Decomposition of complete graphs into tri-cyclic graphs with eight edges

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Abstract

In this paper, we use standard graph labeling techniques to prove that each tri-cyclic graph with eight edges decomposes the complete graph K_n if and only if $n \equiv 0, 1 \pmod{16}$. We apply ρ-tripartite labelings and 1-rotational ρ-tripartite labelings.

Keywords: Graph decomposition, G-design, ρ-labeling
Mathematics Subject Classification: 05C51, 05C78

1 Introduction

Decomposition of complete graphs into isomorphic graphs is a classical graph theory problem and has been studied extensively. One of the most efficient decomposition methods is based on graph labelings. Literature on graph labelings began with Rosa's paper [10]. Since then, much work has been done. Recently, significant progress has been made to classify all decompositions of complete graphs into graphs with eight edges; see [1, 4, 5, 6, 7, 8]. In this paper, we consider tri-cyclic graphs with eight edges, both connected and disconnected.

2 Background

In this section, we provide all necessary definitions and tools needed to find the decompositions mentioned above.

Definition 2.1. Let H be a graph. A *decomposition* of the graph H is a collection of pairwise edge disjoint subgraphs $\mathcal{D} = \{G_1, G_2, \ldots, G_s\}$ such that every edge of H appears in exactly one subgraph $G_i \in \mathcal{D}$.

We say that the collection forms a *G-decomposition* of H (also known as an *(H, G)-design*) if each subgraph G_r is isomorphic to a given graph G. If H is the complete graph K_n, then we can use just the term *G-design*.

As we are only interested in decompositions of complete graphs, we only use the term G-decomposition or G-design.

Definition 2.2. A G-decomposition of the complete graph K_n is *cyclic* if there exists an ordering $(x_0, x_1, \dots, x_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \dots, n-1$ inducing an automorphism on $\mathcal{D}$ (where the addition is performed modulo n).

Definition 2.3. A G-decomposition of the complete graph K_n is 1-*rotational* if there exists an ordering $(x_0, x_1, \dots, x_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \dots, n-2$ and $\varphi(x_{n-1}) = x_{n-1}$ inducing an automorphism on $\mathcal{D}$ (where the addition is performed modulo $n-1$).

A finite graph G with no loops or multiple edges is called *tri-cyclic* if it contains exactly three cycles. Because we only consider graphs with eight edges, no two cycles can be edge-disjoint. Hence, we always have one cycle of length 4, 5, 6 or 7 with exactly one chord, or a cycle of length 4 or 5 with two non-adjacent vertices connected by a path of length 2.

Our main tools in this paper are Rosa-type labelings, which are special cases of the original ρ-labeling defined by Rosa [10] in 1966.

In what follows, for the set of consecutive integers $\{k, k+1, k+2, \dots, n\}$ we use the interval notation $[k, n]$.

Definition 2.4 (Rosa [10]). Let G be a graph with n edges. A *ρ-labeling* of G consists of an injective function $f\colon V(G) \to [0, 2n]$ that induces a *length function* $\ell\colon E(G) \to [1, n]$ which is defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)\}$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [1, n].$$

The following labeling was first used by Huang and Rosa in [9].

Definition 2.5 (Huang, Rosa [9]). Let G be a graph with n edges and edge ww' where $\deg(w) = 1$. A 1-*rotational ρ-labeling* of G consists of an injective function $f\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ such that $f(w) = \infty$ that induces a *length function* $\ell\colon E(G) \to [1, n-1] \cup \{\infty\}$ which is defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n - 1 - |f(u) - f(v)|\}$$

for $u, v \neq w$ and

$$\ell(ww') = \infty$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [1, n-1] \cup \{\infty\}.$$

Definition 2.6 (Bunge et al. [3]). Let G be a tripartite graph with n edges and vertex tripartition $\{A, B, C\}$. A *ρ-tripartite labeling* of G is an injective function $h\colon V(G) \to [0, 2n]$ that satisfies the following conditions:

(a) h is a ρ-labeling of G;

(b) if $ab \in E(G)$ with $a \in A$ then $h(a) < h(v)$;

(c) if $e = bc \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = b'c' \in E(G)$ with $b' \in B$ and $c' \in C$ such that

$$|h(c') - h(b')| + |h(c) - h(b)| = 2n;$$

(d) if $b \in B$ and $c \in C$, then $|h(b) - h(c)| \neq 2n$.

Definition 2.7 (Bunge [2]). Let G be a tripartite graph with n edges, vertex tripartition $\{A, B, C\}$, and edge uw where $\deg(w) = 1$. A *1-rotational ρ-tripartite labeling* of a graph G is a 1-rotational ρ-labeling that satisfies the following:

(a) $f(w) = \infty$;

(b) $f(a) < f(v)$ for all $av \in E(G) \setminus \{uw\}$ with $a \in A$; and

(c) for every edge $bc \in E(G)$ with $b \in B$ and $c \in C$ there exists an edge $e' = b'c'$ with $b' \in B$ and $c' \in C$ such that $|h(c) - h(b)| + |h(c') - h(b')| = 2n$.

We apply the next two theorems to the situation where the graph G is tricyclic with eight edges. The first theorem is from [3, Theorem 6] and the second can be found in [2, Theorem 8].

Theorem 2.8 (Bunge et al. [3]). *If a tripartite graph G with n edges has a ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nk+1} for every positive integer k.*

Theorem 2.9 (Bunge [2]). *Let G be a tripartite graph with n edges and a vertex of degree 1. If G admits a 1-rotational ρ-tripartite labeling, then there exists a 1-rotational G-decomposition of K_{2nk} for any positive integer k.*

3 Main Result

The proof of the main theorem lies in the graph labelings in Section 4. First we describe the layout of the catalog of graphs. The left column contains ρ-tripartite labelings. The vertices are labeled with the numbers 0 through 16. Each edge is labeled with the result of the length function $\ell : E(G) \to [1, 8]$ as defined in Definition 2.4.

The right column contains 1-rotational ρ-tripartite labelings for the graphs. The vertices are labeled with the numbers 0 through 14 and one vertex is labeled with infinity. Each edge is labeled with the result of the length function $\ell : E(G) \to [1,7] \cup \{\infty\}$ as defined in Definition 2.5.

We catalog the graphs by the labels $H_k(a,b;j_1,\ldots,j_a)$, $H_k^*(a,b;j_1,\ldots,j_a)$, $D_k(a,b;j_1,\ldots,j_a)$, and $D_k^*(a,b;j_1,\ldots,j_a)$.

The letter H denotes a connected graph while the letter D denotes a disconnected graph. We let a denote the number of vertices in the longest cycle $v_1, v_2, \ldots, v_a, v_1$, and b the shortest distance around the longest cycle between the end-vertices of the chord $v_1 v_{b+1}$ (in graphs H or D) or the path v_1, v', v_{b+1} (in graphs H^* or D^*). We let j_i denote the number of edges in the tree adjacent to vertex v_i in the longest cycle where the vertex labeled 0 is always vertex v_1 and the vertices are labeled clockwise around the longest cycle. The subscript k is just distinguishing different mutually non-isomorphic graphs with the same set of parameters $(a,b;j_1,\ldots,j_a)$.

The existence of G-decompositions of K_n for graphs $H_1^*(6,2;0,0,0,0,0,0)$, $H_1^*(6,3;0,0,0,0,0,0)$, $H_1(7,2;0,0,0,0,0,0,0)$ and $H_1(7,3;0,0,0,0,0,0,0)$, was completely settled by Blinco [1].

Theorem 3.1 (Blinco [1]). *The tri-cyclic connected graphs* $H_1^*(6,2;0,0,0,0,0,0)$, $H_1^*(6,3;0,0,0,0,0,0)$, $H_1(7,2;0,0,0,0,0,0,0)$ *and* $H_1(7,3;0,0,0,0,0,0,0)$ *decompose the complete graph* K_n *for every* $n \equiv 0, 1 \pmod{16}$.

Next we present our main theorem. Notice that because our graphs have eight edges, the number of edges in K_n must be a multiple of eight. It is easy to check that this is only satisfied when $n \equiv 0 \pmod{16}$ or $n \equiv 1 \pmod{16}$.

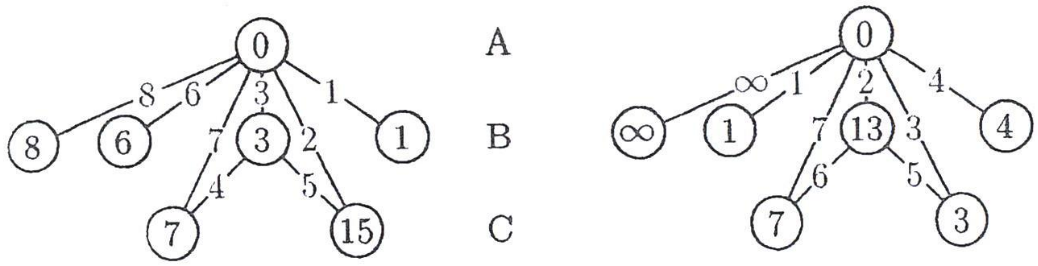
Theorem 3.2. *A tri-cyclic graph G with eight edges decomposes the complete graph K_n if and only if* $n \equiv 0, 1 \pmod{16}$

Proof. Given the graph labelings in Section 4 and Theorems 2.8, 2.9, and 3.1, the result follows.

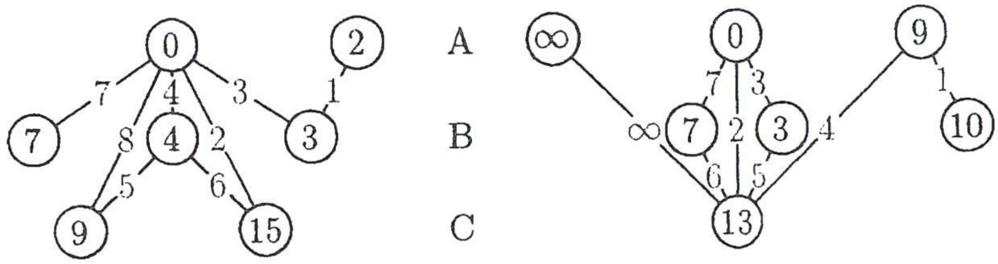
The graph vertices are organized in three rows, where each row is denoted by one of the letters A, B, and C, each denoting the partite set to which all vertices in that row belong.

Notice that in some cases when the graph in question is bipartite, we leave the set C empty while satisfying all requirements of the appropriate labeling. In this case we sometimes split the partite set A into the first and last row to obtain a drawing of the graph with fewer of edge intersections. □

4 Catalog of Graphs and Labelings



$H_1(4,2;3,0,0,0)$



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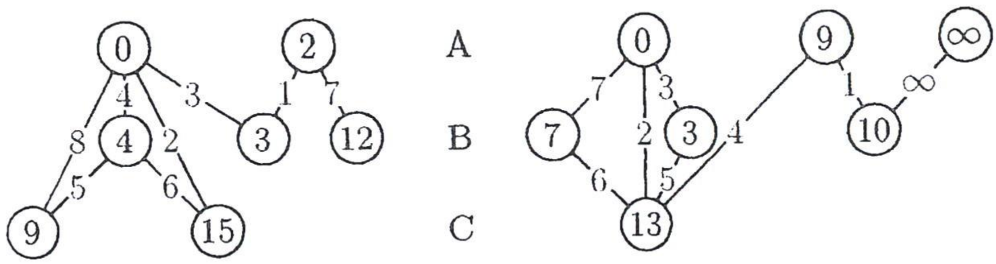
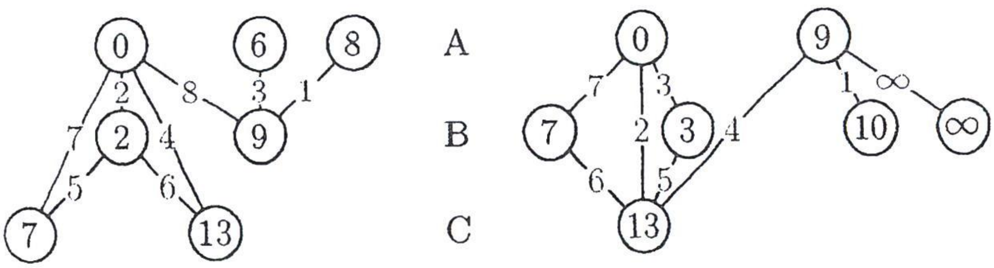


$H_3(4,2;3,0,0,0)$



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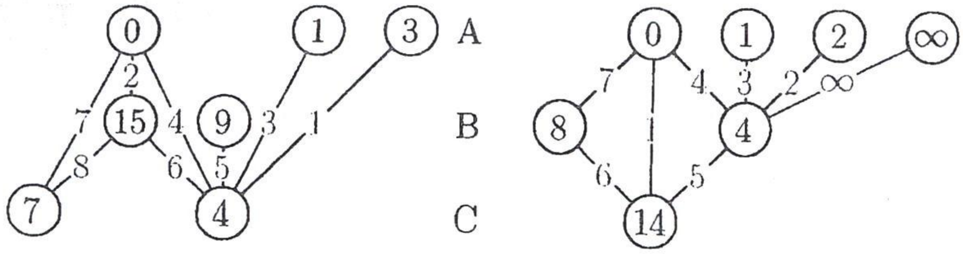
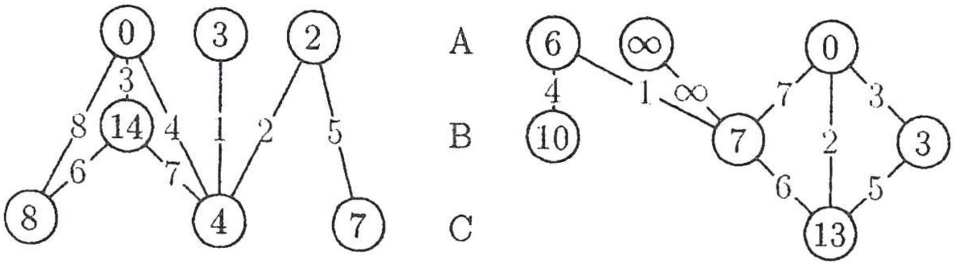
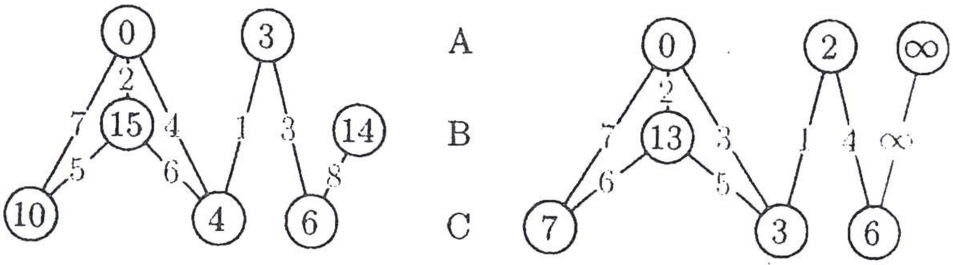


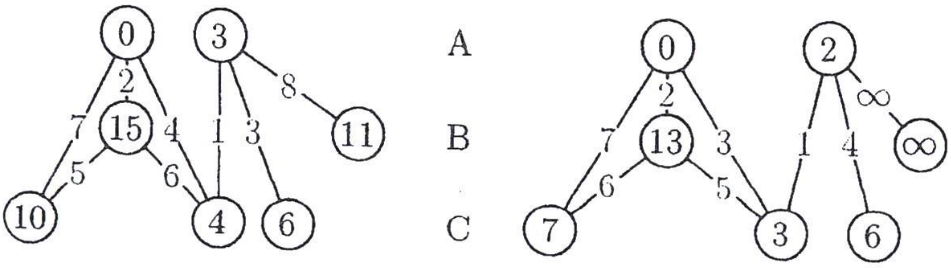
$H_1(4,2;0,3,0,0)$



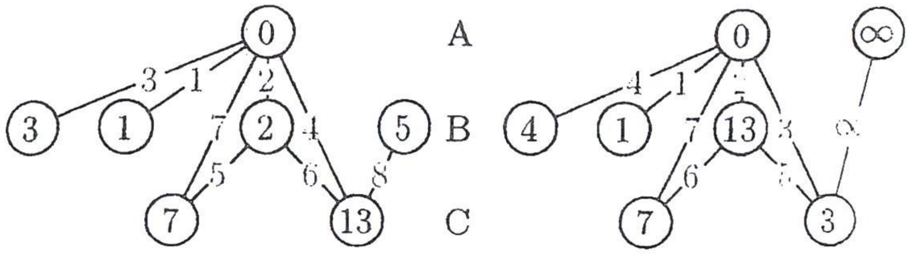
$H_2(4,2;0,3,0,0)$



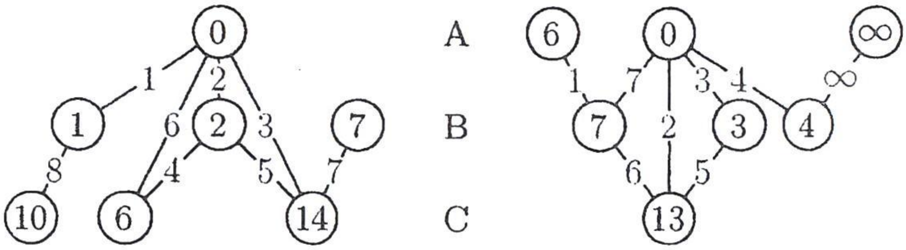
$H_3(4,2;0,3,0,0)$



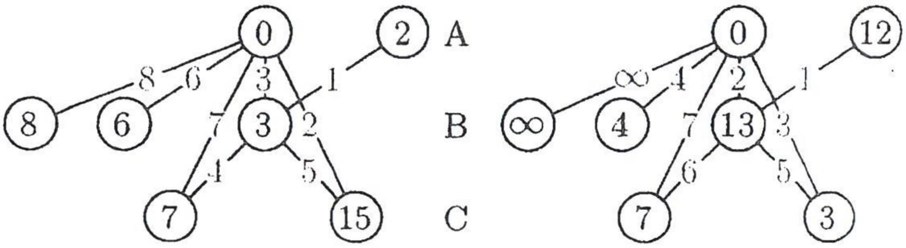
$H_4(4,2;0,3,0,0)$



$H_1(4,2;2,1,0,0)$



$H_2(4,2;2,1,0,0)$



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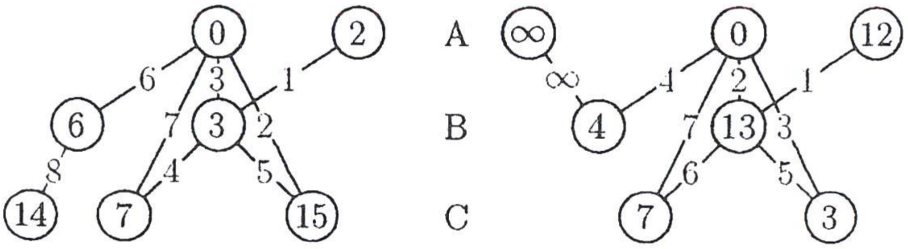
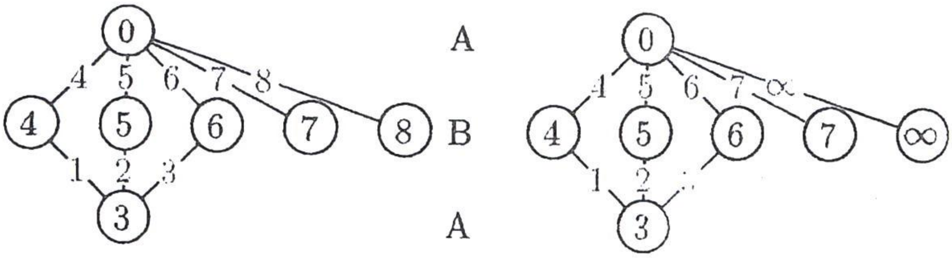
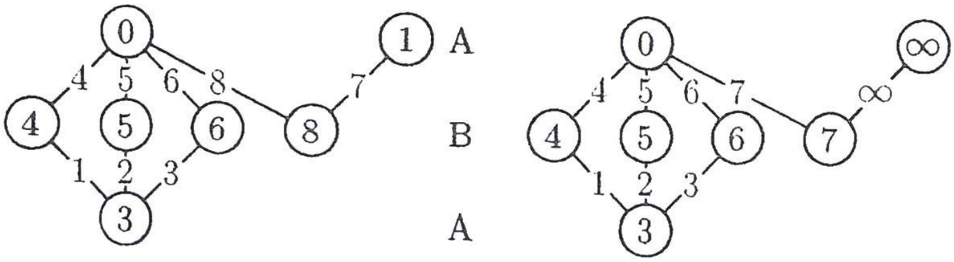


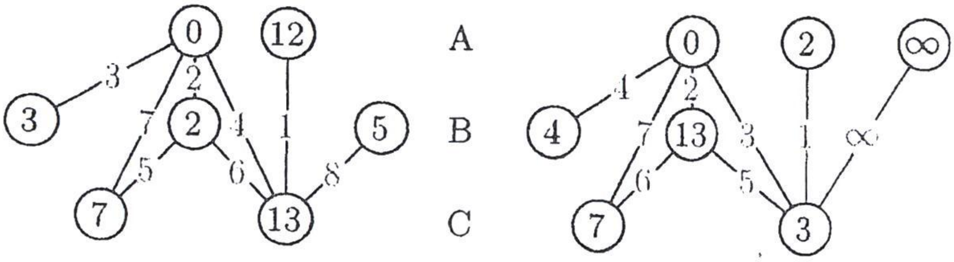
$H_2(4,2;2,0,1,0)$



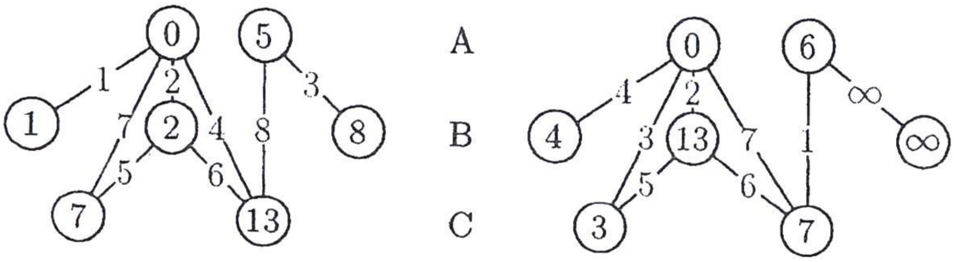
$H_1^*(4,2;2,0,0,0)$



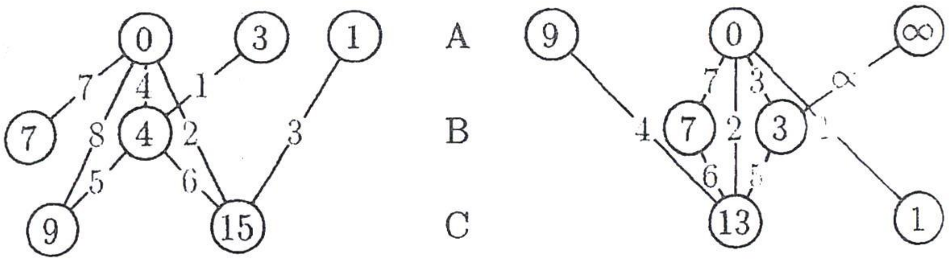
$H_2^*(4,2;2,0,0,0)$



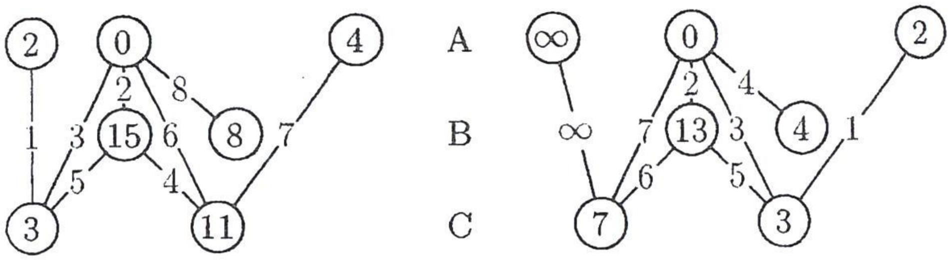
$H_1(4,2;1,2,0,0)$



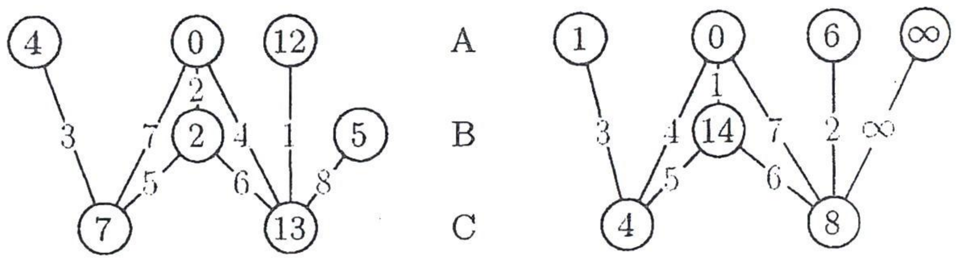
$H_2(4,2;1,2,0,0)$



$H_1(4,2;1,1,1,0)$



$H_1(4,2;1,1,0,1)$



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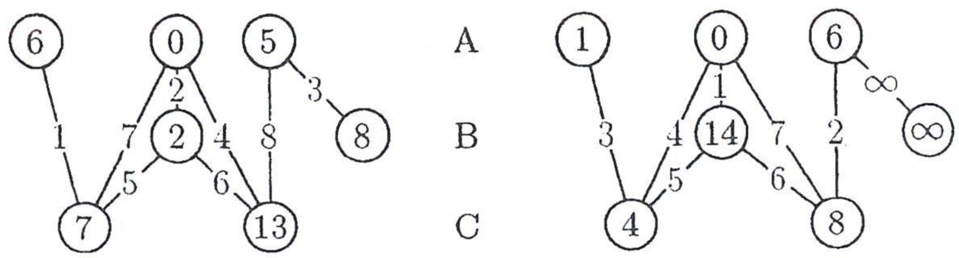
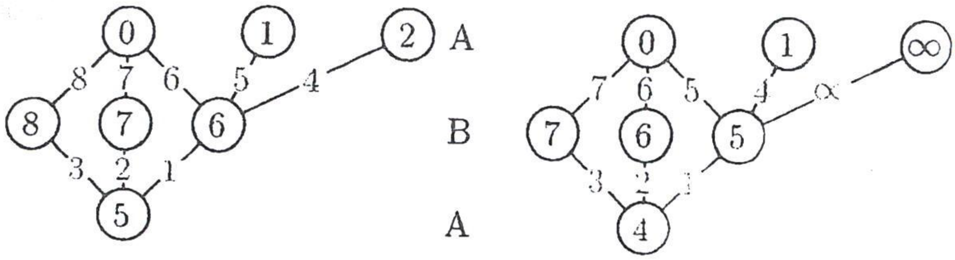
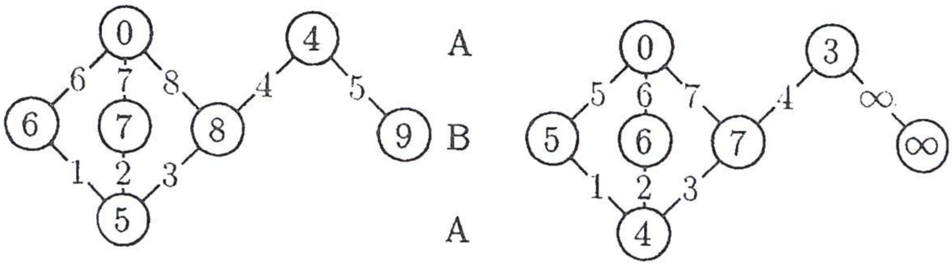


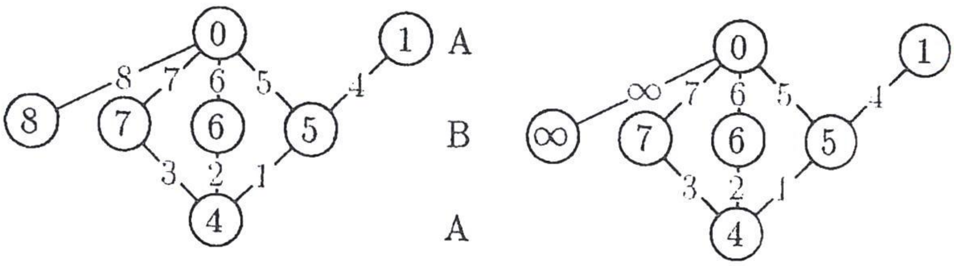
$H_2(4,2;0,2,0,1)$



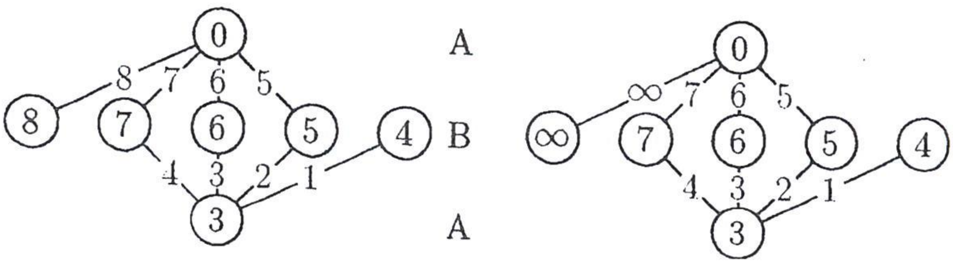
$H_1^*(4, 2; 0, 2, 0, 0)$



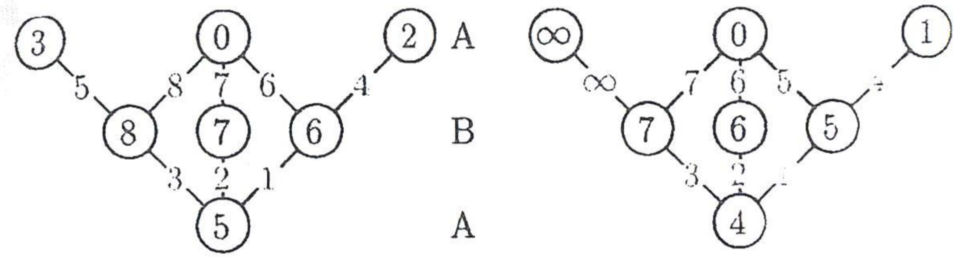
$H_2^*(4, 2; 0, 2, 0, 0)$



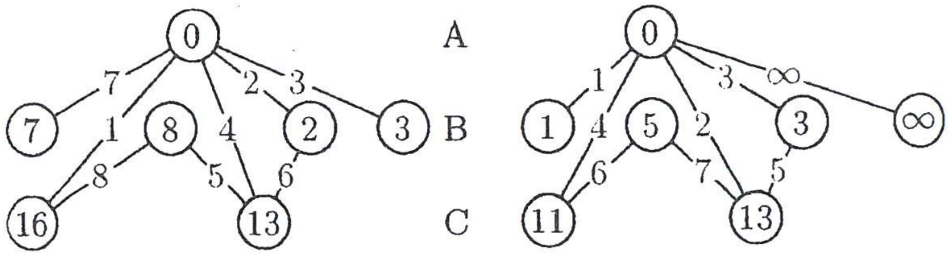
$H_1^*(4, 2; 1, 1, 0, 0)$



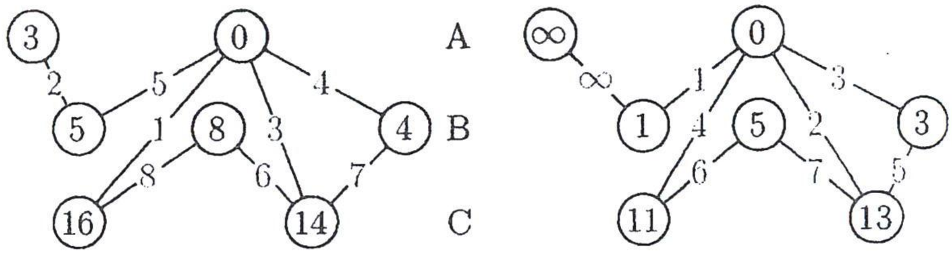
$H_1^*(4, 2; 1, 0, 1, 0)$



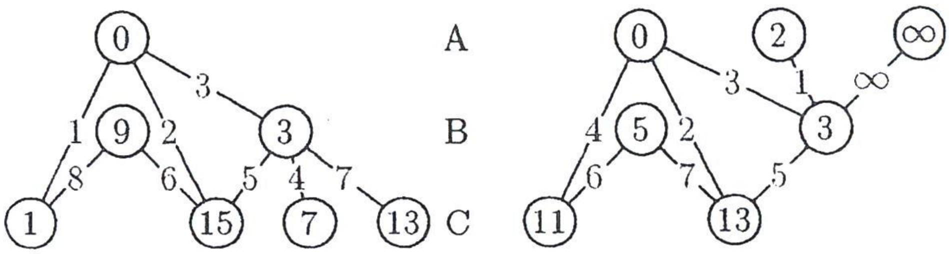
$H_1^*(4,2;0,1,0,1)$



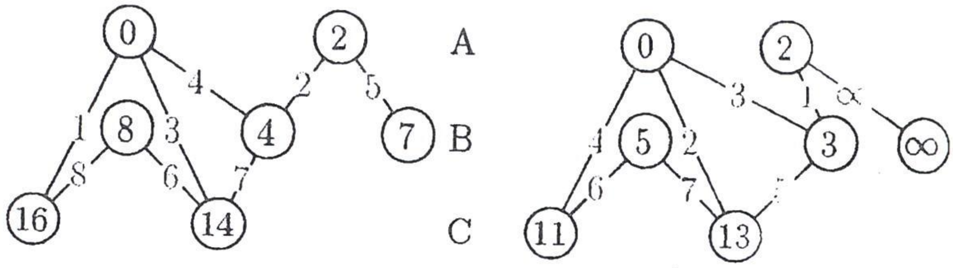
$H_1(5,2;2,0,0,0,0)$



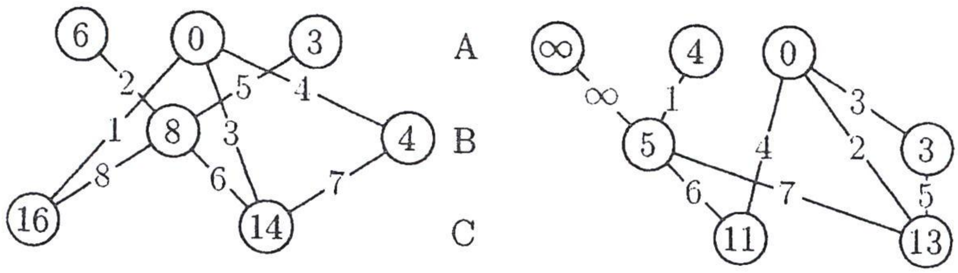
$H_2(5,2;2,0,0,0,0)$



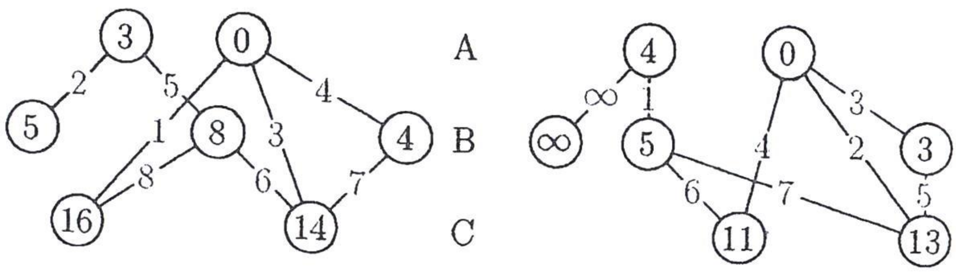
$H_1(5,2;0,2,0,0,0)$



$H_2(5,2;0,2,0,0,0)$



$H_1(5,2;0,0,2,0,0)$



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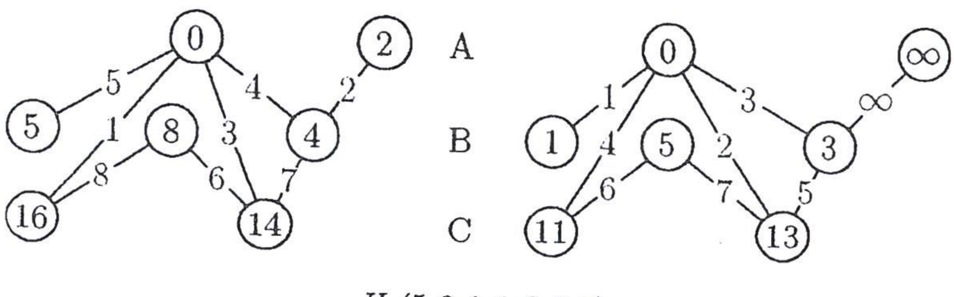
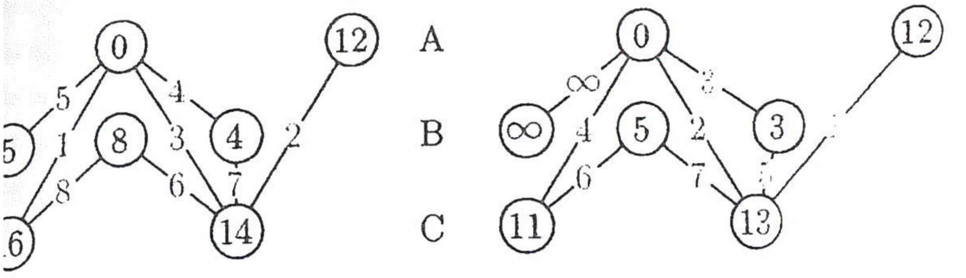
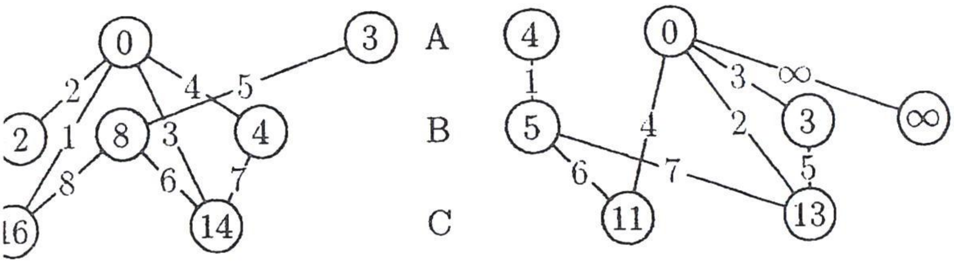


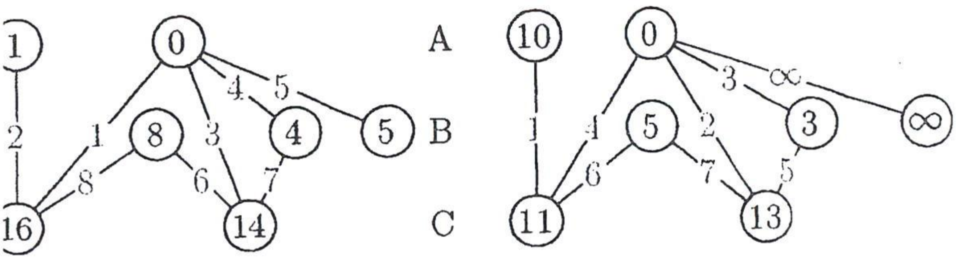
$H_1(5,2;1,1,0,0,0)$



$H_1(5,2;1,0,1,0,0)$



$H_1(5,2;1,0,0,1,0)$



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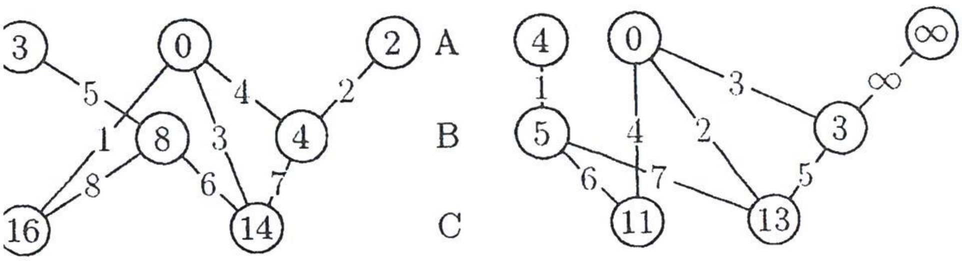
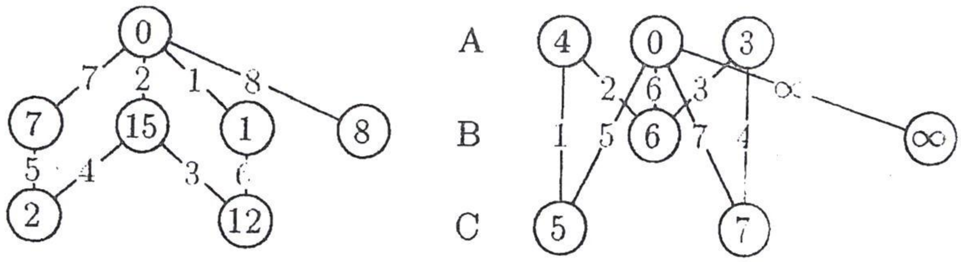
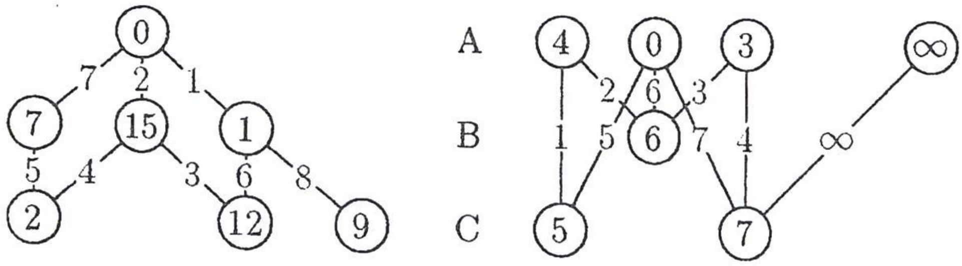


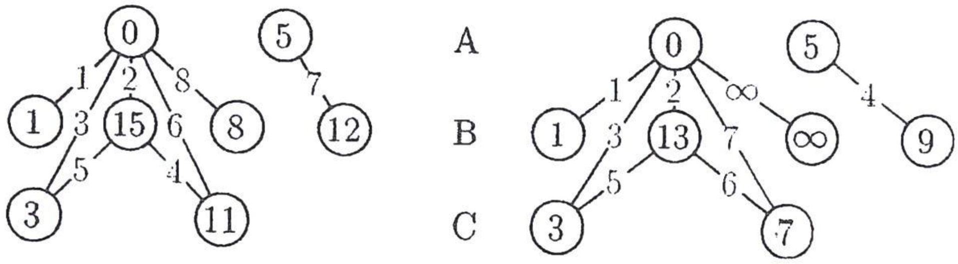
$H_1(5,2;0,1,0,1,0)$



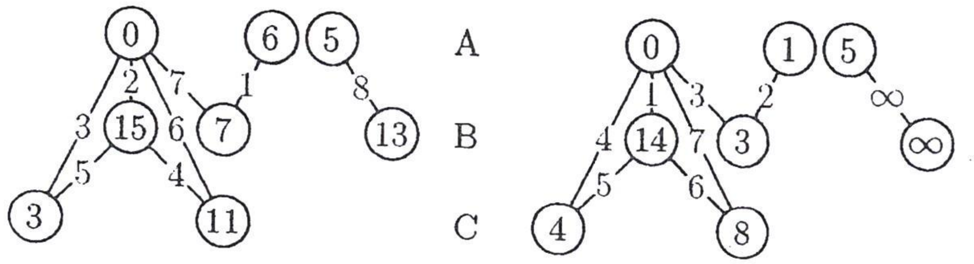
$H_1(6,3;1,0,0,0,0,0)$



$H_1(6,3;0,1,0,0,0,0)$



$D_1(4,2;2,0,0,0)$



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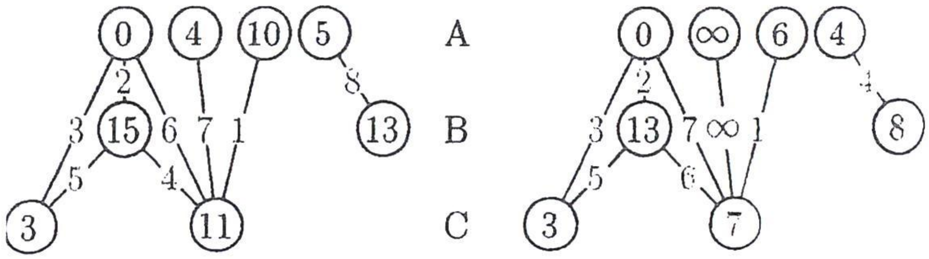

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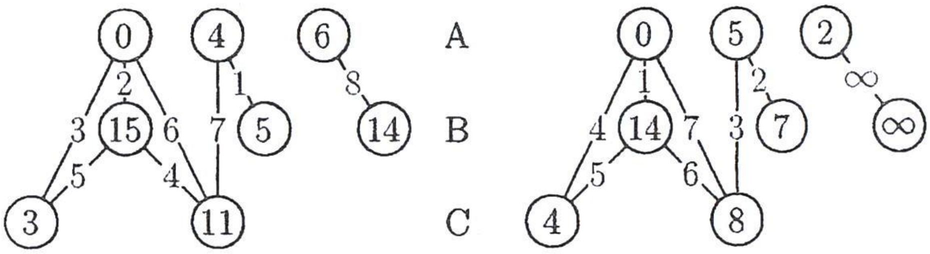

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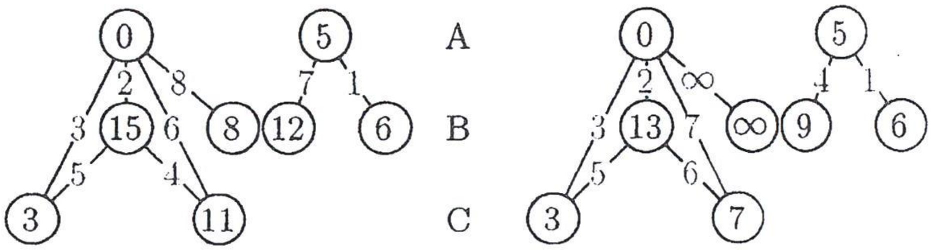
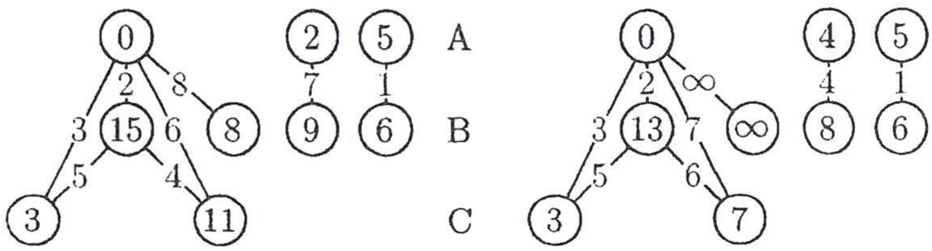


$D_1(4,2;1,0,0,0)$



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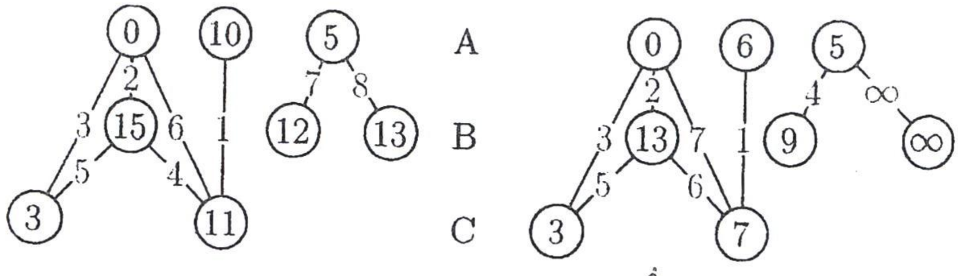
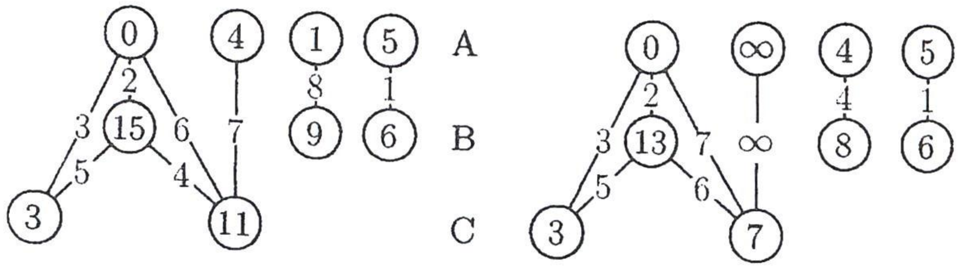
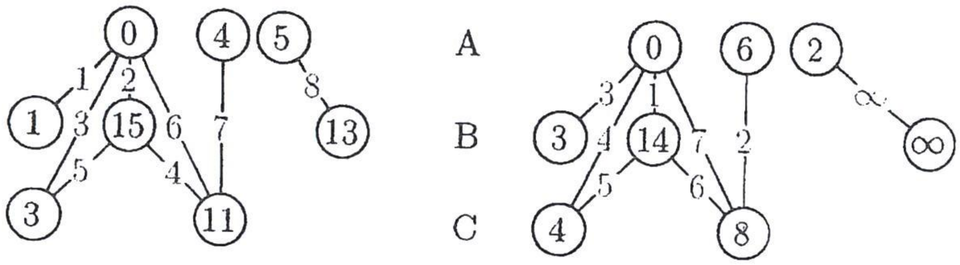


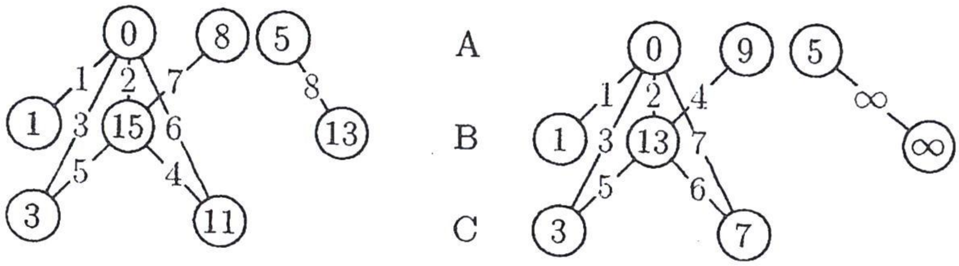
$D_1(4,2;0,1,0,0)$



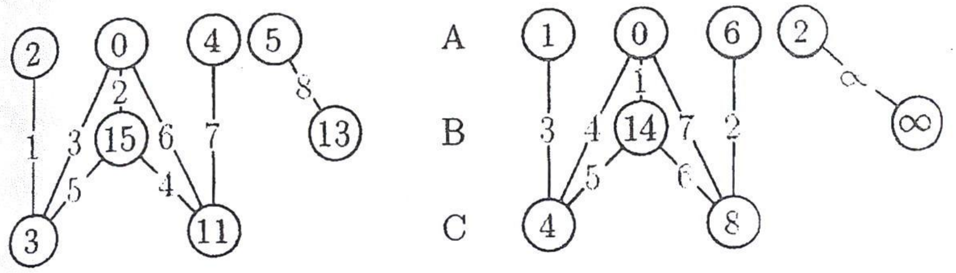
$D_2(4,2;0,1,0,0)$



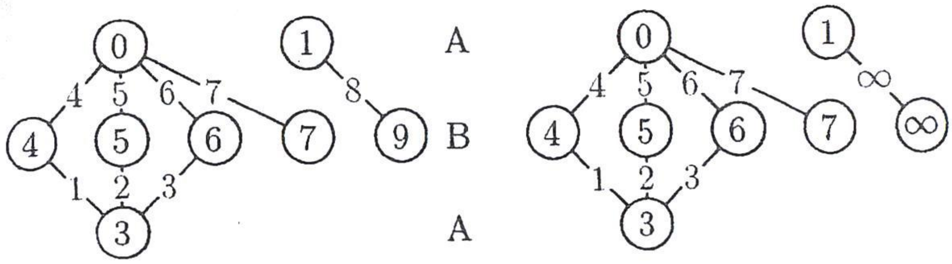
$D_1(4,2;1,1,0,0)$



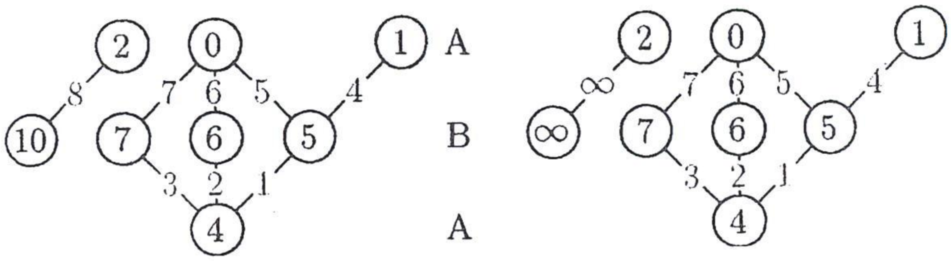
$D_1(4,2;1,0,1,0)$



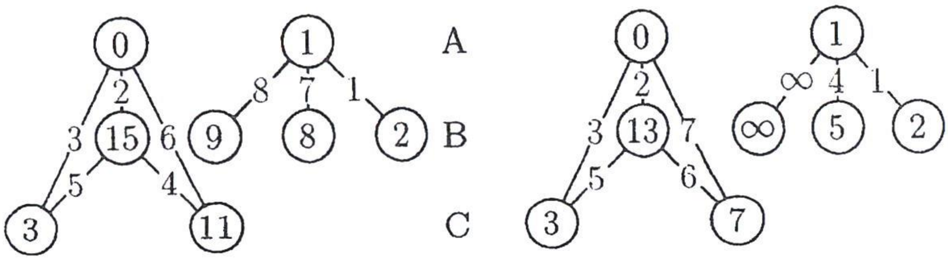
$D_1(4,2;0,1,0,1)$



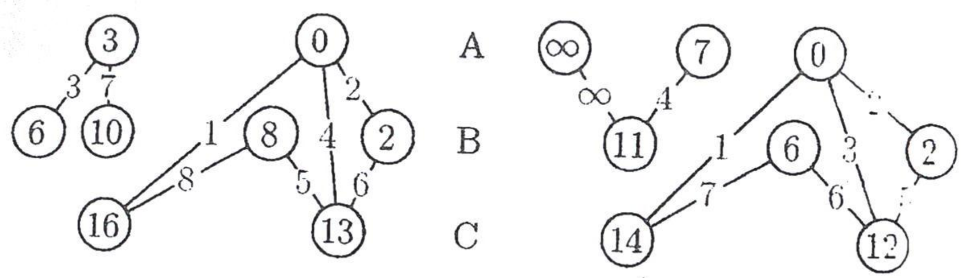
$D_1^*(4,2;1,0,0,0)$



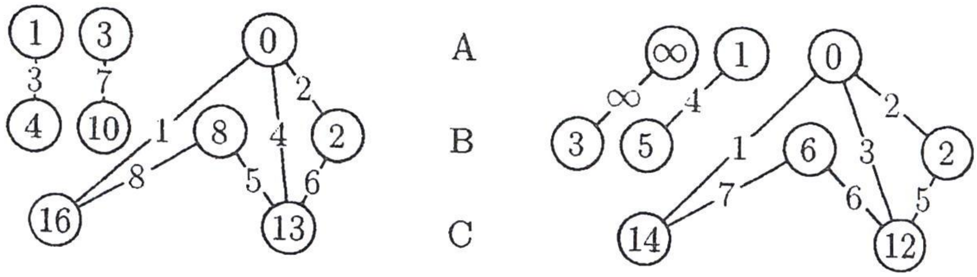
$D_1^*(4,2;0,1,0,0)$



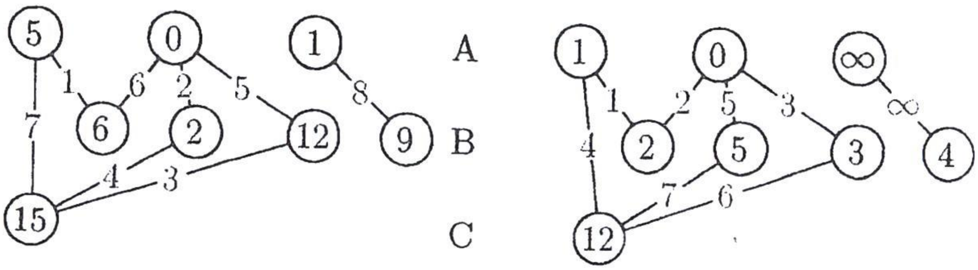
$D_1(4,2;0,0,0,0)$



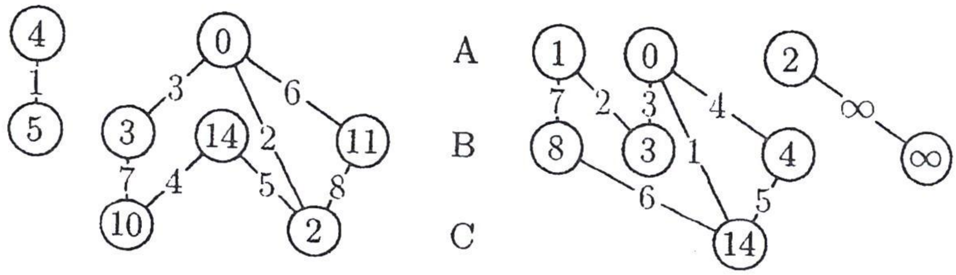
$D_1(5,2;0,0,0,0,0)$



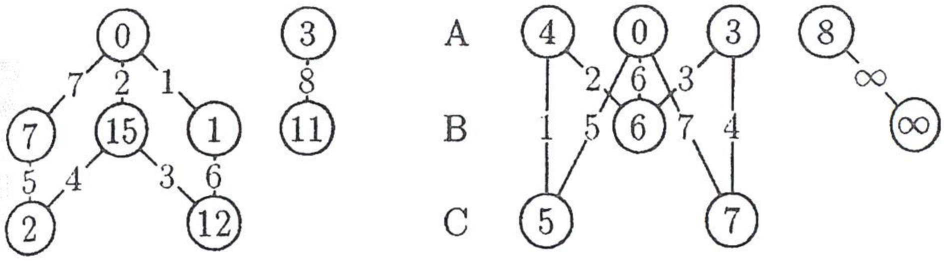
$D_2(5,2;0,0,0,0,0)$



$D_1^*(5,2;0,0,0,0,0)$



$D_1(6,2;0,0,0,0,0,0)$



$D_1(6,3;0,0,0,0,0,0)$

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