Decomposition of complete graphs into unicyclic bipartite graphs with eight edges

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Abstract

We introduce a variation of σ-labeling to prove that every disconnected unicyclic bipartite graph with eight edges decomposes the complete graph K_n whenever the necessary conditions are satisfied. We combine this result with known results in the connected case to prove that every unicyclic bipartite graph with eight edges other than C_8 decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.

1 Introduction

A *decomposition* of the complete graph K_n is a set $\mathcal{G} = \{G_1, G_2, ..., G_t\}$ of pairwise edge-disjoint subgraphs of K_n which partitions the edges of K_n. If each subgraph in $\mathcal{G}$ is isomorphic to the same graph G, then we call the decomposition a *G-decomposition* or *G-design of order n*. If we take the vertex set of K_n to be $\mathbb{Z}_n$ and the permutation $\pi : v \mapsto v + 1$ is an automorphism of the design, we say the decomposition is *cyclic*. If instead we take the vertex set of K_n to be $\mathbb{Z}_{n-1} \cup \{\infty\}$ and π is an automorphism of the design (with $\infty + 1 = \infty$ by definition), we call the decomposition *one-rotational*.

All graphs considered in this article are simple. A graph G is *unicyclic* if it contains exactly one cycle. An attempt is underway to classify the complete graphs which allow a G-decomposition where G is a unicyclic graph with eight edges (see [4], [5], [8]). In this article, we introduce a new graph labeling and apply it, along with other Rosa-type labelings, to show that every unicyclic disconnected bipartite graph with eight edges decomposes the complete graph whenever the necessary conditions are met.

2 Tools and related results

We seek to classify integers n such that a G-decomposition of K_n exists for a unicyclic bipartite graph G with eight edges. The necessary conditions are that 8 must divide $|E(K_n)| = \binom{n}{2}$ which is true whenever $n \equiv 0, 1 \pmod{16}$. The exceptional case is $G \cong C_8$. Since C_8 is 2-regular and K_n is $n-1$-regular, C_8 does not decompose K_n when $n \equiv 0 \pmod{16}$. However, Rosa used α-labelings to prove the following in [9].

Theorem 2.1. [9] *The cycle C_8 decomposes K_{16n+1} for all positive integers n.*

Froncek, along with his students and colleagues have made significant progress towards a complete classification. They proved the following theorems over a series of articles [4], [5], [8].

Theorem 2.2. [4] *Let G be a connected unicyclic bipartite graph with eight edges. If $G \ncong C_8$, then there exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

Theorem 2.3. [5] *Let G be a unicyclic graph with eight edges which contains a 3-cycle. There exists a G-decomposition K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

Theorem 2.4. [8] *Let G be a connected unicyclic graph with eight edges which contains a 5-cycle. There exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

Theorem 2.5. [7] *A bi-cyclic graph G with eight edges decomposes the complete graph K_n if and only if*

- *there is a vertex of an odd degree and $n \equiv 0, 1 \pmod{16}$, or*
- *all vertices have even degrees and $n \equiv 1 \pmod{16}$.*

Theorem 2.6. [6] *A tri-cyclic graph G with eight edges decomposes the complete graph K_n if and only if $n \equiv 0, 1 \pmod{16}$*

Rosa introduced the graph labelings in Definitions 2.7 and 2.8 as a tool to attack the problem of decomposing complete graphs in the late 1960's [9]. We will use them, along with their variations, to prove the main result.

Definition 2.7. Let G be a graph with n edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, ..., 2n\}$ inducing the *length function* $\ell : E(G) \to \{1, 2, ..., n\}$ defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\}$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = \{1, 2, ..., n\}.$$

Definition 2.8. A *σ-labeling* of a graph G is a ρ-labeling such that $\ell(uv) = |f(u) - f(v)|$.

Observing that K_{2n+1} has exactly $2n+1$ edges of each length $1, 2, ..., n$. nd the cyclic permutation $v \to v+1$ preserves edge lengths, Rosa proved the ollowing.

heorem 2.9. [9] *Let G be a graph with n edges. A cyclic decomposition of* K_{2n+1} *exists if and only if G admits a* ρ*-labeling.*

To address the problem of decomposing the complete graph K_{2nx+1} into omorphic copies of a graph with n edges, El-Zanati et al. introduced the idea f ordered labelings in [1] and [3].

efinition 2.10. A ρ- or σ-labeling of a bipartite graph G with bipartition $X, Y)$ is called an *ordered* ρ- or σ-labeling and denoted ρ^+, σ^+, respectively, if $(x) < f(y)$ for each edge xy with $x \in X$ and $y \in Y$.

efinition 2.11. A ρ^+- or σ^+-labeling of a bipartite graph G with bipartition $X, Y)$ is called a *uniformly ordered* ρ- or σ-labeling and denoted ρ^{++}, σ^{++}, espectively, if $f(x) < f(y)$ for $x \in X$ and $y \in Y$.

Notice that Definition 2.10 requires the labeling to be only locally ordered, hereas Definition 2.11 demands a global ordering of the labeled vertices. El-anati et al. used these labelings to prove the following in [3].

heorem 2.12. [3] *Let G be a graph with n edges which has a* ρ^+ *labeling. hen G decomposes* K_{2nx+1} *for all positive integers x.*

The labelings defined here can also be useful in finding isomorphic decomositions of complete graphs of even order.

heorem 2.13. [2] *Let G be a graph with n edges and a vertex v of degree 1. f* $G - v$ *has a* ρ*-labeling, then G decomposes* K_{2n}.

To extend this result to decomposing K_{2nx} for positive integers x, we introuce the following labeling which is more restrictive than σ^+ but less restrictive han σ^{++}.

efinition 2.14. A σ^{+-}*-labeling* of a bipartite graph G with n edges and ipartition (X, Y) is a σ^+ labeling with the property that $f(x) - f(y) \neq n$ for ll $x \in X$ and $y \in Y$.

heorem 2.15. *Let G be a graph with n edges and a* σ^{+-}*-labeling such that he edge of length n is a pendant edge e. Then there exists a graph* H^- *on* $cn-1$ *dges that has a* ρ*-labeling and can be decomposed into* $c-1$ *copies of G and ne copy of* $G - e$.

roof. Let G have bipartition (X, Y_0), a pendant edge $e = uv$ where the degree f v is 1, and σ^{+-}-labeling f' such that $\ell(e) = n$. Construct c isomorphic opies of G denoted $G_0, G_1, ..., G_{c-1}$, with $V(G_i)$ having bipartition (X, Y_i). Let $H = G_0 \cup G_1 \cup ... \cup G_{c-1}$ and define $f : V(H) \to \{0, 1, ..., 2cn\}$ such that

$$f(v) = \left\{ \begin{array}{ll} f'(v), & v \in X \\ f'(v) + ni, & v \in Y_i \end{array} \right\}.$$

Notice that there is no conflict with the labels since $f'(x) - f'(y) \neq n$ for all $x \in X$ and $y \in Y_0$. Also, each induced subgraph G_i of H contains lengths $\{in+1, in+2, ..., (i+1)n\}$, so H has exactly one edge of each of the lengths $\{1, 2, ..., cn\}$. Now remove v from $V(G_{c-1})$ and call the resulting graph H^-. Notice that this removes the edge e of length cn from H, leaving the graph H^- with exactly one edge of each length $1, 2, ..., cn-1$. Therefore, f is a ρ-labeling of H^-. The fact that H^- may be decomposed into $c-1$ copies of G and one copy of $G-e$ is clear by reversing the construction of H^-. □

Theorem 2.16. *Let G be a graph with n edges and a σ^{+-}-labeling such that the edge of length n is a pendant edge e. Then there exists a cyclic G-decomposition of K_{2nx} for every positive integer x.*

Proof. By Theorem 2.15, there exists a ρ-labeling of a graph H^- which decomposes into $x-1$ copies of G and one copy of $G-e$ where $e = uv$ is a pendant edge of G. Let H be the graph obtained by adding the edge $e = uv$ to H^-. By Theorem 2.13, there exists an H-decomposition of K_{2nx}. Since G decomposes H, we have proved the theorem. □

Notice that the proof technique of Theorem 2.15 does not necessarily extend to a σ^+-labeling of the graph G. However, the theorem does of course apply to a σ^{++}-labeling of G since a σ^{++}-labeling is a σ^{+-}-labeling. The next theorem provides the motivation for σ^{+-}-labeling.

Theorem 2.17. *If G is the vertex-disjoint union of C_4 and four isolated edges, then G does not admit a σ^{++}-labeling.*

Proof. Let G have bipartition (X, Y) and suppose a σ^{++}-labeling $f : V \to \{0, 1, ..., 16\}$ of G exists with $f(x) < f(y)$ for all $x \in X$ and $y \in Y$. Notice that $|X| = |Y| = 6$. Since there exists an edge of length 8, it must be the case that $f(y) \geq 8$ and $f(x) < 8$ for all $x \in X$ and $y \in Y$. This implies the edge of length 1 has vertices labeled 7 and 8, which in turn implies the edge of length 2 either has vertices labeled 8 and 6, or 7 and 9. Therefore, the length 1 edge labeled $\{7, 8\}$ is on the 4-cycle.

Case 1: Suppose the vertices of the 4-cycle are labeled $\{7, 8, x, 9\}$ around the cycle. Notice that $x \in X$ and $f(x) \leq 5$, since if $f(x) = 6$, there would be two edges of length 2. Therefore, each of the remaining four isolated edges have length at least 5. This implies $x = 5$ since the 4-cycle must contain the lengths 1, 2, 3, and 4. The remaining four isolated edges x_iy_i must have the property that $f(x_i) \leq 4$ and $f(y_i) \geq 10$. But this is a contradiction, since there is no edge of length 5.

Case 2: Suppose the vertices of the 4-cycle are $\{7, 8, 6, y\}$ around the cycle. Observing that $y \in Y$ and $f(y) \geq 10$ leads to the same contradiction as in the previous case. Therefore, a σ^{+-}-labeling of G does not exist. □

Catalog of graphs

et G and H be graphs. We will use the notation $G+H$ to represent the graph 'hich is the vertex-disjoint union of G and H. For example, $P_2 + P_3$ is shown ı Figure 1.

Figure 1: $P_2 + P_3$

There are (up to isomorphism) 32 disconnected unicyclic bipartite graphs ,ith eight edges. Let $G \cong C + F$ be one of these graphs where C is the largest onnected component of G containing the cycle and F is a forest. We establish ɔur cases, one for each of the possible values of $|E(C)| \in \{4, 5, 6, 7\}$.

If $|E(C)| = 4$, then $C \cong C_4$ and F is one of the eight forests on four edges hown in Figure 2.

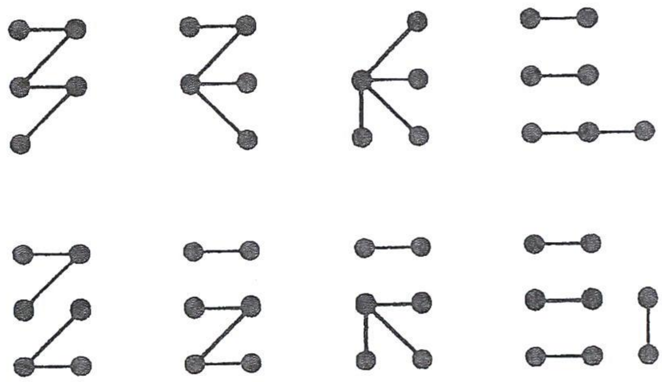
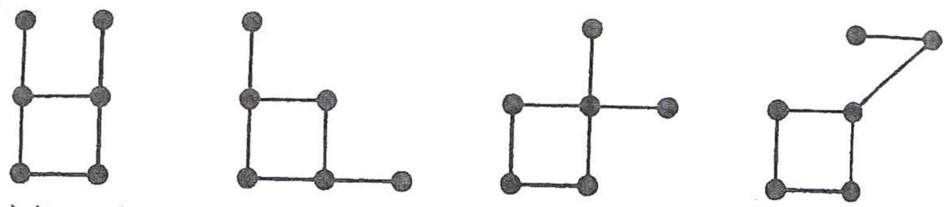


Figure 2: All the forests on four edges

Before we examine the remaining cases, we introduce some notation. Let C_t ɔe the cycle contained in C. We define the *type* of C by the t-tuple $(i_1, i_2, ..., i_t)$ where i_j is the number of edges in the tree attached to vertex v_j of the cycle. The non-zero entries of the t-tuple will always be non-increasing from left to ight. For example, Figure 3 shows graphs of types $(1,1,0,0)$, $(1,0,1,0)$, and $(2,0,0,0)$. Notice that there are two non-isomorphic graphs of type $(2,0,0,0)$.

If $|E(C)| = 5$, then C is the unique graph of type $(1,0,0,0)$, (the four cycle with one pendant edge) and F is congruent to one of the four graphs in the set $\{P_4, K_{1,3}, P_2 + P_3, P_2 + P_2 + P_2\}$.

If $|E(C)| = 6$, then C is either congruent to C_6 or is of type (1,1,0,0), (1,0,1,0), or (2,0,0,0). Notice that there is only one graph of each of the first three types and two graphs of type (2,0,0,0) (see Figure 3). The forest, F is either P_3 or $P_2 + P_2$. Therefore, there are 10 non-isomorphic graphs in this case.



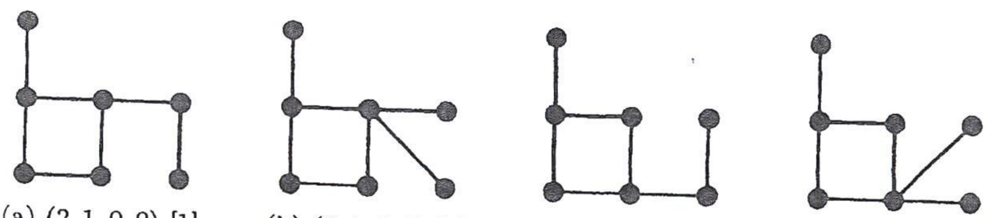
(a) (1,1,0,0) (b) (1,0,1,0) (c) (2,0,0,0) [1] (d) (2,0,0,0) [2]

Figure 3: Non-isomorphic graphs C such that $|E(C)| = 6$

If $|E(C)| = 7$, then C is type (1,1,1,0), (2,1,0,0), (2,0,1,0), (3,0,0,0), or (1,0,0,0,0,0). Notice that there is only one graph of each type (1,1,1,0) and (1,0,0,0,0,0) (see Figure 4); two graphs of each type (2,1,0,0) and (2,0,1,0) (see Figure 5); and four graphs of type (3,0,0,0) (see Figure 6). Since the forest F is the edge P_2, we count 10 non-isomorphic graphs in this case.

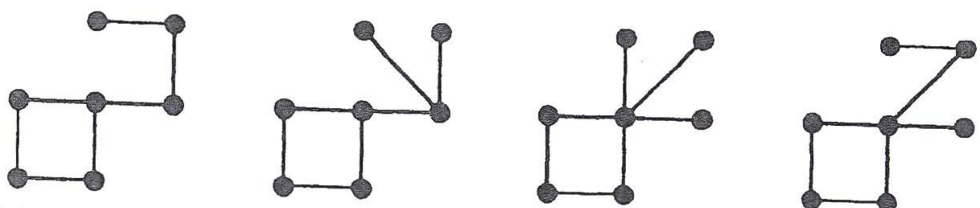
(a) $(1,1,1,0)$ (b) $(1,0,0,0,0,0)$

Figure 4: $|E(C)| = 7$; C type (1,1,1,0) or (1,0,0,0,0,0)



(a) $(2,1,0,0)$ [1] (b) $(2,1,0,0)$ [2] (c) $(2,0,1,0)$ [1] (d) $(2,0,1,0)$ [2]

Figure 5: $|E(C)| = 7$; C type $(2,1,0,0)$ or $(2,0,1,0)$



(a) $(3,0,0,0)$ [1] (b) $(3,0,0,0)$ [2] (c) $(3,0,0,0)$ [3] (d) $(3,0,0,0)$ [4]

Figure 6: $|E(C)| = 7$; C type $(3,0,0,0)$

4 Labelings

If $|E(C)| = 4$, then $C \cong C_4$. Apply the labels $\{0, 2, 1, 4\}$ consecutively around the cycle. This uses lengths $1, 2, 3$, and 4. Then we label F as shown in Figure 7, which induces edge lengths $5, 6, 7$, and 8, completing the desired σ^{+-}-labeling of $G \cong C + F$.

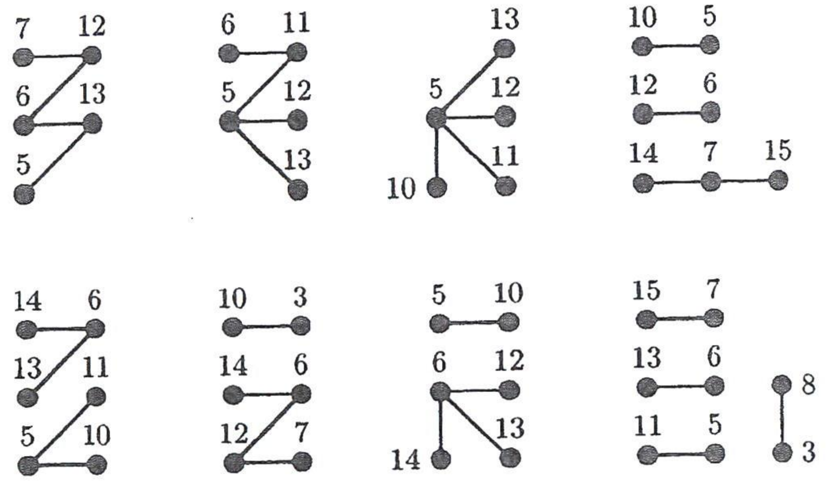
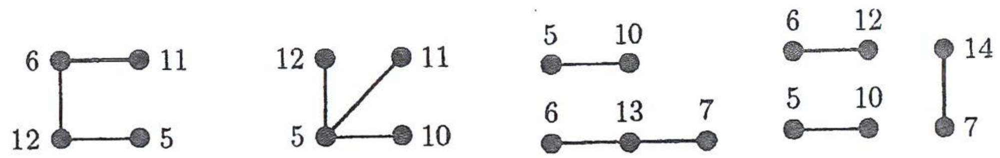


Figure 7: Labels of F when $|E(C)| = 4$

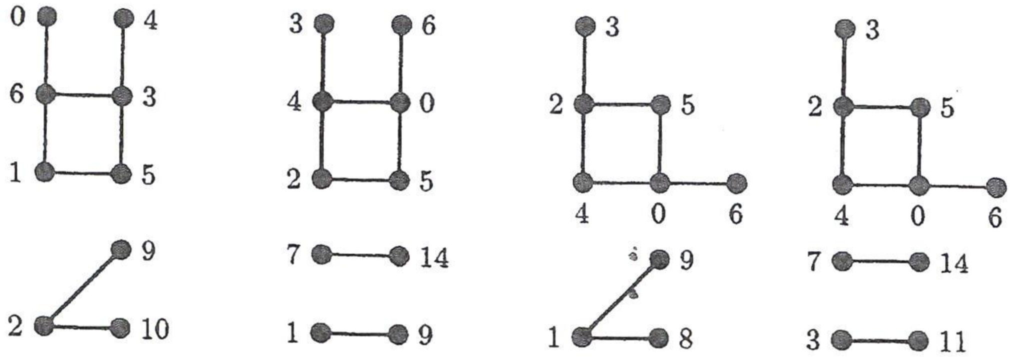
If $|E(C)| = 5$, then C is the unique graph of type $(1, 0, 0, 0)$. We apply the labels $\{0, 2, 1, 4\}$ consecutively around the 4-cycle so that the vertex adjacent to the vertex of degree 1 is labeled 0 and the vertex of degree 1 receives the label 8. This uses lengths $1, 2, 3, 4$ (on the cycle), and 8 (on the pendant edge). Then we label F as shown in Figure 8, which induces edge lengths $5, 6$ and 7, completing the desired σ^{+-}-labeling of $G \cong C + F$.



(a) P_4 (b) $K_{1,3}$ (c) $P_2 + P_3$ (d) $P_2 + P_2 + P_2$

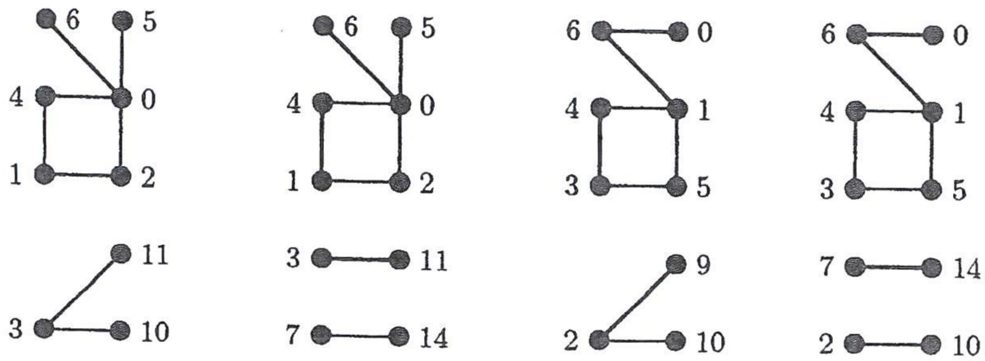
Figure 8: Labels of F when $|E(C)| = 5$

For the case $|E(C)| = 6$ or 7, Figures 9 through 14 show a σ^{+-}-labeling for each of the 20 non-isomorphic graphs $G \cong C + F$.



(a) (1, 1, 0, 0) (b) (1, 1, 0, 0) (c) (1, 0, 1, 0) (d) (1, 0, 1, 0)

Figure 9: $|E(C)| = 6$; C type $(1,1,0,0)$ or $(1,0,1,0)$



(a) (2, 0, 0, 0) [1] (b) (2, 0, 0, 0) [1] (c) (2, 0, 0, 0) [2] (d) (2, 0, 0, 0) [2]

Figure 10: $|E(C)| = 6$; C type $(2,0,0,0)$

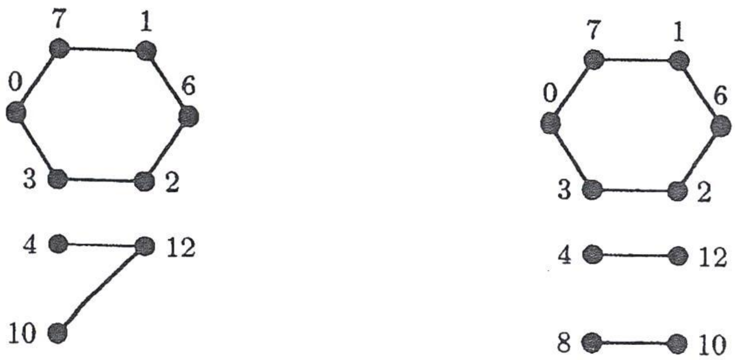
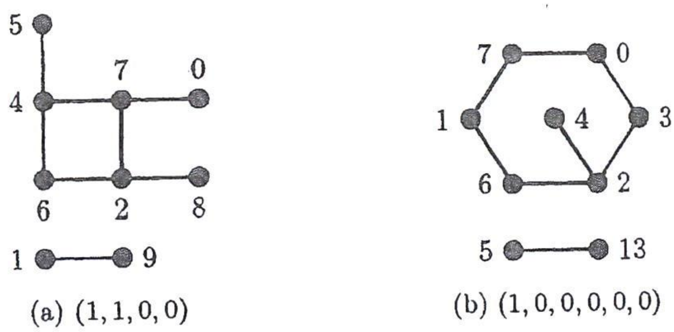
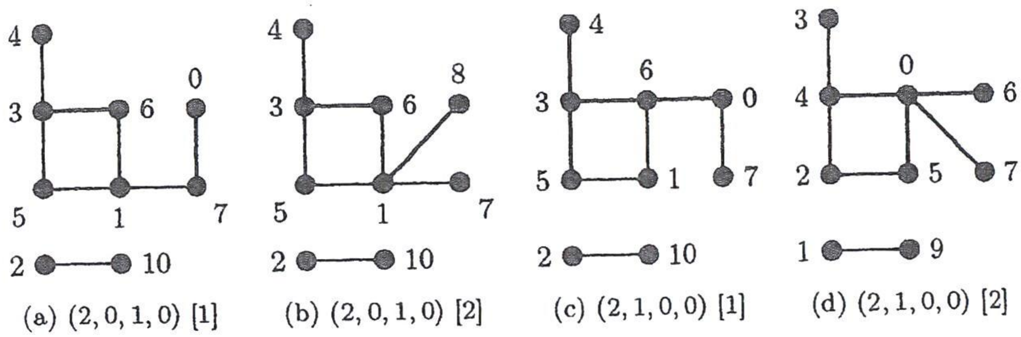


Figure 11: $|E(C)| = 6$; C type $(0,0,0,0,0,0)$



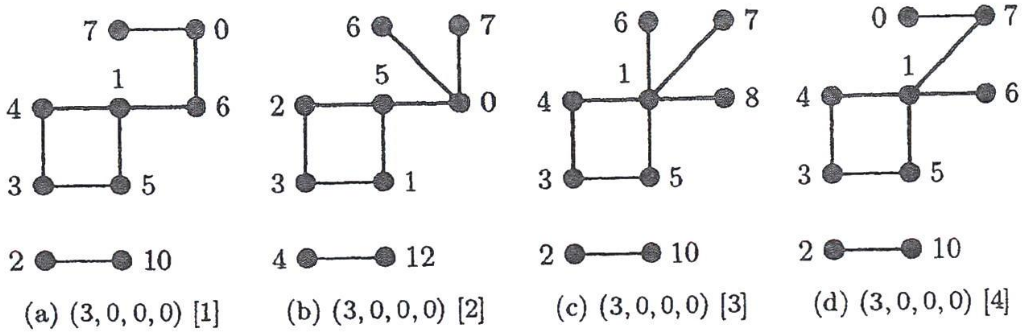
(a) $(1,1,0,0)$ (b) $(1,0,0,0,0,0)$

Figure 12: $|E(C)| = 7$; C type $(1,1,1,0)$ or $(1,0,0,0,0,0)$



(a) $(2,0,1,0)$ [1] (b) $(2,0,1,0)$ [2] (c) $(2,1,0,0)$ [1] (d) $(2,1,0,0)$ [2]

Figure 13: $|E(C)| = 7$; C type $(2,0,1,0)$ or $(2,1,0,0)$



(a) $(3,0,0,0)$ [1] (b) $(3,0,0,0)$ [2] (c) $(3,0,0,0)$ [3] (d) $(3,0,0,0)$ [4]

Figure 14: $|E(C)| = 7$; C type $(3,0,0,0)$

5 Main result

We seek to classify integers n such that a G-decomposition of K_n exists for a unicyclic bipartite graph G with eight edges. We conclude with the main theorem.

Theorem 5.1. *Let G be a bipartite unicyclic graph with eight edges which is not C_8. Then there exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

Proof. The necessary conditions are obvious since 8 divides $|E(K_n)| = \binom{n}{2}$ if and only if $n \geq 16$ and $n \equiv 0, 1 \pmod{16}$. If G is connected, we are done by Theorem 2.2. So assume from now on that G is disconnected. Then G is one of the 32 graphs cataloged in Section 3. Observe that Section 4 provides a σ^{+-}-labeling of G with the property that the edge of length 8 is a pendant edge. If $n \equiv 1 \pmod{16}$, then a G-decomposition of K_n exists by Theorem 2.12, since a σ^{+-}-labeling is a ρ^{+}-labeling. If $n \equiv 0 \pmod{16}$, then the result follows from Theorem 2.16. □

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