2)$ does not exist when $\Lambda_2 \equiv 1,5 \pmod 6$. □

Remark 3. *If $\Lambda_2 \equiv 0 \pmod 4$ and $\Lambda_1 \equiv 0 \pmod 4$ then $n \equiv 0 \pmod 3$. The reason is as follows: If $n \equiv 1,2 \pmod 3$ and a 3-GDD$(n,3,4;0,4)$ exists, we can use a 3-$(n,4,4)$ on each group and construct a 3-$(3n,4,4)$. But for $v \equiv 0 \pmod 3$, a 3-$(v,4,4)$ does not exist.*

Remark 4. *If a 3-(n,k,λ) exists then a 3-GDD$(n,m,k;\lambda,0)$ exists.*

Though it is possible to use the conditions and the divisibility requirements to give a general table for parameters n, Λ_1 and Λ_2; for organization purpose and simplicity, we consider the conditions for n odd and n even in the next two sections. In this short note, our main purpose is to obtain families of 3-GDDs for even n though we have given above results for odd n as well.

3 n odd

From Equations 5 and 6 we have $n\Lambda_1 \equiv 0 \pmod 2$ which implies that if n is odd then Λ_1 is even. Also, $(3n-2)\Lambda_2 \equiv 0 \pmod 2$ implies that if n is odd then Λ_2 is even. For n odd, from Equation 8,

i) for $n \equiv 1 \pmod 8$, $\Lambda_2 \equiv 0 \pmod 4$.

ii) for $n \equiv 5 \pmod 8$, $2\Lambda_1 + \Lambda_2 \equiv 0 \pmod 4$. As Λ_1 is even, we have $\Lambda_2 \equiv 0 \pmod 4$.

iii) for $n \equiv 3 \pmod 8$, $\Lambda_1 \equiv \Lambda_2 \pmod 4$.

iv) for $n \equiv 7 \pmod 8$, $\Lambda_1 \equiv \Lambda_2 \pmod 4$.

Based on the above requirements and Remark 3, we have the following necessary conditions for the existence of a 3-GDD$(n, 3, 4; \Lambda_1, \Lambda_2)$ when n is odd. The values of Λ_1 and Λ_2 are given modulo 12.

$\Lambda_1 \backslash \Lambda_2$	0	2	4	6	8	10
0	all n	none	all n	none	all n	none
2	$1, 5 \pmod 8$	$3, 7 \pmod 8$	$1, 5 \pmod 8$	$3, 7 \pmod 8$	$1, 5 \pmod 8$	$3, 7 \pmod 8$
4	all n	none	all n	none	all n	none
6	$1, 5 \pmod 8$	$3, 7 \pmod 8$	$1, 5 \pmod 8$	$3, 7 \pmod 8$	$1, 5 \pmod 8$	$3, 7 \pmod 8$
8	all n	none	all n	none	all n	none
10	$1, 5 \pmod 8$	$3, 7 \pmod 8$	$1, 5 \pmod 8$	$3, 7 \pmod 8$	$1, 5 \pmod 8$	$3, 7 \pmod 8$

Table 1

For Λ_1 and Λ_2 given in modulo 4, the above table can be written as follows:

$\Lambda_1 \backslash \Lambda_2$	0	2
0	all n	none
2	$1 \pmod 4$	$3 \pmod 4$

Table 2

4 n even

Using Equation 9,

i) for $n \equiv 0 \pmod 6$, $\Lambda_1 \equiv 0 \pmod 3$, and

ii) for $n \equiv 2, 4 \pmod 6$, $\Lambda_2 \equiv 0 \pmod 3$.

Based on the above requirements from Equation 9, and as other equations do not give any further restrictions, we have the following necessary conditions on even n for the existence of a 3-GDD$(n, 3, 4; \Lambda_1, \Lambda_2)$. The values of Λ_1 and Λ_2 are given modulo 3.

$\Lambda_1 \backslash \Lambda_2$	0	1	2
0	$0, 2, 4 \pmod 6$	$0 \pmod 6$	$0 \pmod 6$
1	$2, 4 \pmod 6$	no even	no even
2	$2, 4 \pmod 6$	no even	no even

Table 3

Theorem 5. *If a* 3*-GDD*$(n, 3, 4; 0, 3)$ *exists, then there exists a* 3*-GDD*$(n, 3, 4; \Lambda_1, \Lambda_2)$ *when* Λ_1 *and* Λ_2 *are congruent to* $0 \pmod 3$.
Similarly, when $\Lambda_1 \equiv 1, 2 \pmod 3$, *if a* 3*-GDD*$(n, 3, 4; 0, 3)$ *exists then the necessary conditions will be sufficient for the existence of a* 3*-GDD*$(n, 3, 4; \Lambda_1, \Lambda_2)$, *where* $\Lambda_1 \equiv 1, 2 \pmod 3$ *and* $\Lambda_2 \equiv 0 \pmod 3$.

Proof. Let a 3-GDD$(n, 3, 4; 0, 3)$, say X, exists, then as a 3-$(n, 4, 3)$, say Y, exists for n even. If $\Lambda_1 = 3s$ and $\Lambda_2 = 3l$, the blocks of s copies of Y on each group and l copies of X together provide the blocks of 3-GDD$(n, 3, 4; \Lambda_1 = 3s, \Lambda_2 = 3l)$. Hence for n even, the necessary conditions for the existence of a 3-GDD$(n, 3, 4; \Lambda_1, \Lambda_2)$ will be sufficient when both Λ_1 and Λ_2 are congruent to $0 \pmod 3$ if a 3-GDD$(n, 3, 4; 0, 3)$ exists.
Similarly, for $n \equiv 2, 4 \pmod 6$, a 3-$(n, 4, 1)$ exists and hence if $\Lambda_1 = 3s + 1$ and $\Lambda_2 = 3l$ (or if $\Lambda_1 = 3s + 2$ and $\Lambda_2 = 3l$ respectively), the blocks of a 3-GDD$(n, 3, 4; \Lambda_1 = 3s, \Lambda_2 = 3l)$ together with the blocks of a 3-$(n, 4, 1)$ (or blocks of a 3-$(n, 4, 2)$ respectively) provide the blocks of the required 3-GDD. □

4.1 $\Lambda_2 \equiv 0 \pmod 3$

Example 4.1. *A* 3*-GDD*$(2, 3, 4; 0, 3)$ *with* $X = \{1, 2, a, b, x, y\}$, $G_1 = \{1, 2\}$, $G_2 = \{a, b\}$, $G_3 = \{x, y\}$. *Blocks are written as columns:*

1	1	1	1	a	a	a	a	x	x	x	x	1	a	x
2	2	2	2	b	b	b	b	y	y	y	y	2	b	y
a	b	a	b	1	2	1	2	1	2	1	2	a	x	1
x	y	y	x	x	y	y	x	a	b	a	b	b	y	2

Hence the necessary conditions are sufficient for the existence of a 3-GDD$(2,3,4;\Lambda_1,\Lambda_2)$ when $\Lambda_2 \equiv 0 \pmod 3$.

Example 4.2. *A 3-GDD$(4,3,4;0,3)$.*
Note that there are 156 blocks in the design if it exists. Let $G_1=\{1,2,3,4\}$, $G_2=\{a,b,c,d\}$ and $G_3=\{w,x,y,z\}$ be three groups. Partition each group in two subsets: $G_{11}=\{1,2\}$, $G_{12}=\{3,4\}$, $G_{21}=\{a,b\}$, $G_{22}=\{c,d\}$, $G_{31}=\{w,x\}$ and $G_{32}=\{y,z\}$. For these six groups a 3-PBIBD$(2,6,4;0,0,3)$ exists by Hanani [1]. There are $3\binom{6}{3}2^3=120\cdot 4$ triples of the type $(1,1,1)$ and they are covered in 120 blocks in a 3-PBIBD$(2,6,4;0,0,3)$. If two elements are from same group G_{ij} and third element is from another group then a triple must occur from the remaining 36 blocks of the type $(2,2)$. This is achieved by taking 3 copies of the following twelve blocks, written as columns below.

1	1	1	1	3	3	3	3	a	a	c	c
2	2	2	2	4	4	4	4	b	b	d	d
a	c	w	y	a	c	w	y	w	y	w	y
b	d	x	z	b	d	x	z	x	z	x	z

This example can be generalized as follows:

Theorem 6. *3-GDD$(n,3,4;0,3)$ exists for $n \equiv 0 \pmod 4$.*

Proof. Let $n=4t$ for some t. Let G_{i1} and G_{i2} be two subsets of G_i of size $2t$; $i=1,2,3$ and $G_{i1}\cup G_{i2}=G_i$. A 3-PBIBD$(2,6,4;0,0,3)$ exists by Hanani [1], so Theorem 2 implies a 3-PBIBD$(2t,6,4;0,0,3)$ exists on groups, G_{ij}, $i=1,2,3$; $j=1,2$. As in the example above, the triples, where exactly 2 elements are from one of the G_{ij}'s (and one element from another group) are not contained in the blocks of 3-PBIBD$(2t,6,4;0,0,3)$. These triples are obtained by taking 3 copies of a 3-GDD$(2t,2,4;0,1)$ on G_{ij},G_{kl}, where $i<k$; $i,k\in\{1,2,3\}$; $j,l\in\{1,2\}$. Recall, the Fundamental Construction (Theorem 1) gives a 3-GDD$(2t,2,4;0,1)$. □

Corollary 6.1. *Necessary conditions are sufficient for the existence of a 3-GDD$(n,3,4;\Lambda_1,\Lambda_2)$ for $n\equiv 0 \pmod 4$ when $\Lambda_2\equiv 0 \pmod 3$.*

4.2 $\Lambda_2 \equiv 1, 2 \pmod 3$

When $\Lambda_2 \equiv 1, 2 \pmod 3$, from the necessary conditions, $\Lambda_1 \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 6$. Theorem 2 and a 3-PBIBD$(3, 6, 4; 0, 0, 1)$ given in Hanani [Page 709, [1]], imply that a 3-PBIBD$(3t, 6, 4; 0, 0, 1)$ exists. Now we can construct a 3-GDD$(6t, 3, 4; 0, 1)$ for t even and a 3-GDD$(6t, 3, 4; 0, 2)$ for t odd as follows:
We partition the groups G_i into subsets G_{i1} and G_{i2} of size $3t$, $i = 1, 2, 3$. Then form a 3-PBIBD$(3t, 6, 4; 0, 0, 1)$ on groups G_{ij}, $i = 1, 2, 3$ and $j = 1, 2$. On the other hand, we use the Fundamental Construction (Theorem 1), to construct a 3-GDD$(3t, 2, 4; 0, 2)$ if t is odd or a 3-GDD$(3t, 2, 4; 0, 1)$ if t is even on the following pairs of groups:

$$G_{11} \text{ with } G_{j1} \text{ and } G_{j2},\ j = 2, 3;$$
$$G_{12} \text{ with } G_{j1} \text{ and } G_{j2},\ j = 2, 3;$$
$$G_{2i} \text{ with } G_{31} \text{ and } G_{32},\ i = 1, 2.$$

Let t be even: Putting together three copies of the blocks of 3-PBIBD$(3t, 6, 4; 0, 0, 1)$ and three copies of, just constructed, 3-GDD$(3t, 2, 4; 0, 1)'s$, we have a 3-GDD$(6t, 3, 4; 0, 3)$. Now as n is 0 (mod 6), 3-$(n, 4, 3)$ also exists, hence one copy of a 3-GDD$(6t, 3, 4; 0, 1)$, s copies of a 3-$(n, 4, 3)$ and l copies of a 3-GDD$(6t, 3, 4; 0, 3)$ together give a 3-GDD$(n, 3, 4; 3s, 3l{+}1)$. Similarly, two copies of a 3-GDD$(6t, 3, 4; 0, 1)$, s copies of a 3-$(n, 4, 3)$ and l copies of a 3-GDD$(6t, 3, 4; 0, 3)$ together give a 3-GDD$(n, 3, 4; 3s, 3l + 2)$.

Let t be odd: Putting together six copies of the blocks of 3-PBIBD$(3t, 6, 4; 0, 0, 1)$ and three copies of, just constructed, 3-GDD$(3t, 2, 4; 0, 2)'s$, we have a 3-GDD$(6t, 3, 4; 0, 6)$. Now as n is 0 (mod 6), 3-$(n, 4, 3)$ also exists, therefore one copy of a 3-GDD$(6t, 3, 4; 0, 2)$, s copies of a 3-$(n, 4, 3)$ and l copies of a 3-GDD$(6t, 3, 4; 0, 6)$ together give a 3-GDD$(n, 3, 4; 3s, 6l + 2)$. Hence we have the following theorem.

Theorem 7. *Necessary conditions are sufficient for the existence of a 3-GDD*$(6t, 3, 4; 3s, 3l + 1)$ *when* $t \equiv 0 \pmod 2$ *and 3-GDD*$(6t, 3, 4; 3s, 6l + 2)$ *for all* t*, where* s *and* l *are any positive integers.*

All together, we have:

Theorem 8. *Necessary conditions are sufficient for the existence of a 3-GDD*$(n, 3, 4; \Lambda_1, \Lambda_2)$ *for* $\Lambda_2 \equiv 0 \pmod 3$ *for* $n \equiv 0 \pmod 4$ *and are sufficient for the existence of 3-GDD*$(n, 3, 4; \Lambda_1, \Lambda_2)$ *for* $\Lambda_2 \equiv 1, 2 \pmod 3$ *except possibly for* $n \equiv 6 \pmod{12}$*, with* $\Lambda_2 \equiv 1, 4, 5 \pmod 6$.

What remains to prove is the existence of a 3-GDD($6t, 3, 4; 0, 1$) for all odd, but in [3] authors have generalized a design given in [1], to obtain ıe following theorem for all t.

'heorem 9. *A 3-GDD($n = 6t, 3, 4; 0, \Lambda$) exists for any Λ.*

As n is even, there exists a 3-($n, 4, 3$) and hence a 3-($n, 4, \Lambda_1$) exists if $_1 \equiv 0 \pmod 3$. The blocks of a 3-GDD($n = 6t, 3, 4; 0, \Lambda_2$) together with ıe blocks of 3-($n, 4, \Lambda_1$) on each group gives the blocks of a 3-GDD($n =$ $t, 3, 4; \Lambda_1, \Lambda_2$) and hence we have:

'heorem 10. *The necessary conditions are sufficient for the existence of 3-GDD($n, 3, 4; \Lambda_1, \Lambda_2$) for all even n.*

References

[1] H. Hanani, *On Some Tactical Configurations*, Canad. J. Math. 15 (1963), 702-722.

[2] D. G. Sarvate and W. Bezire, *3-GDDs with Block Size* 4, Bulletin of the ICA, 82 (2018), 50-65.

[3] D. G. Sarvate and P. N. Shinde, *Missing Blocks of Two Designs in Hanani's Paper*, submitted.