Color-Induced Graph Colorings

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Abstract

For a positive integer k, let $\mathcal{P}^*([k])$ denote the set of nonempty subsets of $[k] = \{1, 2, \ldots, k\}$. For a graph G without isolated vertices, let $c : E(G) \to \mathcal{P}^*([k])$ be an edge coloring of G where adjacent edges may be colored the same. The induced vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is defined by $c'(v) = \bigcap_{e \in E_v} c(e)$, where E_v is the set of edges incident with v. If c' is a proper vertex coloring of G, then c is called a regal k-edge coloring of G. The minimum positive integer k for which a graph G has a regal k-edge coloring is the regal index of G. If c' is vertex-distinguishing, then c is a strong regal k-edge coloring of G. The minimum positive integer k for which a graph G has a strong regal k-edge coloring is the strong regal index of G. The regal index (and, consequently, the strong regal index) is determined for each complete graph and for each complete multipartite graph. Sharp bounds for regal indexes and strong regal indexes of connected graphs are established. Strong regal indexes are also determined for several classes of trees. Other results and problems are also presented.

Key Words: color-induced coloring, edge coloring, regal and strong regal colorings, regal and strong regal indexes.

AMS Subject Classification: 05C15, 05C05.

1 Introduction

For a graph G without isolated vertices, an edge coloring of G is an assignment of colors to the edges of G. An edge coloring c is *unrestricted* if no condition is placed on how the edges may be colored; in particular, adjacent edges may be colored the same by c. If every two adjacent edges of G are colored differently, then c is a *proper edge coloring* and the minimum number of colors required of a proper edge coloring of G is its *chromatic index*

$\chi'(G)$. A vertex coloring c' of G is an assignment of colors to the ve of G. A vertex coloring c' of a graph G is *neighbor-distinguishing* or *p* if adjacent vertices are colored differently. The minimum number of required of a proper vertex coloring of G is its *chromatic number* $\chi($ vertex coloring c' of a graph G is *vertex-distinguishing* or *rainbow* if n vertices are colored the same by c'.

During the past three decades, several types of edge colorings of g have been described that give rise to vertex colorings defined in a vari manners (see [1, 2, 4, 9, 10, 11] for example). Among the vertex colc c' of a graph G obtained from an edge coloring c of G in which the are selected from a set $[k] = \{1, 2, \ldots, k\}$ for some positive integer k most commonly studied are those where the color $c'(v)$ of a vertex v o either (1) the set of colors of those edges incident with v, (2) the multi colors of the edges incident with v, or (3) the sum of the colors of the incident with v. In most cases, the induced vertex coloring c' is requir be proper or rainbow.

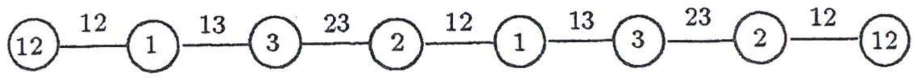
While an edge coloring c of a graph G typically uses colors from th $[k]$ for some positive integer k, resulting in $c(e) = i \in [k]$ for $e \in E(G)$ can define $c(e) = \{i\}$ instead. That is, in (1), both the edge coloring the induced vertex coloring c' assign subsets of $[k]$ to the edges and ve of G. This suggests the idea of studying edge colorings c where both its derived vertex coloring c' assign subsets of $[k]$ to the elements (edges vertices) of a graph G. A number of unrestricted edge colorings of a g have been studied that use subsets of $[k]$ as colors and give rise to pr or rainbow vertex colorings by means of set union (see [5, 6, 7, 8, 9, 10 example). Here, set intersection is the operation. We refer to the boo for graph theory notation and terminology not described in this pape

2 Regal Colorings

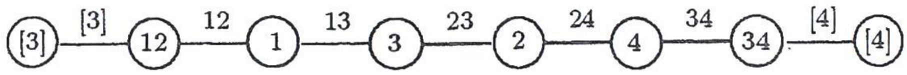
For a positive integer k, let $\mathcal{P}^*([k])$ denote the set of nonempty sub of $[k]$. For a graph G without isolated vertices, let $c : E(G) \to \mathcal{P}^*$ be an unrestricted edge coloring of G, where then adjacent edges ma colored the same. The vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is defined b

$$c'(v) = \bigcap_{e \in E_v} c(e),$$

where E_v is the set of edges incident with a vertex v of G. That is, $c'($ the intersection of the sets of colors of those edges incident with v and consists of all elements of $[k]$ belonging to the color of every edge inci with v. Furthermore, the coloring c has the property that requires $c'(v)$ for every vertex v of G. If c' is a proper vertex coloring of G, then c is ca a *regal k-edge coloring* of G. An edge coloring of G is a *regal coloring* if it is a regal k-edge coloring for some positive integer k. The minimum positive integer k for which a graph G has a regal k-edge coloring is called the *regal index* $\text{reg}(G)$ of G. If c' is vertex-distinguishing, then c is called a *strong regal k-edge coloring* of G. An edge coloring of G is a strong regal coloring if it is a strong regal k-edge coloring for some integer $k \geq 2$. The minimum positive integer k for which a graph G has a strong regal k-edge coloring is called the *strong regal index* $\text{sreg}(G)$ of G. While no regal coloring exists for the graph K_2, such a coloring exists for every connected graph of order at least 3. Since every strong regal coloring is also a regal coloring, it follows that $\text{reg}(G) \leq \text{sreg}(G)$ for every connected graph G of order at least 3. For example, Figure 1 shows a regal 3-edge coloring and a strong regal 4-edge coloring of the path P_8 of order 8. (We write the set $\{a\}$ as a, $\{a,b\}$ as ab, and $\{a,b,c\}$ as abc for simplicity.) In fact, $\text{reg}(P_8) = 3$ and $\text{sreg}(P_8) = 4$, as we will see later.



A regal 3-coloring of P_8



A strong regal 4-coloring of P_8

Figure 1: A regal 3-coloring and a strong 4-coloring of P_8

We mentioned that every connected graph of order 3 or more has a regal coloring. To show this, we first present a lemma dealing with strong regal colorings.

Lemma 2.1 *Let H be a connected spanning subgraph of a graph G of order at least 3. If H has a strong regal k-edge coloring for some positive integer k, then so does G. Consequently,* $\text{sreg}(G) \leq \text{sreg}(H)$.

Proof. Suppose that H has a strong regal coloring and that $\text{sreg}(H) = k$. Let $c_H : E(H) \to \mathcal{P}^*([k])$ be a strong regal k-edge coloring of H. Then $c'_H(x) \neq c'_H(y)$ for every two distinct vertices x and y. The edge coloring c_H is extended to an edge coloring $c_G : E(G) \to \mathcal{P}^*([k])$ of G by defining

$$c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ [k] & \text{if } e \in E(G) - E(H). \end{cases}$$

Since $c'_G(x) = c'_H(x)$ for each $x \in V(G)$ and c'_H is vertex-distinguishing, it follows that c'_G is vertex-distinguishing. Therefore, c_G is a strong : k-edge coloring of G and so $\text{sreg}(G) \leq k = \text{sreg}(H)$.

Theorem 2.2 *Every connected graph of order* 3 *or more has a strong coloring and therefore a regal coloring.*

Proof. By Lemma 2.1, it suffices to show that every tree of order 3 or r has a strong regal coloring. We proceed by induction on the order $n \geq 3$ tree T to show that there exists a strong regal coloring $c : E(T) \to \mathcal{P}^*($ For $n = 3$, the path P_3 is the only tree of order 3. Assigning the colors $\{$ and $\{1, 3\}$ to the two edges of P_3 produces a strong regal 3-edge colo of P_3. Thus, $\text{sreg}(P_3) \leq 3$ (in fact, $\text{sreg}(P_3) = 3$) and so the base ste the induction holds. Now, suppose that every tree of order $n - 1 \geq 3$ h strong regal coloring whose edges are colored with elements of $\mathcal{P}^*([n -$ Let T be a tree of order n. Let v be an end-vertex of T and let $T_0 = T$ By the induction hypothesis, T_0 has a strong regal $(n-1)$-edge colo $c_0 : E(T_0) \to \mathcal{P}^*([n-1])$. Let u be the vertex of T adjacent to v. Supp that $c'_0(u) = S \subseteq [n-1]$. Then $c'_0(u) \neq c'_0(x)$ for all $x \in V(T_0) -$ Define an edge coloring $c : E(T) \to \mathcal{P}^*([n])$ by

$$c(e) = \begin{cases} c_0(e) & \text{if } e \in E(T_0) \\ [n] & \text{if } e = uv. \end{cases}$$

Thus, $c'(x) = c'_0(x) \subseteq [n-1]$ for all $x \in V(T_0)$ and $c'(v) = [n]$. Si $c'(v) \neq c'(x)$ for all vertices $x \in V(T_0)$ and $c'(x) \neq c'(y)$ for every distinct vertices x and y of T, it follows that c' is vertex-distinguishing so c is a strong regal n-edge coloring of T. Therefore, $\text{sreg}(G)$ exists an does $\text{reg}(G)$.

A consequence of the proof of Theorem 2.2 is that if G is a connec graph of order $n \geq 3$, then $\text{reg}(G) \leq \text{sreg}(G) \leq n$. Also, observe tha c is an edge coloring of a connected graph G of order at least 3 such t $c(e)$ is a singleton set for some edge $e = uv$ of G, then the induced ver coloring c' of c satisfies $c'(u) = c'(v)$ and so c cannot be not regal. T observation yields the following useful lemma.

Lemma 2.3 *If c is a regal coloring of a connected graph G of orde least* 3, *then* $|c(e)| \geq 2$ *for each* $e \in E(G)$ *and so* $\text{reg}(G) \geq 2$.

Even Lemma 2.3 can be improved, however. The following result gi a lower bound for the regal index (and the strong regal index) of a gra in terms of its chromatic number.

Theorem 2.4 *If G is a connected graph of order* 3 *or more, then*

$$\max\{3, \lceil \log_2(\chi(G) + 1) \rceil\} \leq \text{reg}(G).$$

Proof. Suppose that $\operatorname{reg}(G) = k$. Then $k \geq 2$ by Lemma 2.3. However, if there were a regal 2-coloring of G using the colors in $\mathcal{P}^*([2])$, then each edge e of G must be colored $\{1,2\}$ by Lemma 2.3, but then the induced vertex coloring assigns $\{1,2\}$ to every vertex of G, which is impossible. Thus, $k \geq 3$. Next, let $c : E(G) \to \mathcal{P}^*([k])$ be a regal k-edge coloring of G where $k \geq 3$. Since $c' : V(G) \to \mathcal{P}^*([k])$ is a proper vertex coloring of G, it follows that $\chi(G) \leq |\mathcal{P}^*([k])| = 2^k - 1$. Therefore, $k \geq \log_2(\chi(G)+1)$ and so $k \geq \lceil \log_2(\chi(G)+1) \rceil$. Thus, $\operatorname{reg}(G) \geq \max\{3, \lceil \log_2(\chi(G)+1) \rceil\}$. ■

Since $\chi(K_n) = n$, it follows by Theorem 2.4 that $\operatorname{reg}(K_n) = \operatorname{sreg}(K_n) \geq \lceil \log_2(n+1) \rceil$ for $n \geq 4$. We show that equality holds here.

Theorem 2.5 *For each integer $n \geq 4$,*

$$\operatorname{reg}(K_n) = \operatorname{sreg}(K_n) = \lceil \log_2(n+1) \rceil.$$

Proof. Let $k = \lceil \log_2(n+1) \rceil \geq 3$. Hence, $2^{k-1} \leq n \leq 2^k - 1$. We have already observed that $\operatorname{reg}(G) \geq k$. It remains to show that $\operatorname{reg}(K_n) \leq k$, namely that there is a regal k-edge coloring of K_n. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let $S_1, S_2, \ldots, S_{2^k-1}$ be the $2^k - 1$ elements of $\mathcal{P}^*([k])$ such that $1 = |S_1| \leq |S_2| \leq |S_3| \leq \cdots \leq |S_{2^k-1}| = k$. Therefore, $|S_i| = 1$ for $1 \leq i \leq k$, $|S_i| = 2$ for $k+1 \leq i \leq k + \binom{k}{2}$, $|S_i| = 3$ for $k + \binom{k}{2} + 1 \leq i \leq k + \binom{k}{2} + \binom{k}{3}$, and so on. We may assume that $S_i = \{i\}$ for $1 \leq i \leq k$. First, we define a labeling f of the vertices of K_n by $f(v_i) = S_i$ for $1 \leq i \leq n$. Since $n \geq 2^{k-1}$ and $k \geq 3$, it follows that $n > k$ and so $S_1, S_2, \ldots, S_k$ are assigned to the vertices of K_n by f. We now use the vertex labeling f to define an edge coloring of K_n. In particular, we define $c : E(K_n) \to \mathcal{P}^*([k])$ by $c(v_i v_j) = f(v_i) \cup f(v_j)$ for each pair i, j of integers with $1 \leq i < j \leq n$. This coloring is illustrated in Figure 2 for $n = 4, 5$. The vertex coloring $c' : V(K_n) \to \mathcal{P}^*([k])$ induced by c is then defined by

$$c'(v_i) = \bigcap_{\substack{1 \leq j \leq n \\ j \neq i}} c(v_i v_j). \tag{1}$$

From the manner in which $c(v_i v_j)$ is defined, it follows that $f(v_i) \subseteq c'(v_i)$ for $1 \leq i \leq n$. We claim that $c'(v_i) = f(v_i)$ for $1 \leq i \leq n$. First, suppose that $f(v_i) = [k]$. Then $c(v_i v_j) = [k]$ for all integers j with $1 \leq j \leq n$ and $j \neq i$ and so $c'(v_i) = [k]$. Next, suppose that $f(v_i) \subset [k]$. For each integer $\ell \in [k] - f(v_i)$, let $t \in [k] - \{i, \ell\}$. Then $f(v_t) = \{t\}$. It follows by (1) that $c'(v_i) \subseteq c(v_i v_t) = f(v_i) \cup f(v_t) = f(v_i) \cup \{t\}$. Since $\ell \notin f(v_i) \cup \{t\}$, it follows that $\ell \notin c'(v_i)$. Because $f(v_i) \subseteq c'(v_i)$ and, for each $\ell \in [k] - f(v_i)$, we have $\ell \notin c'(v_i)$, it follows that $c'(v_i) = f(v_i)$ for $1 \leq i \leq n$. Hence, c is a regal k-edge coloring of K_n and so $\operatorname{reg}(K_n) \leq k$. Therefore, $\operatorname{reg}(K_n) = k$. ■

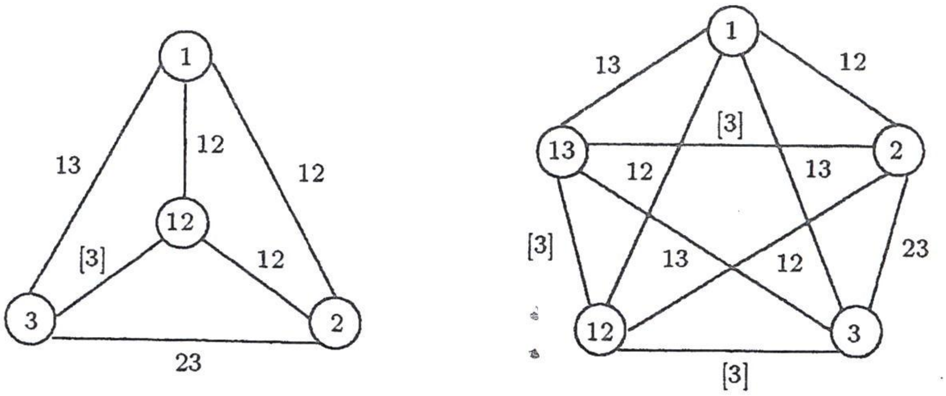


Figure 2: Regal 3-colorings of K_4 and K_5

With the aid of the proof of Theorem 2.5, we are able to determine tl regal indexes of all complete multipartite graphs.

Corollary 2.6 *If G is a complete ℓ-partite graph of order 3 or more f some integer $\ell \geq 2$, then*

$$\operatorname{reg}(G) = \max\{3, \lceil \log_2(\ell+1) \rceil\}.$$

Proof. Since $\chi(G) = \ell$, it follows that $\operatorname{reg}(G) \geq \max\{3, \lceil \log_2(\ell+1) \rceil\}$ t Theorem 2.4. Thus, it remains to show that $\operatorname{reg}(G) \leq \max\{3, \lceil \log_2(\ell+1) \rceil$ First, suppose that $\ell = 2$ and we show that $\operatorname{reg}(G) = 3$. Let the parti sets of G be $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ where $r \leq$ and $r + s \geq 3$. When $r = 1$, define c such that $c(u_1w_1) = \{1, 2\}$ an $c(u_1w_j) = \{1, 3\}$ for all $2 \leq j \leq s$. Then $c'(u_1) = \{1\}$ and $|c'(w_j)| \geq$ for all $2 \leq j \leq s$. Now, when $r \geq 2$, define c such that $c(u_1w_j) = \{1,$ for $1 \leq j \leq s$ and $c(e) = \{1, 3\}$ for all edges $e \in E(K_{r,s})$ that are n incident with u_1. Then $c'(u_1) = \{1, 2\}$, $c'(u_i) = \{1, 3\}$ for $2 \leq i \leq r$, an $c'(w_j) = \{1\}$ for all $1 \leq j \leq s$. Thus, c' is a proper vertex coloring of and therefore c is a regal 3-edge coloring of G. Therefore, $\operatorname{reg}(G) = 3$.

Next, suppose that $\ell \geq 3$. Let $V_1, V_2, \ldots, V_\ell$ be the partite sets of and let $H = K_\ell$ where $V(H) = \{v_1, v_2, \ldots, v_\ell\}$. For $\ell = 3$, let $c_0 : E(H)$ - $\mathcal{P}^*([3])$ be the regal 3-edge coloring of K_3 defined by $c_0(v_iv_j) = \{i, j\}$ fc $1 \leq i < j \leq 3$. For $\ell \geq 4$, let $k = \operatorname{reg}(K_\ell) = \lceil \log_2(\ell+1) \rceil$ and le $c_0 : E(H) \to \mathcal{P}^*([k])$ be the regal k-edge coloring of K_ℓ described in tl proof of Theorem 2.5. We now use this regal k-edge coloring c_0 of $H = K$ $(\ell \geq 3)$ to define a regal k-edge coloring c of G. In particular, we defir $c : E(G) \to \mathcal{P}^*([k])$ by $c(u_iu_j) = c_0(v_iv_j)$ if $u_i \in V_i$ and $u_j \in V_j$ fc each pair i, j of integers with $1 \leq i < j \leq \ell$. Since (i) $c'(u_i) = c_0(v$ if $u_i \in V_i$ for $1 \leq i \leq \ell$ and (ii) c_0 is a regal coloring of $H = K_\ell$, follows that c' is a proper vertex coloring of G. Hence, c is a regal k-edg coloring of G. Therefore, if $\ell = 3$, then $\text{reg}(G) = 3$, while if $\ell \geq 4$, then $\text{reg}(G) \leq \text{reg}(K_\ell) = \lceil \log_2(\ell+1) \rceil\}$ ■

3 Strong Regal Colorings of Trees

We have seen that if H is a connected spanning subgraph of a graph G of order at least 3, then $\text{sreg}(G) \leq \text{sreg}(H)$. In particular, if T is a spanning tree of a graph G of order at least 3, then $\text{sreg}(G) \leq \text{sreg}(T)$. Thus, for each integer $n \geq 3$, trees of order n have the largest strong regal index among all connected graphs of order n. Hence, strong regal indexes of trees play an important role in studying strong regal colorings of connected graphs in general. Therefore, our emphasis in this section is on the strong regal indexes of trees. While the strong regal index of every complete graph of order $n \geq 4$ is $\lceil \log_2(n+1) \rceil$, the strong regal index of every star of order $n \geq 3$ is $1 + \lceil \log_2 n \rceil$.

Theorem 3.1 *For every integer $n \geq 3$,*

$$\text{sreg}(K_{1,n-1}) = 1 + \lceil \log_2 n \rceil.$$

Proof. Let $G = K_{1,n-1}$ be a star of order $n \geq 3$, where $V(G)=\{v, v_1, v_2, \ldots, v_{n-1}\}$ and v is the central vertex of G, and let $k = 1 + \lceil \log_2 n \rceil$. Thus, $2^{k-2} < n \leq 2^{k-1}$ and so

$$2^{k-2} - 1 < n - 1 \leq 2^{k-1} - 1. \tag{2}$$

First, we show that $\text{sreg}(G) \leq k$. Let $S_1, S_2, \ldots, S_{2^{k-1}-1}$ be the distinct nonempty subsets of the set $[2,k] = [k] - \{1\} = \{2, 3, \ldots, k\}$ such that $S_i = \{i+1\}$ for $1 \leq i \leq k-1$. Now, let $T_i = \{1\} \cup S_i$ for $1 \leq i \leq 2^{k-1}-1$. Since $n - 1 \leq 2^{k-1} - 1$ by (2), we can define an edge coloring $c : E(G) \to \mathcal{P}^*([k])$ of G by $c(vv_i) = T_i$ for $1 \leq i \leq n-1$. Then $c'(v) = \{1\}$ and $c'(v_i) = c(vv_i) = T_i$ for $1 \leq i \leq n-1$. Since c' is vertex-distinguishing, it follows that c is a strong regal k-edge coloring of G and so $\text{sreg}(G) \leq k$.

Next, we show that $\text{sreg}(G) \geq k$. Assume, to the contrary, that $\text{sreg}(G) = \ell \leq k-1$. Let $c_0 : E(G) \to \mathcal{P}^*([\ell])$ be a strong ℓ-regal coloring of G where $c_0(vv_i) = X_i$ for $1 \leq i \leq n-1$. Then $|X_i| \geq 2$ for $1 \leq i \leq n-1$ and $X_1, X_2, \ldots, X_{n-1}$ are distinct subsets of $[\ell]$. Since $c_0'(v) \neq \emptyset$, we may assume that $\ell \in c'(v)$. This implies that $\ell \in X_i$ for each integer i with $1 \leq i \leq n-1$. Let $Y_i = X_i - \{\ell\}$ for $1 \leq i \leq n-1$. Then $Y_1, Y_2, \ldots, Y_{n-1}$ are distinct nonempty subsets of $[\ell - 1]$. However then, $n - 1 \leq 2^{\ell-1} - 1 \leq 2^{k-2} - 1$, which is impossible by (2). Hence, $\text{sreg}(G) \geq k$ and so $\text{sreg}(G) = k$. ■

We now turn to another class of trees of interest, namely the paths. We will soon see that we have a special interest in the strong regal index of the path P_7.

Proposition 3.2 $\text{sreg}(P_7) = 4$.

Proof. Let $P_7 = (v_1, v_2, \ldots, v_7)$ where $e_i = v_i v_{i+1}$ for $1 \le i \le 6$. Figure 3 shows a strong regal 4-edge coloring of P_7; consequently, $\text{sreg}(P_7) \le 4$.

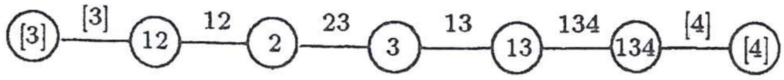


Figure 3: A strong regal 4-edge coloring of P_7

Next, we show that $\text{sreg}(P_7) \ge 4$. Assume, to the contrary, that $\text{sreg}(P_7) = 3$. Let $c : E(P_7) \to \mathcal{P}^*([3])$ be a strong regal 3-edge coloring of P_7. Then

$$\{c'(v_i) : 1 \le i \le 7\} = \mathcal{P}^*([3]). \tag{3}$$

Since $|c'(v_i)| \ge 2$ for $i = 1, 7$, there are integers r, s, t such that $2 \le r < s < t \le 6$ such that $|c'(v_r)| = |c'(v_s)| = |c'(v_t)| = 1$. We may assume that $c'(v_r) = \{1\}$, $c'(v_s) = \{2\}$, and $c'(v_t) = \{3\}$. Since $|c(e)| \ge 2$ for each edge e of P_7, no edge incident with v_r, v_s, or v_t can be colored $[3]$.

First, suppose that $\{r, s, t\} = \{2, 4, 6\}$. This implies that $c(e) \ne [3]$ for every edge e of P_7. However then, $c'(v) \ne [3]$ for any vertex v of P_7, which is impossible by (3). Thus, either $s = r + 1$ or $t = s + 1$. By the symmetry of P_7, we may that either $|c'(v_2)| = |c'(v_3)| = 1$ or $|c'(v_3)| = |c'(v_4)| = 1$. We consider these two cases.

Case 1. $|c'(v_2)| = |c'(v_3)| = 1$, *where* $c'(v_2) = \{1\}$ *and* $c'(v_3) = \{2\}$. Thus, $|c(v_1v_2)| = |c(v_2v_3)| = |c(v_3v_4)| = 2$. Since $c'(v_2) = \{1\}$ and $c'(v_3) = \{2\}$, it follows that $c(v_1v_2) = \{1, 3\}$, $c(v_2v_3) = \{1, 2\}$, and $c(v_3v_4) = \{2, 3\}$. Thus, $c'(v_1) = \{1, 3\}$ and $c'(v_4) \in \{\{3\}, \{2, 3\}\}$.

- ⋆ If $c'(v_4) = \{3\}$, then $c(v_4v_5) = \{1, 3\}$. Hence, either $c'(v_5) = c'(v_1) = \{1, 3\}$ or $c'(v_5) = c'(v_4) = \{3\}$, which is impossible.
- ⋆ If $c'(v_4) = \{2, 3\}$, then $c(v_4v_5) = \{2, 3\}$ or $c(v_4v_5) = [3]$. First, suppose that $c(v_4v_5) = \{2, 3\}$. Since $c'(v_3) = \{2\}$ and $c'(v_4) = \{2, 3\}$, it follows that $c'(v_5) = \{3\}$. Hence, $c(v_5v_6) = \{1, 3\}$. However then, $c'(v_6) = c'(v_1) = \{1, 3\}$, a contradiction. Next, suppose that $c(v_4v_5) = [3]$. Since $c'(v_1) = \{1, 3\}$ and $c'(v_4) = \{2, 3\}$, it follows that $c'(v_5) = c(v_5v_6) = \{1, 2\}$. However then, $c'(v_6) \in \{\{1\}, \{2\}, \{1, 2\}\} = \{c'(v_2), c'(v_3), c'(v_5)\}$, which is a contradiction.

Case 2. $|c'(v_3)| = |c'(v_4)| = 1$. By Case 1, we may assume that $|c'(v_2)| \ge 2$ and so $c'(v_3) = \{1\}$ and $c'(v_4) = \{2\}$. Thus, $|c(v_2v_3)| = |c(v_3v_4)| = |c(v_4v_5)| = 2$. Since $c'(v_3) = \{1\}$ and $c'(v_4) = \{2\}$, it follows that $c(v_2v_3) = \{1, 3\}$, $c(v_3v_4) = \{1, 2\}$, and $c(v_4v_5) = \{2, 3\}$. Thus, $c'(v_2) = \{1, 3\}$ or $c'(v_2) = \{3\}$. Since $|c'(v_2)| \ge 2$, it follows that $c'(v_2) = \{1, 3\}$. Then $c(v_1v_2) = [3]$ and so $c'(v_1) = [3]$. Thus, $c'(v_5) \in \{\{3\}, \{2, 3\}\}$.

⋆ If $c'(v_5) = \{3\}$, then $c(v_5v_6) = \{1,3\}$. However then, $c'(v_6) \in \{\{1\},\{3\},\{1,3\}\}$, which is a contradiction.

⋆ If $c'(v_5) = \{2,3\}$, then $c(v_5v_6) = \{2,3\}$ or $c(v_5v_6) = [3]$. Necessarily, $c'(v_6) = \{3\}$; so $c(v_5v_6) = \{2,3\}$ and $c(v_6v_7) = \{1,3\}$. However then, $c'(v_7) = c'(v_2) = \{1,3\}$, which is a contradiction. ■

Next, we consider the paths P_n where $n \geq 4$ and $n \neq 7$. Observe that if G is a connected graph of order n where $n \geq 2^{k-1} = |\mathcal{P}^*([k-1])| + 1$ for some integer $k \geq 3$, then $\text{sreg}(G) \geq k$. Consequently, if $2^{k-1} \leq n \leq 2^k - 1$, then $\text{sreg}(G) \geq 1 + \lfloor \log_2 n \rfloor$. Since $1 + \lfloor \log_2 n \rfloor = \lceil \log_2(n+1) \rceil$ for each integer $n \geq 4$, this observation is also a consequence of Theorems 2.4 and 2.5.

Corollary 3.3 *If G is a connected graph of order $n \geq 4$, then*

$$\text{sreg}(G) \geq 1 + \lfloor \log_2 n \rfloor.$$

We saw in Proposition 3.2 that equality in Corollary 3.3 does not hold for the path P_7. However, equality holds for all other paths P_n when $n \geq 4$. In order to show this, we first present some useful notation. For $n \geq 4$, let $P_n = (v_1, v_2, \ldots, v_n)$ where $e_i = v_iv_{i+1}$ for $1 \leq i \leq n-1$. For an edge coloring c of P_n and a vertex coloring c' of P_n, let

$$\begin{aligned} \mathcal{S}_c(P_n) &= (c(e_1), c(e_2), \ldots, c(e_{n-1})) \\ \mathcal{S}_{c'}(P_n) &= (c'(v_1), c'(v_2), \ldots, c'(v_n)). \end{aligned}$$

For two integers a and b with $a < b$, let $[a,b] = \{a, a+1, \ldots, b\}$ be the set of integers between a and b.

Theorem 3.4 *If $n \geq 4$ is an integer with $n \neq 7$, then*

$$\text{sreg}(P_n) = 1 + \lfloor \log_2 n \rfloor.$$

Proof. Let $k = 1 + \lfloor \log_2 n \rfloor$, where $n \geq 4$ and $n \neq 7$. Since $\text{sreg}(P_n) \geq k$ by Corollary 3.3, it suffices to show that P_n has a strong regal k-edge coloring. Figure 4 shows that P_n has such a coloring for $n \in [4,6] \cup [12,15]$.

Observe that the induced vertex coloring of each path P_n in Figure 4 for $n \in [4,5] \cup [12,15]$ contains two adjacent vertices whose colors are disjoint. For an integer $n \in [4,5] \cup [12,15]$, let $H = (v_1, v_2, \ldots, v_n)$ be the path P_n of order n and let $H^\star = (v_n, v_{n-1}, \ldots, v_1)$ be the path P_n in reverse order. Let c_H be the edge coloring of H shown in Figure 4. We now define a strong regal $(k+1)$-edge coloring $c_{H^\star} : E(H^\star) \to \mathcal{P}^*([k+1])$ of $H^\star$ by

$$c_{H^\star}(v_{i+1}v_i) = c_H(v_iv_{i+1}) \cup \{k+1\} \text{ for } 1 \leq i \leq n-1.$$

$$
\begin{aligned}
\mathcal{S}_c(P_4) &= (12, 23, 13)\\
\mathcal{S}_{c'}(P_4) &= (\mathbf{12}, \underline{\mathbf{2}, \mathbf{3}}, \mathbf{13})\\
\mathcal{S}_c(P_5) &= ([3], 12, 23, 13)\\
\mathcal{S}_{c'}(P_5) &= ([\mathbf{3}], \mathbf{12}, \underline{\mathbf{2}, \mathbf{3}}, \mathbf{13})\\
\mathcal{S}_c(P_6) &= (12, 13, 13, 23, [3])\\
\mathcal{S}_{c'}(P_6) &= (\mathbf{12}, \mathbf{1}, \mathbf{13}, \mathbf{3}, \mathbf{23}, [\mathbf{3}])\\
\mathcal{S}_c(P_{12}) &= (124, 123, [4], [4], 234, 134, 14, 24, 12, 13, 134)\\
\mathcal{S}_{c'}(P_{12}) &= (\mathbf{124}, \mathbf{12}, \mathbf{123}, [\mathbf{4}], \mathbf{234}, \mathbf{34}, \mathbf{14}, \underline{\mathbf{4}, \mathbf{2}}, \mathbf{1}, \mathbf{13}, \mathbf{134})\\
\mathcal{S}_c(P_{13}) &= (123, 13, 124, 12, 234, 124, [4], 234, 23, 134, 14, 34)\\
\mathcal{S}_{c'}(P_{13}) &= (\mathbf{123}, \mathbf{13}, \mathbf{1}, \mathbf{12}, \mathbf{2}, \mathbf{24}, \mathbf{124}, \mathbf{234}, \mathbf{23}, \underline{\mathbf{3}, \mathbf{14}}, \mathbf{4}, \mathbf{34})\\
\mathcal{S}_c(P_{14}) &= (123, 13, 124, 12, 234, 124, [4], 234, 23, 134, 14, 34, 134)\\
\mathcal{S}_{c'}(P_{14}) &= (\mathbf{123}, \mathbf{13}, \mathbf{1}, \mathbf{12}, \mathbf{2}, \mathbf{24}, \mathbf{124}, \mathbf{234}, \mathbf{23}, \underline{\mathbf{3}, \mathbf{14}}, \mathbf{4}, \mathbf{34}, \mathbf{134})\\
\mathcal{S}_c(P_{15}) &= ([4], 123, 13, 124, 12, 234, 124, [4], 234, 23, 134, 14, 34, 134\\
\mathcal{S}_{c'}(P_{15}) &= ([\mathbf{4}], \mathbf{123}, \mathbf{13}, \mathbf{1}, \mathbf{12}, \mathbf{2}, \mathbf{24}, \mathbf{124}, \mathbf{234}, \mathbf{23}, \underline{\mathbf{3}, \mathbf{14}}, \mathbf{4}, \mathbf{34}, \mathbf{134}
\end{aligned}
$$

Figure 4: Showing that $\text{sreg}(P_n) = 1 + \lfloor \log_2 n \rfloor$ for $n \in [4, 6] \cup [12, 1$

Let G be the path of order $2n$ obtained from H and H^* by joining the vertices v_n in H and H^* by the edge f. The edge coloring $c_G : E(G)$ $\mathcal{P}^*([k+1])$ is defined by

$$
c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ c_{H^*}(e) & \text{if } e \in E(H^*) \\ c_{H^*}(v_n v_{n-1}) & \text{if } e = f. \end{cases}
$$

The coloring c_G is illustrated in Figure 5 for $G = P_{10}$ when $n = 5$. Si this edge coloring is a strong regal $(k+1)$-edge coloring of the path G order $2n$, it follows that $\text{sreg}(G) = 1 + \lfloor \log_2(2n) \rfloor$.

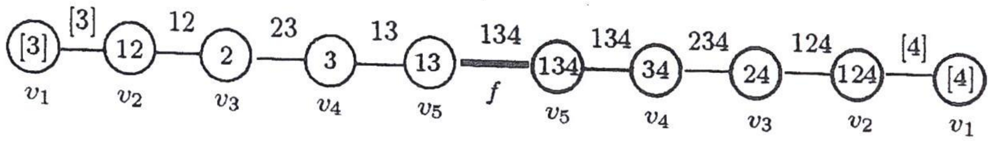


Figure 5: Constructing a strong regal 4-edge coloring of P_{10}

Next, for each $n \in [4, 5] \cup [12, 15]$, let F be the path of order n

1 obtained from H by subdividing an edge $v_j v_{j+1}$ of H where $c'_H(v_j) \cap c'_H(v_{j+1}) = \emptyset$, obtaining the subpath (v_j, u, v_{j+1}). Define an edge coloring c_F of F by

$$c_F(e) = \begin{cases} c'_H(v_j) & \text{if } e = v_j u \\ c'_H(v_{j+1}) & \text{if } e = u v_{j+1} \\ c_H(e) & \text{if } e \neq v_j u, u v_{j+1}. \end{cases}$$

(The edge coloring c_F is not a regal edge coloring since $c'_F(u) = \emptyset$.) Let $F^\star = (v_n, v_{n-1}, \ldots, v_{j+1}, u, v_j, \ldots, v_1)$ be the path F in reverse order. Define the edge coloring $c_{F^\star} : E(F^\star) \to \mathcal{P}^*([k+1])$ of $F^\star$ by

$$c_{F^\star}(e) = c_F(e) \cup \{k+1\} \text{ for each } e \in E(F^\star).$$

Then $c'_{F^\star}(v) = c'_H(v_i) \cup \{k+1\}$ for $1 \leq i \leq n$ and $c'_{F^\star}(u) = \{k+1\}$. Since $c'_{F^\star}$ is vertex-distinguishing, it follows that $c_{F^\star}$ is a strong regal $(k+1)$-edge coloring of $F^\star$. The graphs H, F and $F^\star$ are shown in Figure 6 as well as the corresponding edge colorings. Let G be the path of order $2n+1$ obtained from H and $F^\star$ by joining the vertex v_n in H and $F^\star$ by the edge f. The edge coloring $c_G : E(G) \to \mathcal{P}^*([k+1])$ is defined by

$$c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ c_{F^\star}(e) & \text{if } e \in E(F^\star) \\ c_{F^\star}(v_n v_{n-1}) & \text{if } e = f. \end{cases}$$

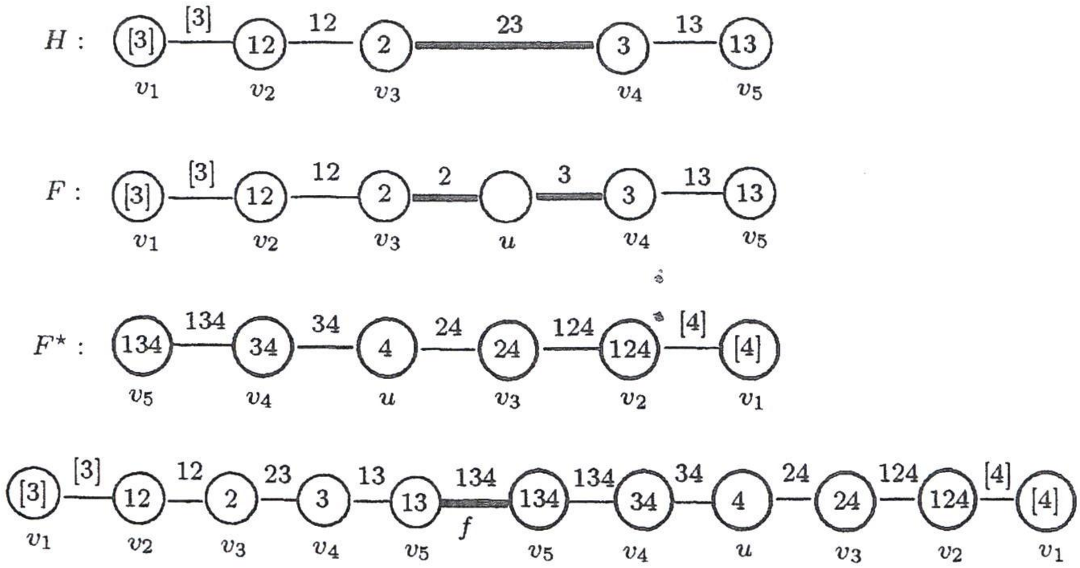
The coloring c_G is illustrated in Figure 6 for $G = P_{11}$ when $n = 5$. Since this edge coloring is a strong regal $(k+1)$-edge coloring of the path G of order $2n+1$, it follows that $\text{sreg}(G) = 1 + \lfloor \log_2(2n+1) \rfloor$.

The colorings defined above show, in particular, that if $n \in [4, 31]$ where $n \neq 7$, then $\text{sreg}(P_n) = 1 + \lfloor \log_2 n \rfloor$. Furthermore, the induced vertex coloring of each such path P_n where $n \in [16, 31]$ has the property that there exist two adjacent vertices whose colors are disjoint. By proceeding as above, we see that if $n \in [32, 63]$, then $\text{sreg}(P_n) = 1 + \lfloor \log_2 n \rfloor$ and the induced vertex coloring of each such path has the property that there exist two adjacent vertices whose colors are disjoint. Consequently, for each integer $\ell \geq 2$ and each integer $n \in [2^\ell, 2^{\ell+1} - 1]$, we have $\text{sreg}(P_n) = 1 + \lfloor \log_2 n \rfloor$ except when $n = 7$. That is, for each integer $n \geq 4$ and $n \neq 7$, it follows that $\text{sreg}(P_n) = 1 + \lfloor \log_2 n \rfloor$. ∎

The following result is a consequence Corollary 3.3 and Theorem 3.4.

Corollary 3.5 *If* $n \geq 4$ *is an integer, then* $\text{sreg}(C_n) = \text{sreg}(P_n)$.

We have seen that if T is a star of order $n \geq 4$, then $\text{sreg}(T) = 1 + \lceil \log_2 n \rceil$; while if T is a path of order $n \geq 4$, then $\text{sreg}(T) = 1 + \lfloor \log_2 n \rfloor$. Next, we show that if T is a double star (a tree of diameter 3) of order $n \geq 4$, then $1 + \lfloor \log_2 n \rfloor \leq \text{sreg}(T) \leq 1 + \lceil \log_2 n \rceil$.



A strong regal 4-coloring of $G = P_{11}$

Figure 6: Constructing a strong regal 4-edge coloring of P_{11}

Theorem 3.6 *If T is a double star of order $n \geq 4$, then*

$$1 + \lfloor \log_2 n \rfloor \leq \text{sreg}(T) \leq 1 + \lceil \log_2 n \rceil .$$

Proof. Since the lower bound is a consequence of Corollary 3.3, we need only to establish the upper bound. Since $\text{sreg}(P_4) = 3$, we may assume that $n \geq 5$. Let $k = 1 + \lceil \log_2 n \rceil \geq 4$. Since $k = 1 + \lceil \log_2 n \rceil \geq 1 + \log_2 n$, it follows that $n \leq 2^{k-1}$. We show that $\text{sreg}(T) \leq k$ or there is a strong k-coloring of T. Let T be a double star of order n whose central vertices u and v have degrees a and b, respectively, where $2 \leq a \leq b$. Then $n = a + b$. Suppose that u is adjacent to the end-vertices $u_1, u_2, \ldots, u_{a-1}$ and v is adjacent to the end-vertices $v_1, v_2, \ldots, v_{b-1}$. Let $X_1, X_2, \ldots, X_{2^{k-2}-1}$ be the distinct nonempty subsets of $\{3, 4, \ldots, k\}$ where $|X_1| \leq |X_2| \leq \ldots \leq |X_{2^{k-2}-1}|$ and $X_i = \{i + 2\}$ for $1 \leq i \leq k - 2$. Since $a \leq b$ and $a + b = n \leq 2^{k-1}$, it follows that $a - 1 \leq \frac{1}{2}\left(2^{k-1} - 2\right) = 2^{k-2} - 1$. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1,2\} & \text{if } e = uv \\ \{1\} \cup X_i & \text{if } e = uu_i,\ 1 \leq i \leq a-1 \\ \{2\} \cup X_j & \text{if } e = vv_j,\ \ 1 \leq j \leq 2^{k-2} - 1 \\ \{1,2\} \cup X_j & \text{if } e = vv_{j+(2^{k-2}-1)},\ \ 1 \leq j \leq b - 2^{k-2}. \end{cases}$$

The induced vertex coloring c' satisfies

$$c'(w) = \begin{cases} \{1\} & \text{if } w = u \\ \{2\} & \text{if } w = v \\ \{1\} \cup X_i & \text{if } w = u_i,\ \ 1 \le i \le a-1 \\ \{2\} \cup X_j & \text{if } w = v_j,\ \ 1 \le j \le 2^{k-2}-1 \\ \{1,2\} \cup X_j & \text{if } w = v_{j+(2^{k-2}-1)},\ \ 1 \le j \le b - 2^{k-2}. \end{cases}$$

Since c' is vertex-distinguishing, it follows that c is a strong k-regal coloring of T and so $\mathrm{sreg}(T) \le k = 1 + \lceil \log_2 n \rceil$. ■

It can be verified that there are infinitely many double stars T of order $n \ge 4$ with $\mathrm{sreg}(T) = 1 + \lfloor \log_2 n \rfloor$ and there are infinitely many double stars T of order $n \ge 4$ with $\mathrm{sreg}(T) = 1 + \lceil \log_2 n \rceil$. In fact, if T is any tree of order n with $4 \le n \le 6$ that is not a star, then $\mathrm{sreg}(T) = 3 = 1 + \lfloor \log_2 n \rfloor$ and if T is any tree of order $n = 7$, then $\mathrm{sreg}(T) = 4 = 1 + \lceil \log_2 n \rceil$. It is not known if there is any tree of order $n \ge 8$ whose strong regal index is neither $1 + \lfloor \log_2 n \rfloor$ nor $1 + \lceil \log_2 n \rceil$. Therefore, we conclude this paper with the following conjecture.

Conjecture 3.7 *For every tree T of order $n \ge 4$,*

$$1 + \lfloor \log_2 n \rfloor \le \mathrm{sreg}(T) \le 1 + \lceil \log_2 n \rceil.$$

Conjecture 3.7, if true, states that if T_1 and T_2 are any two trees of the same order $n \ge 4$, then $|\mathrm{sreg}(T_1) - \mathrm{sreg}(T_2)| \le 1$. Of course, Conjecture 3.7 also states that every two trees of order 2^k for some integer $k \ge 2$ have the same strong regal index. Furthermore, Conjecture 3.7, Proposition 3.3 and Lemma 2.1 give rise to the following conjecture.

Conjecture 3.8 *For every connected graph G of order $n \ge 4$,*

$$1 + \lfloor \log_2 n \rfloor \le \mathrm{sreg}(G) \le 1 + \lceil \log_2 n \rceil.$$

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