Decomposition of complete graphs into unicyclic graphs with eight edges

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Abstract

Let G be a tripartite unicyclic graph with eight edges that either (i) contains a triangle or heptagon, or (ii) contains a pentagon and is disconnected. We prove that G decomposes the complete graph K_n whenever the necessary conditions are satisfied. We combine this result with other known results to prove that every unicyclic graph with eight edges other than C_8 decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.

1 Introduction

A *decomposition* of a graph K is a set $\mathcal{G} = \{G_1, G_2, ..., G_t\}$ of pairwise edge-disjoint subgraphs of K that partitions the edges of K. If each subgraph in $\mathcal{G}$ is isomorphic to the same graph G, then we call the decomposition a *G-decomposition* of K. If $K \cong K_n$, we call the decomposition a *G-design of order n*. If we take the vertex set of K_n to be $\mathbb{Z}_n$ and the permutation $\pi : v \mapsto v + 1$ is an automorphism of the design, we say the decomposition is *cyclic*. If instead we take the vertex set of K_n to be $\mathbb{Z}_{n-1} \cup \{\infty\}$ and π is an automorphism of the design (with $\infty + 1 = \infty$ by definition), we call the decomposition *1-rotational*.

A graph is *unicyclic* if it contains exactly one cycle. Let G be a unicyclic graph with eight edges. If a G-decomposition of K_n exists, then 8 must divide $\binom{n}{2}$. Therefore, the necessary conditions are $n \equiv 0, 1 \pmod{16}$. The authors and their collaborators have shown these conditions to be sufficient when G is bipartite [6], [7], or contains a pentagon and is connected [8]. In this article, we use ρ-tripartite and 1-rotational ρ-tripartite labelings, recently introduced by Bunge et al. in [1] and [2], to find G-decompositions for the remaining cases, completely classifying the complete graphs that allow a G-decomposition.

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2 Related results

Let G be a unicyclic graph with eight edges. The case $G \cong C_8$ is exceptional, so we consider it first. Since C_8 is 2-regular, it cannot decompose K_n for $n \equiv 0 \pmod{16}$, since K_n is odd-regular in that case. This leaves $n \equiv 1 \pmod{16}$ as the only necessary condition. Rosa proved this condition is sufficient in [9].

Theorem 2.1. [9] *The cycle C_8 decomposes K_n if and only if $n \equiv 1 \pmod{16}$.*

The next two theorems, along with the previous theorem, completely settle the case in which G is bipartite.

Theorem 2.2. [6] *Let G be a connected unicyclic bipartite graph with eight edges other than C_8. The graph G decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

Theorem 2.3. [7] *Let G be a disconnected unicyclic bipartite graph with eight edges. The graph G decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

If G is connected and contains a C_5, Froncek and Kingston proved the following in [8].

Theorem 2.4. [8] *Let G be a connected unicyclic graph with eight edges which contains a C_5. The graph G decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

The cases which remain are when G contains a C_3, contains a C_7, or contains a C_5 and is disconnected. We settle each of these cases in separate sections in this paper.

3 Tools

Rosa introduced a number of graph labelings (he called them valuations) as a means to decompose complete graphs. The next two definitions were given in [9].

Definition 3.1. Let G be a graph with n edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, ..., 2n\}$ inducing the *length function* $\ell : E(G) \to \{1, 2, ..., n\}$ defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\}$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = \{1, 2, ..., n\}.$$

A more restrictive ρ-labeling reserved for bipartite graphs is α-labeling.

Definition 3.2. Let G be a bipartite graph with n edges and vertex bipartition (X, Y). A ρ-labeling f of G is an *α-labeling* if there exists an integer λ such that $f(x) \leq \lambda < f(y) \leq n$ for every edge xy with $x \in X$ and $y \in Y$.

Lemma 3.3. *Every tree T on $3 \leq m \leq 6$ vertices with vertex bipartition (X, Y) can be labeled with labels from the set $\{0, 1, \ldots, m\}$ so that the resulting edge lengths are $1, 3, 4, \ldots, m$ and $f(x) < f(y)$ for every $x \in X$ and $y \in Y$.*

Proof. One can easily check that the only tree on seven or fewer vertices that is not a caterpillar is the star-like tree S shown in Figure 1.

Figure 1: S

We pick a diametrical path $v_0, v_1, \ldots, v_d$ with $2 \leq d \leq 5$ in T and assume, without loss of generality, that $v_0 \in X$. We denote $v_0 = x_1$ and $v_1 = y_1$. Now we create a new tree T' on seven vertices by adding a new vertex x_1' to the partite set $X' = X \cup \{x_1'\}$ and join it to y_1. Obviously, $T' \ncong S$, since T' has two vertices of degree one at distance two, while S does not. It is well known that every caterpillar has an α-labeling g such that one of the terminal edges of the diametrical path, v_0v_1 or $v_{d-1}v_d$, obtains labels λ and $\lambda+1$ [9]. We can assume without losing generality that it is v_0v_1. It is also well known that the labeling can be chosen so that v_0 is labeled λ and v_1 is labeled $\lambda + 1$. For the reader's convenience, we sketch a proof of this claim below, constructing a labeling f' with the required property.

If $g(v_1) = \lambda + 1$, we define $f'(v) = g(v)$ for every vertex v in T' and set $\lambda' = \lambda$. Thus $f'(v_0) = f'(x_1) = \lambda'$ and $f'(v_1) = f'(y_1) = \lambda' + 1$.

If $g(v_1) = \lambda$, we define $f'(v) = m - g(v)$ and observe that f' is also an α-labeling with $\lambda' = m - \lambda$. Hence again $f'(v_0) = f'(x_1) = \lambda'$ and $f'(v_1) = f'(y_1) = \lambda' + 1$. (In this case, the labeling f' is known as the *complementary* labeling to the labeling g.)

It follows from Rosa's construction that the remaining neighbors of y_1 of degree one are labeled $\lambda' - 1, \lambda' - 2, \ldots$ and we can set $f'(x_1') = \lambda' - 1$. Then the edge $x_1'y_1$ has length 2.

Removing vertex x_1' from T', we obtain T with the desired labeling f defined as $f(x_i) = f'(x_i)$ and $f(y_j) = f'(y_j)$ for all vertices $x_i, y_j \in T$. □

We will call the labeling described in Lemma 3.3 a *2-gap α-labeling*.

As a tool for decomposing complete graphs into tripartite graphs, Bunge et al. introduced the following labeling in [2].

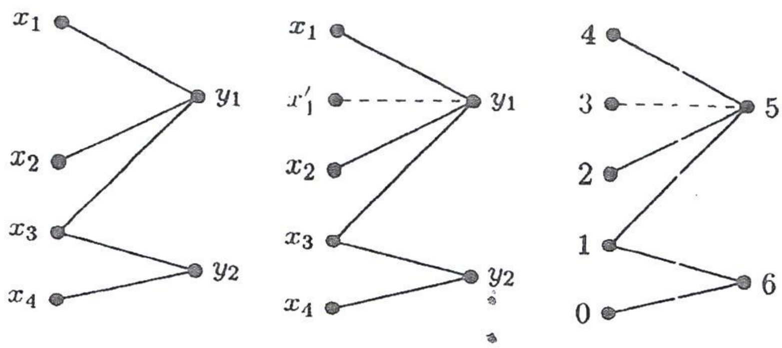


Figure 2: 2-gap α-labeling

Definition 3.4. Let G be a tripartite graph with vertex tripartition (A, B, C) and n edges. A *ρ-tripartite labeling* of G is a ρ-labeling f with the additional properties:

- $f(a) < f(x)$ for every edge ax where $a \in A$.
- For every edge bc with $b \in B$ and $c \in C$, there exists an edge $b'c'$ with $b' \in B$ and $c' \in C$ such that $|f(b) - f(c)| + |f(b') - f(c')| = 2n$.
- For every $b \in B$ and $c \in C$, $|f(b) - f(c)| \neq 2n$.

In Section 6, we refer to the second property in 3.4 as the *complement* property of ρ-tripartite labeling. They went on to prove the following.

Theorem 3.5. [2] *Let G be a tripartite graph with n edges. If G admits a ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nk+1} for every positive integer k.*

To decompose complete graphs of even order into tripartite graphs, Bunge introduced the following variation of ρ-tripartite labeling in [1].

Definition 3.6. Let G be a tripartite graph with n edges, vertex tripartition (A, B, C), and an edge uv where the degree of v is 1. A *1-rotational ρ-tripartite labeling* f of G is an injection $f : V(G) \to \{0, 1, ..., 2n-2, \infty\}$ inducing the *length function* $\ell : E(G) \to \{1, 2, ..., n-1, \infty\}$ defined as

$$\ell(xy) = \begin{cases} \min\{|f(x) - f(y)|, 2n - 1 - |f(x) - f(y)|\} & \text{if } x, y \in V(G) \setminus \{\infty\}; \\ \infty & \text{otherwise.} \end{cases}$$

with the additional properties:

- $\{\ell(xy) : xy \in E(G)\} = \{1, 2, ..., n-1, \infty\}$.
- $f(v) = \infty$.
- $f(a) < f(x)$ for every edge $ax \in E(G) \setminus \{uv\}$ where $a \in A$.
- For every edge bc with $b \in B$ and $c \in C$, there exists an edge $b'c'$ with $b' \in B$ and $c' \in C$ such that $|f(b) - f(c)| + |f(b') - f(c')| = 2n$.

Theorem 3.7. [1] *Let G be a tripartite graph with n edges. If G admits a 1-rotational ρ-tripartite labeling, then there exists a 1-rotational G-decomposition of K_{2nk} for every positive integer k.*

When we give a ρ-tripartite or a 1-rotational ρ-tripartite labeling of a graph G with vertex tripartition (A, B, C) in a figure, we will place the vertices belonging to A in the left-most column of the figure, the vertices belonging to B in the next column to the right, and the vertices belonging to C in the third column (if C is non-empty). In this way, we avoid unnecessary notation and clutter, although this restriction in drawing may lead to a less natural rendition of the graph.

To simplify the cataloging of unicyclic graphs, we use the following notation. Let G be a unicyclic graph containing the cycle C_n. We say that G is of *type* $(i_1, i_2, ..., i_n)$ if a tree containing i_j edges is attached to vertex j of the cycle. We will distinguish between non-isomorphic graphs of the same type by appending a parameter t in brackets after the n-tuple.

4 Connected graphs containing a triangle

In this section, we assume G is a connected unicyclic graph with eight edges containing a C_3. We call such a graph a *unicyclic triangular graph.* In Subsection 4.1, we find a ρ-tripartite labeling of G, proving that a G-decomposition of K_n exists for every $n \equiv 1 \pmod{16}$. In Subsection 4.2, we find a 1-rotational ρ-tripartite labeling of G, proving that a G-decomposition of K_n exists for every $n \equiv 0 \pmod{16}$.

4.1 $n \equiv 1 \pmod{16}$

Lemma 4.1. *Every unicyclic triangular graph G on eight vertices of type* $(5, 0, 0)$ *has a ρ-tripartite labeling.*

Proof. Label the vertices of the triangle of G with $0, 7$, and 15, and set $0 \in A$, $7 \in B$, and $15 \in C$. We have $c = c' = 15$, $b = b' = 7$, so $|15-7|+|15-7| = 2 \cdot 8$, and the edge lengths used in the triangle are 2, 7, and 8. Let T be the tree with vertex bipartition (X, Y) attached to the triangle at vertex y_0. Without loss of generality, we may assume that $y_0 \in Y$. Find a 2-gap α-labeling f of T such that $f(y_0) > \lambda$. Such a labeling exists by Lemma 3.3. Now increase all labels in T by $7 - f(y_0)$. This preserves all edge lengths, namely $1, 3, 4, 5, 6$, and since $1 \leq f(y_0) \leq 6$, the labels of T do not conflict with the labels of the triangle. By placing $X \subset A$ and $Y \subset B$, we obtain the desired ρ-tripartite labeling of G. □

Lemma 4.2. *Every unicyclic triangular graph G on eight vertices of type* $(4, 1, 0)$ *has a ρ-tripartite labeling.*

Proof. Label the vertices of the triangle of G with $0, 7$, and 15, and set $0 \in A$, $7 \in B$, and $15 \in C$. We have $c = c' = 15$, $b = b' = 7$, so $|15-7|+|15-7| = 2 \cdot 8$, and the edge lengths used in the triangle are 2, 7, and 8. Join vertices 0 and 1 to obtain an edge of length 1 and place $16 \in B$.

Let T be the tree on five vertices with vertex bipartition (X, Y) attached t the triangle at vertex $y_0 \in Y$. Find an α-labeling f of T such that $f(y_0) > \lambda$ Now increase all labels in Y by $7 - f(y_0)$ and in X by $5 - f(y_0)$. This provide edge lengths 3, 4, 5, and 6, and by placing $X \subset A$ and $Y \subset B$, we obtain th desired ρ-tripartite labeling of G. □

Lemma 4.3. *Every unicyclic triangular graph G on eight vertices of type* (3, 2, 0 *has a ρ-tripartite labeling.*

Proof. Label the vertices of the triangle of G with 0, 7, and 15, and set $0 \in A$ $7 \in B$, and $15 \in C$. We have $c = c' = 15$, $b = b' = 7$, so $|15 - 7| + |15 - 7| = 2 \cdot 8$ and the edge lengths used in the triangle are 2, 7, and 8.

Let R be the tree with two edges, and T the tree with three edges. Then is the path r_0, r_1, r_2. First identify r_0 with 0 and label r_1 with 12 and r_2 wit 1, and set $r_2 \in A$ and $r_1 \in B$. In this way we obtain edges of lengths 5 and 6.

Now T with vertex bipartition (X, Y) is attached to the triangle at verte 7 which we identify with $y_0 \in Y$. We find a 2-gap α-labeling f of T suc that $f(y_0) > \lambda$. Now we increase all labels in T by $7 - f(y_0)$. This provide edge lengths 1, 3, 4 and by placing $X \subset A$ and $Y \subset B$, we obtain the desire ρ-tripartite labeling of G. Notice that the lowest label used in T is at least and the highest label used in T does not exceed 10, so we do not have a conflic between vertex labels of R and T.

Second, identify r_1 with 0 and label r_0 by 16 and r_2 by 14 and place $r_0, r_2 \in$ B. This way we obtain edges of lengths 1 and 3. Again, T is the tree on fou vertices with vertex bipartition (X, Y) attached to the triangle at vertex 7 whic we identify with $y_0 \in Y$. We find an α-labeling f of T such that $f(y_0) > \lambda$ an increase all labels in Y by $7 - f(y_0)$ and all labels in X by $4 - f(y_0)$. Thi provides edge lengths 4, 5, 6 and by placing $X \subset A$ and $Y \subset B$, we obtain th desired ρ-tripartite labeling of G. □

Lemma 4.4. *Every unicyclic triangular graph G on eight vertices of type* (3, 1, 1 *has a ρ-tripartite labeling.*

Proof. Label the vertices of the triangle of G with 0, 7, 15 and set $0 \in A$, $7 \in B$, and $15 \in C$. We have $c = c' = 15$, $b = b' = 7$, and the edge lengths used in the triangle are 2, 7, and 8. Join vertices 0 and 16 to obtain an edge of length 1 and place $16 \in B$. Also, join vertices 15 and 12 to obtain an edge of length 3 and place $12 \in A$.

Let T be the tree on four vertices with vertex bipartition (X, Y) attached to the triangle at vertex $y_0 \in Y$. Find an α-labeling f of T such that $f(y_0) > \lambda$. Now increase all labels in Y by $7 - f(y_0)$ and in X by $4 - f(y_0)$. This provides edge lengths 4, 5, and 6, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired ρ-tripartite labeling of G. □

Lemma 4.5. *Every unicyclic triangular graph G on eight vertices of type* (2, 2, 1 *has a ρ-tripartite labeling.*

Proof. Label the vertices of the triangle of G with $0, 7, 15$ and set $0 \in A$, $7 \in B$, and $15 \in C$. We have $c = c' = 15$, $b = b' = 7$, and the edge lengths used in the triangle are $2, 7$, and 8. Join vertices 0 and 16 to obtain an edge of length 1 and place $16 \in B$.

Let R and T be trees with two edges. Let T be the tree on three vertices with vertex bipartition (X, Y) attached to the triangle at vertex 7 which we identify with $y_0 \in Y$. We find an α-labeling f of T such that $f(y_0) > \lambda$ and increase all labels in Y by $7 - f(y_0)$ and increase all labels in X by $5 - f(y_o)$. This provides edge lengths 3 and 4.

Let R be the tree on three vertices attached to the triangle at vertex 15. Then R is the path r_0, r_1, r_2. If r_0 is identified with vertex 15, we label the path $15, 9, 14$ and place $r_1 \in A$ and $r_2 \in B$. If r_1 is identified with vertex 15, we label the path $10, 15, 9$ and place $r_0, r_2 \in A$. This produces edge lengths 5 and 6, and since the largest label used in T is at most 8, there is no conflict between the labels of T and R. By placing $X \subset A$ and $Y \subset B$, we obtain the desired ρ-tripartite labeling of G. □

4.2 $n \equiv 0 \pmod{16}$

First we choose the edge of length ∞. To minimize the number of different cases, we select it so that we reduce the cases of type $(5, 0, 0)$ or type $(4, 1, 0)$ to type $(4, 0, 0)$; type $(3, 2, 0)$ and type $(3, 1, 1)$ to type $(3, 1, 0)$; and type $(2, 2, 1)$ to type $(2, 1, 1)$.

Lemma 4.6. *Every unicyclic triangular graph G on eight vertices of type* $(5, 0, 0)$ *or type* $(4, 1, 0)$ *has a 1-rotational ρ-tripartite labeling.*

Proof. First label a vertex of degree one ∞ so that the graph reduces to type $(4, 0, 0)$. Then label the vertices of the triangle of G by $0, 6, 14$ and set $0 \in A$, $6 \in B$, and $14 \in C$. We have $c = c' = 14$, $b = b' = 6$, and the edge lengths used in the triangle are $1, 6$, and 7.

Let T be the tree on five vertices with vertex bipartition (X, Y) attached to the triangle at vertex $y_0 \in Y$. Find an α-labeling f of T such that $f(y_0) > \lambda$. Now increase all labels in in Y by $6 - f(y_0)$ and in X by $5 - f(y_0)$. This provides edge lengths $2, 3, 4$, and 5, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired 1-rotational ρ-tripartite labeling of G. □

Lemma 4.7. *Every unicyclic triangular graph G on eight vertices of type* $(3, 2, 0)$ *or type* $(3, 1, 1)$ *has a 1-rotational ρ-tripartite labeling.*

Proof. First label a vertex of degree one ∞ so that the graph reduces to type $(3, 1, 0)$. Then label the vertices of the triangle of G by $0, 6, 14$ and set $0 \in A$, $6 \in B$, and $14 \in C$. We have $c = c' = 14$, $b = b' = 6$, and the edge lengths used in the triangle are $1, 6$, and 7. Join vertex 0 to 13 to obtain an edge of length 2 and place $13 \in B$.

Let T be the tree on four vertices with vertex bipartition (X, Y) attached to the triangle at vertex $y_0 \in Y$. Find an α-labeling f of T such that $f(y_0) > \lambda$.

Now increase all labels in Y by $6 - f(y_0)$ and in X by $4 - f(y_0)$. This provides edge lengths $3, 4$, and 5, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired 1-rotational ρ-tripartite labeling of G. □

Lemma 4.8. *Every unicyclic triangular graph G on eight vertices of type* $(2, 2, 1)$ *has a 1-rotational ρ-tripartite labeling.*

Proof. First label a vertex of degree one ∞ so that the graph reduces to type $(2, 1, 1)$. Then label the vertices of the triangle of G by $0, 6, 14$ and set $0 \in A$, $6 \in B$, and $14 \in C$. We have $c = c' = 14$, $b = b' = 6$, and the edge lengths used in the triangle are $1, 6$, and 7. Join vertex 0 to 13 to obtain an edge of length 2, and 14 to 11 to obtain an edge of length 3 and place $11 \in A$ and $13 \in B$.

Let T be the tree on three vertices with vertex bipartition (X, Y) attached to the triangle at vertex $y_0 \in Y$. Find an α-labeling f of T such that $f(y_0) > \lambda$. Now increase all labels in Y by $6 - f(y_0)$ and in X by $3 - f(y_0)$. This provides edge lengths 4 and 5, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired 1-rotational ρ-tripartite labeling of G. □

Theorem 4.9. *Let G be a connected unicyclic graph with eight edges that contains a C_3. The graph G decomposes K_n if and only if $n \equiv 0, 1$* (mod 16).

Proof. The proof is by Lemmas 4.1 through 4.8 and Theorems 3.5 and 3.7. □

5 Disconnected graphs containing a triangle

Let G be a disconnected unicyclic triangular graph on eight edges. We may write $G \cong H_3 + F$ where H_3 is the component containing a C_3 and F is a forest. From here on, "+" denotes the vertex-disjoint union of two graphs. We categorize the graphs in this section by the number of edges in H.

5.1 $n \equiv 1 \pmod{16}$

We aim to show that G has a ρ-tripartite labeling.

Case 1. $|E(H_3)| = 3$.

We have $H_3 \cong C_3$ and F is a forest with 5 edges. We may write $F \cong T_1 + T_2 + \ldots + T_k$ where T_i is a tree and the trees are indexed in non-decreasing order by size. Apply the labels $0, 7$, and 15 to the vertices of the triangle, and set $0 \in A$, $7 \in B$, and $15 \in C$. We have $c = c' = 15$, $b = b' = 7$, and the edge lengths used in the triangle are $2, 7$ and 8. We establish five subcases based on k.

Subcase 1.1. $k = 1$.

Let T_1 be the tree with vertex bipartition (X, Y). Find a 2-gap α-labeling f of T_1 such that $f(y) > \lambda$ with $y \in Y$. Now increase all labels in T_1 by 8. This preserves all edge lengths, namely $1, 3, 4, 5$, and 6, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired ρ-tripartite labeling of G.

Subcase 1.2. $k = 2$.

Suppose $T_1 \cong P_2$ and T_2 is a tree with four edges. Give to the vertices of T_1 the labels 8 and 14 and set $8 \in A$ and $14 \in B$. This edge has length 6. Let T_2 be the tree with four edges with vertex bipartition (X, Y). Find a 2-gap α-labeling f of T_2 such that $f(y) > \lambda$. Now increase all labels in T_2 by 1. This preserves all edge lengths, namely $1, 3, 4$, and 5, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired ρ-tripartite labeling of G.

On the other hand, suppose $T_1 \cong P_3$. Label the vertices of the path $11, 6, 12$, consecutively, setting $6 \in A$ and $\{11, 12\} \subset B$. This induces lengths 5 and 6. Let T_2 be the tree with three edges with vertex bipartition (X, Y). Find a 2-gap α-labeling f of T_2 such that $f(y) > \lambda$ for $y \in Y$. Now increase all labels in T_2 by 1. This preserves all edge lengths, namely $1, 3$, and 4, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired ρ-tripartite labeling of G.

Subcase 1.3. $k = 3$.

If $T_1 \cong T_2 \cong P_2$, label the vertices of T_1 with 6 and 11 and label the vertices of T_2 with 8 and 14, setting $\{6, 8\} \subset A$ and $\{11, 14\} \subset B$. This induces lengths 5 and 6. Let T_3 be the tree with three edges with vertex bipartition (X, Y). Find a 2-gap α-labeling f of T_3 such that $f(y) > \lambda$ for $y \in Y$. Now increase all labels in T_3 by 1. This preserves all edge lengths, namely $1, 3$, and 4, and by placing $X \subset A$ and $Y \subset B$, we obtain the desired ρ-tripartite labeling of G.

If rather $T_1 \cong P_2$ and $T_2 \cong T_3 \cong P_3$, label the vertices of T_1 with 5 and 16, setting $5 \in A$ and $16 \in B$. The length of this edge is 6. Consecutively label the vertices along each path with $1, 4, 3$, and $9, 13, 8$, respectively. This induces lengths $1, 3, 4$, and 5. By placing $\{1, 3, 8, 9\} \subset A$, and $\{4, 13\} \subset B$, we obtain the desired ρ-tripartite labeling of G.

Subcase 1.4. $k = 4$.

We have $F \cong P_2 + P_2 + P_2 + P_3$. Label the vertices of the three isolated edges 1 and 4; 2 and 6; 3 and 8, respectively, setting the smaller number of each pair in A and the larger number in each pair in B. This induces lengths $3, 4$, and 5. Then label the vertices of P_3 sequentially 11, 10 and 16, setting $10 \in A$ and $\{11, 16\} \subset B$. This induces lengths 1 and 6, so we have described a ρ-tripartite labeling of G.

Subcase 1.5. $k = 5$.

In this case $F \cong P_2 + P_2 + P_2 + P_2 + P_2$. Label the vertices of the five isolated edges 1 and 4; 2 and 6; 3 and 8; 12 and 13; and 5 and 11, respectively, setting the smaller number in each pair in A and the larger number in each pair in B. This induces lengths $3, 4, 5, 1$, and 6, respectively, so we have obtained a ρ-tripartite labeling of G.

Case 2. $|E(H_3)| = 4$.

If $|E(H_3)| = 4$, then H_3 is congruent to C_3 with a hanging edge. Apply the labels $0, 7, 15$ to the vertices of the cycle so that the vertex of degree 3 receives the label 0. Apply the label 16 to the vertex of degree 1. Place $0 \in A$, $15 \in C$, and $\{7, 16\} \subset B$. We have $b = b' = 7$, $c = c' = 15$, and lengths $2, 7, 8$, (on the cycle) and 1 (on the hanging edge). Then we label F as shown in Figure 3, which induces edge lengths $3, 4, 5$, and 6, completing the desired ρ-tripartite labeling of $G \cong H_3 + F$.

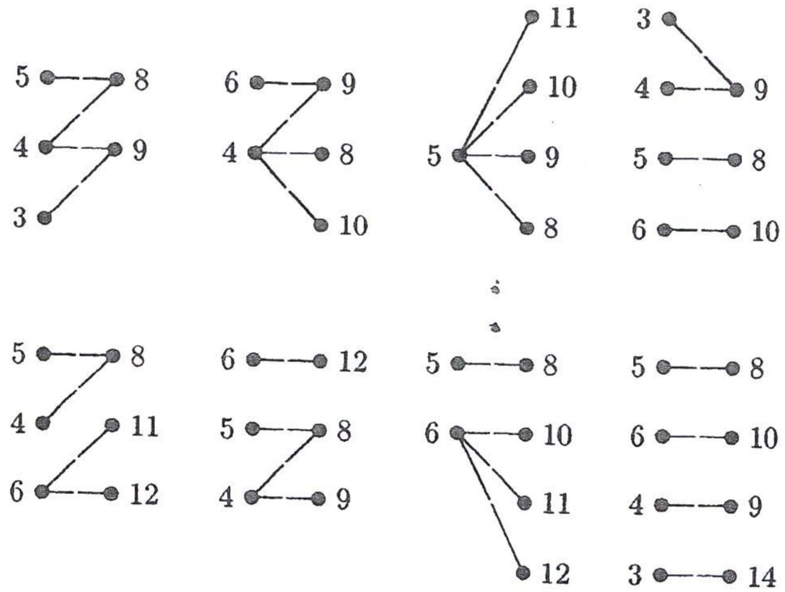
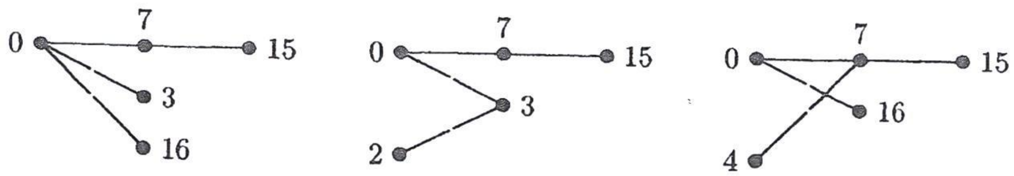


Figure 3: Labels of F when $|E(H_3)| = 4$

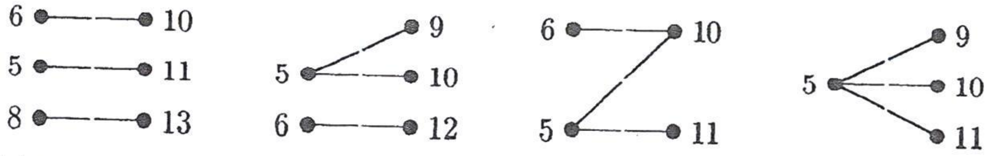
Case 3. $|E(H_3)| = 5$.

If $|E(H_3)| = 5$, then H_3 is either type $(2,0,0)$ or $(1,1,0)$. Label H_3 as shown in Figure 4. We have $b = b' = 7$, $c = c' = 15$, and the edge lengths used are $1, 2, 3, 7$, and 8. Then label F as shown in Figure 5 which induces lengths $4, 5$, and 6, giving a ρ-tripartite labeling of $G \cong H_3 + F$.



(a) $(2,0,0)$ [1]　(b) $(2,0,0)$ [2]　(c) $(1,1,0)$

Figure 4: $|E(H_3)| = 5$

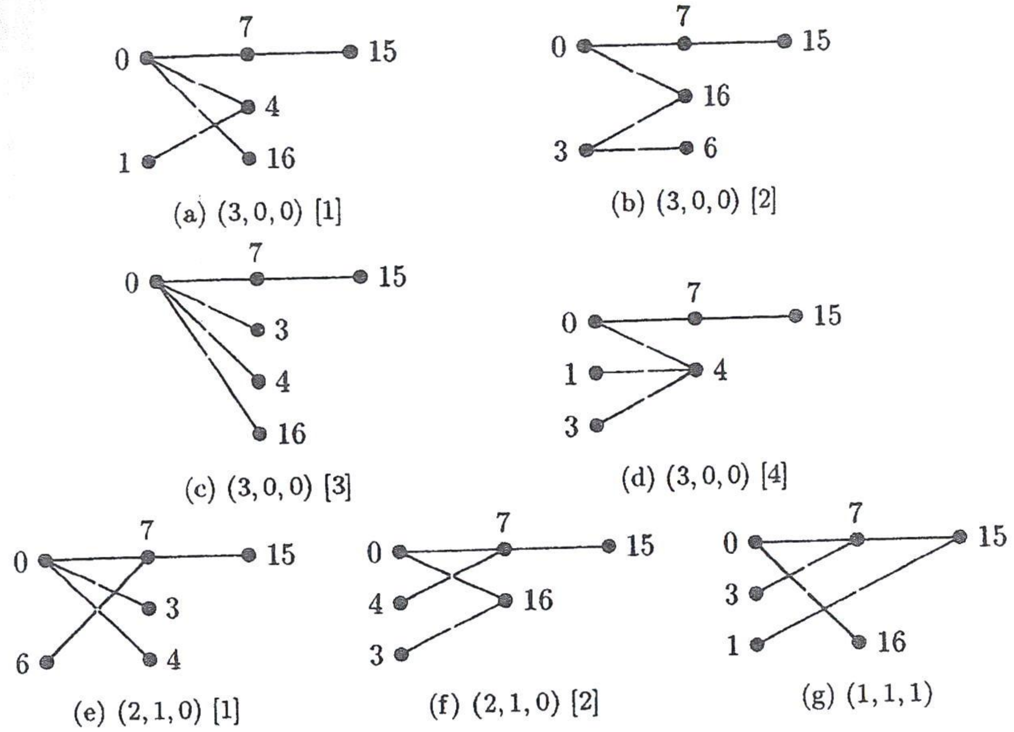


(a) $P_2 + P_2 + P_2$　(b) $P_2 + P_3$　(c) P_4　(d) $K_{1,3}$

Figure 5: All forests with 3 edges

Case 4. $|E(H_3)| = 6$.

If $|E(H_3)| = 6$, then H_3 is either of type $(3,0,0)$, type $(2,1,0)$, or type $(1,1,1)$. Label H_3 as shown in Figure 6. We have $b = b' = 7$, $c = c' = 15$, and the edge lengths used are $1, 2, 3, 4, 7$, and 8. Then label F, which is either P_3 or P_2+P_2 as shown in Figure 7 which induces lengths 5 and 6, giving a ρ-tripartite labeling of $G \cong H_3 + F$.



(a) $(3,0,0)$ [1] (b) $(3,0,0)$ [2] (c) $(3,0,0)$ [3] (d) $(3,0,0)$ [4] (e) $(2,1,0)$ [1] (f) $(2,1,0)$ [2] (g) $(1,1,1)$

Figure 6: $|E(H_3)| = 6$

(a) $P_2 + P_2$ (b) P_3

Figure 7: All forests with 2 edges

Case 5. $|E(H_3)| = 7$.

Arbitrarily add a vertex u and edge uv to the graph H_3 where $v \in V(H_3)$ and call this new graph H_3^+. Find a ρ-tripartite labeling f of H_3^+, which exists by Lemmas 4.1 - 4.8, and say $\ell(uv) = l$. Removing uv from H_3^+ leaves the graph H_3 with all necessary lengths except l. Notice that the forest, F is simply an edge in this subcase. Amongst the 10 remaining numbers in the set $\{0, 1, ..., 16\}$, we need to find two labels a, b such that $\ell(a, b) = l$. To see that such a pair exists, let $m_1 < m_2 < ... < m_{10}$ be the unused numbers in $\{0, 1, ..., 16\}$ and set $n_i = m_i + l$ for $i = 1, 2, ..., 10$, where the arithmetic is performed modulo 17. Let $M = \{m_i : i = 1, 2, ..., 10\}$ and $N = \{n_i : i = 1, 2, ..., 10\}$. Then we have 20 elements of $M \cup N$ in $\{0, 1, ..., 16\}$. By the Pigeonhole Principle, two elements of $M \cup N$ are equal. But $m_i \neq m_j$ and $n_{i'} \neq n_{j'}$. Therefore, $m_i = n_j$ for some i, j, so we have found the desired pair a, b with difference l. Label the two vertices of F with this pair, setting the smaller label in A and the larger label in B. We have described a ρ-tripartite labeling of $G \cong H_3 + F$.

The next theorem follows directly from the labelings provided in this subsection and Theorem 3.5.

Theorem 5.1. *Let G be a disconnected unicyclic graph with eight edges that contains a C_3. The graph G decomposes K_n if $n \equiv 1 \pmod{16}$.*

5.2 $n \equiv 0 \pmod{16}$

We proceed by looking for a 1-rotational ρ-tripartite labeling of G. First we choose the edge of length ∞. To minimize the number of different cases, we select it so that we reduce the cases $|E(H_3)| = 3$ and $|E(H_3)| = 4$ to $|E(H_3)| = 3$; and $|E(H_3)| = 5$ and $|E(H_3)| = 6$ to $|E(H_3)| = 5$. For the case $|E(H_3)| = 7$, we choose the isolated edge to be the edge of length ∞.

Case 1. $|E(H_3)| = 3$ or 4.

Label a vertex of degree one with ∞ so that only a triangle and a forest with 4 edges remains to be labeled. Apply the labels 0, 6, and 14 to the vertices of the triangle, and set $0 \in A$, $6 \in B$, and $14 \in C$. We have $c = c' = 14$, $b = b' = 6$, and the edge lengths used in the triangle are $1, 6$, and 7. Label F as shown in Figure 8, inducing lengths 2, 3, 4, and 5, and completing the desired 1-rotational ρ-tripartite labeling of $G \cong H_3 + F$.

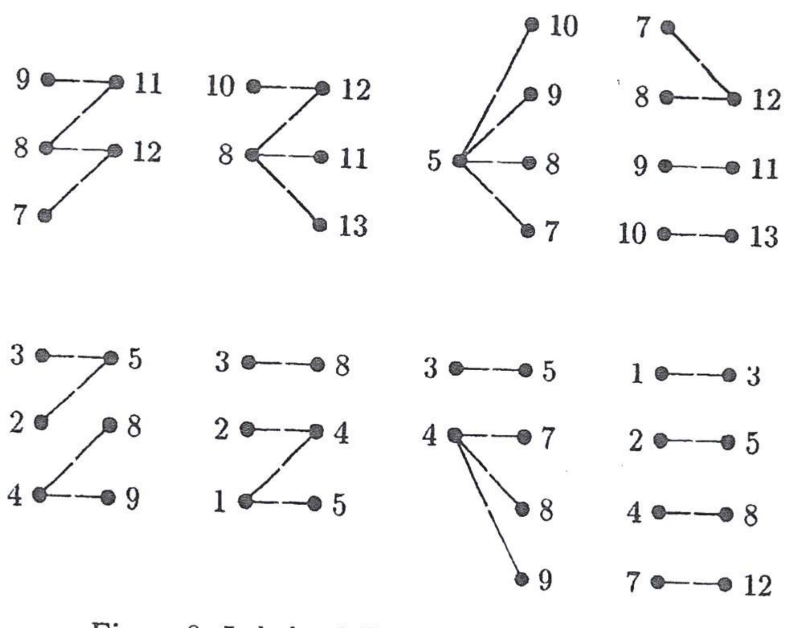
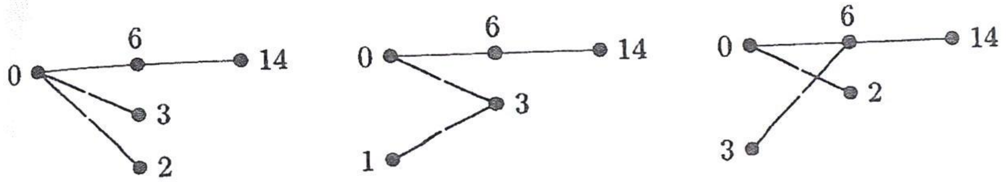


Figure 8: Labels of F when $|E(H_3)| = 3$ or 4

Case 2. $|E(H_3)| = 5$ or 6.

Label a vertex of degree one with ∞ so that the subgraph of H_3 which mains to be labeled is of type $(2,0,0)$ or type $(1,1,0)$. Label the remaining rtices as shown in Figure 9. We have $b = b' = 6$, $c = c' = 14$, and the ge lengths used are $1, 2, 3, 6$, and 7. Then label F as shown in Figure 10 which luces lengths 4 and 5, giving a 1-rotational ρ-tripartite labeling of $G \cong H_3 + F$.



(a) $(2,0,0)$ [1] (b) $(2,0,0)$ [2] (c) $(1,1,0)$

Figure 9: $|E(H_3)| = 5$ or 6

(a) $P_2 + P_2$ (b) P_3

Figure 10: All forests with 2 edges

Case 3. $|E(H_3)| = 7$.

We establish three subcases by the type of H_3. In each subcase, we reserve he isolated edge for the edge of length ∞. It remains to find a labeling of H_3 vhich uses lengths $\{1, 2, ..., 7\}$.

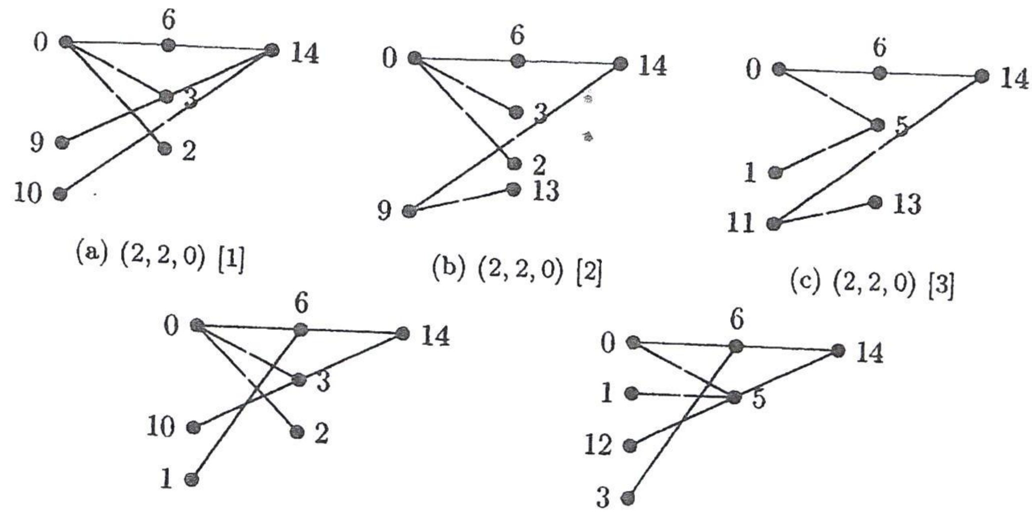
Subcase 3.1. H_3 is of type $(4,0,0)$

Label the vertices of the triangle of with $0, 6, 14$ and set $0 \in A$, $6 \in B$, and .4 $\in C$. We have $c = c' = 14$, $b = b' = 6$, and the edge lengths used in the riangle are $1, 6$, and 7. Let T be the tree with 4 edges and vertex bipartition (X, Y) attached to the triangle at vertex 6 which we identify with $y_0 \in Y$. Find n α-labeling f of T such that $f(y_0) > \lambda$ and increase all labels in Y by $6 - f(y_0)$ nd increase all labels in X by $5 - f(y_o)$. This provides edge lengths $2, 3, 4$ and 5. Find a label $l \in \{0, 1, ..., 14\}$ such that l has not been used thus far. By applying he labels ∞ and l to the isolated edge and placing $X \cup \{l\} \subset A$, $Y \subset B$, we have found the desired 1-rotational ρ-tripartite labeling of $G \cong H_3 + F$.

Subcase 3.2. H_3 is of type $(3,1,0)$

Label the vertices of the triangle of by $0, 6, 14$, so that the vertex labeled 14 is associated with the hanging edge, and set $0 \in A$, $6 \in B$, and $14 \in C$. Label the vertex of degree 1 on the hanging edge 12 and place it in A. We have $c = c' = 14$, $b = b' = 6$, and the edge lengths used so far are $1, 2, 6$, and 7. Let T be the tree with 3 edges and vertex bipartition (X, Y) attached to the triangle at vertex 6 which we identify with $y_0 \in Y$. Find an α-labeling f of T

such that $f(y_0) > \lambda$ and increase all labels in Y by $6 - f(y_0)$ and increase all labels in X by $4 - f(y_o)$. This provides edge lengths 3, 4, and 5. Find a label $l \in \{0, 1, ..., 14\}$ such that l has not been used thus far. By applying the labels ∞ and l to the isolated edge and placing $X \cup \{l\} \subset A$, $Y \subset B$, we have found the desired 1-rotational ρ-tripartite labeling of $G \cong H_3 + F$.



(a) $(2, 2, 0)$ [1] (b) $(2, 2, 0)$ [2] (c) $(2, 2, 0)$ [3]

(d) $(2, 1, 1)$ [1] (e) $(2, 1, 1)$ [2]

Figure 11: Type $(2, 2, 0)$ or $(2, 1, 1)$

Subcase 3.3. H_3 is of type $(2, 2, 0)$ or type $(2, 1, 1)$

In this subcase, H_3 is one of the five graphs shown in Figure 11. Label the isolated edge with 4 and ∞ and set $4 \in A$ and $\infty \in B$. We leave it to the reader to check that the labelings described result in 1-rotational ρ-tripartite labelings of $G \cong H_3 + F$.

We conclude this section with the following theorem which follows directly from the labelings provided in this subsection and Theorems 3.7 and 5.1.

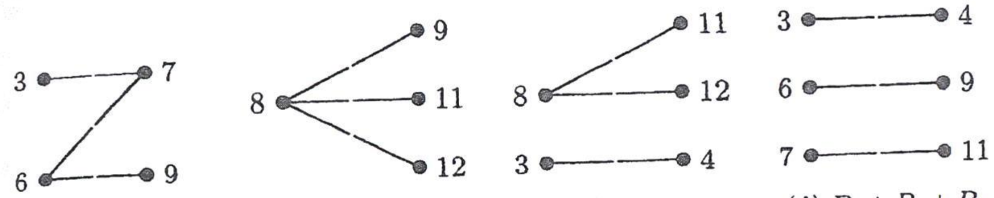
Theorem 5.2. *Let G be a disconnected unicyclic graph with eight edges that contains a C_3. The graph G decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

6 Disconnected graphs containing a pentagon

Let G be a disconnected unicyclic graph on eight edges which contains a C_5, and similar to the previous section, we will write $G \cong H_5 + F$ where H_5 is the component containing the pentagon and F is a forest. Then $|E(H_5)| \in \{5, 6, 7\}$.

6.1 $n \equiv 1 \pmod{16}$

We aim to find a ρ-tripartite labeling of G. If $|E(H_5)| = 5$, then $H_5 \cong C_5$ and $F \in \{P_4, K_{1,3}, P_2+P_3, P_2+P_2+P_2\}$, the set of forests containing 3 edges. Label he vertices of H_5 with $0, 2, 13, 5$, and 10 around the cycle with $0 \in A$, $\{2,5\} \subset B$, nd $\{10, 13\} \subset C$. This induces edge lengths 2, 6, 8, 5, and 7, respectively. Notice hat $|13-2| + |10-5| = 16$ and $2|13-5| = 16$, so the edges between partite ets B and C satisfy the complement property of a ρ-tripartite labeling. Then abel F as shown in Figure 12 which induces the remaining lengths 1, 3, and 4, nd completes the desired ρ-tripartite labeling of $G \cong H_5 + F$.



(a) P_4 (b) $K_{1,3}$ (c) $P_2 + P_3$ (d) $P_2 + P_2 + P_2$

Figure 12: Labels of F when $H_5 \cong C_5$

If $|E(H_5)| = 6$, then H_5 is congruent to C_5 with a hanging edge and F is either $P_2 + P_2$ or P_3. Label H_5 as shown in Figure 13. This induces edge lengths $1, 2, 5, 6, 7$, and 8. Notice that $|13-2| + |10-5| = 16$ and $2|13-5| = 16$, so the edges between partite sets B and C satisfy the complement property of a ρ-tripartite labeling. If $F \cong P_2 + P_2$, then label one edge with 3 and 6, and the other edge with 4 and 8, setting $\{3,4\} \subset A$, and $\{6,8\} \subset B$. On the other hand, if $F \cong P_3$, label the vertices $7, 4, 8$ consecutively along the path and set $4 \in A$ and $\{7,8\} \subset B$. This induces the remaining lengths 3 and 4, and completes the desired ρ-tripartite labeling of $G \cong H_5 + F$.

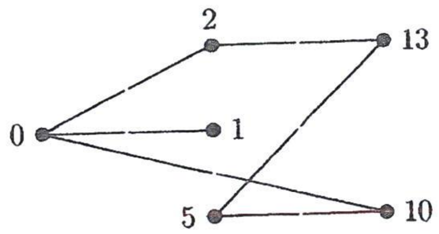
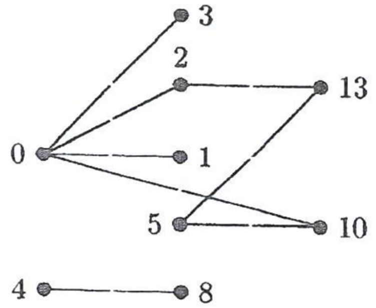
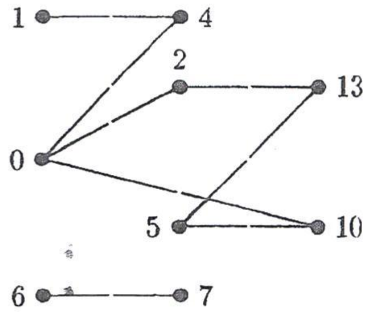


Figure 13: Labels of H_5 when $|E(H_5)| = 6$

If $|E(H_5)| = 7$, then $F \cong P_2$ and H_5 is of type $(2,0,0,0,0)$, type $(1,1,0,0,0)$, or type $(1,0,1,0,0)$. Note that there are two non-isomorphic graphs H_5 which are of the first type, but only one each of the latter two types. Label $G \cong H_5 + F$ as shown in Figures 14 and 15. Notice that all edge lengths are present and the the edges between partite sets B and C in Figure 14 satisfy the complement property of a ρ-tripartite labeling since $|13-2| + |10-5| = 16$ and $2|13-5| = 16$. Similarly, $|15-5| + |15-9| = 16$ and $|13-9| + |13-1| = 16$, so this property is also present in the labeling of Figure 15. Therefore, we have provided the desired ρ-tripartite labeling of the graph G.

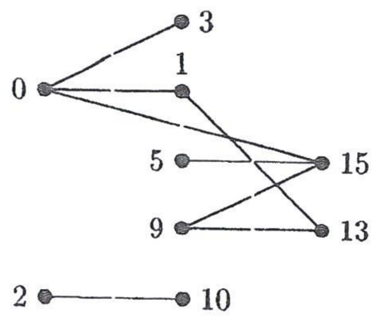


(a) (2, 0, 0, 0, 0) [1]

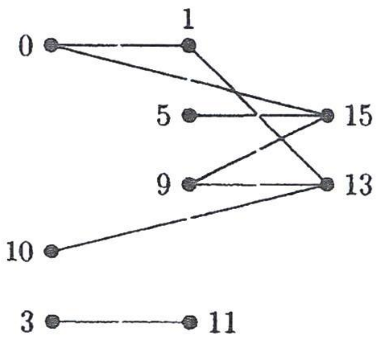


(b) (2, 0, 0, 0, 0) [2]

Figure 14: Labels of G; H_5 of type $(2, 0, 0, 0, 0)$



(a) (1, 1, 0, 0, 0)



(b) (1, 0, 1, 0, 0)

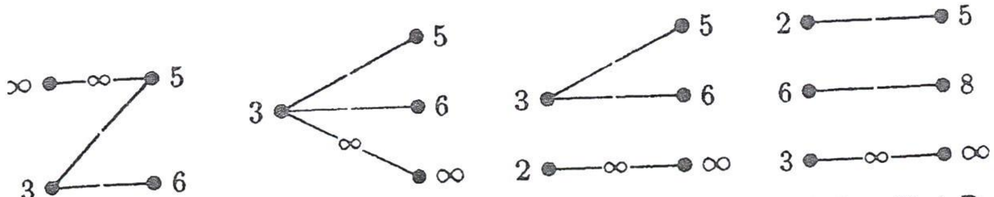
Figure 15: Labels of G; H_5 of type $(1, 1, 0, 0, 0)$ or type $(1, 0, 1, 0, 0)$

We have found a ρ-tripartite labeling of every disconnected unicyclic graph with eight edges which contains C_5. Therefore, the next theorem follows directly from Theorem 3.5.

Theorem 6.1. *Let G be a disconnected unicyclic graph with eight edges that contains a C_5. The graph G decomposes K_n if $n \equiv 1 \pmod{16}$.*

6.2 $n \equiv 0 \pmod{16}$

We proceed by finding a 1-rotational ρ-tripartite labeling of G. If $|E(H_5)| = 5$, label the vertices of $H_5 \cong C_5$ with $0, 1, 12, 4$, and 9 around the cycle with $0 \in A$, $\{1, 4\} \subset B$, and $\{9, 12\} \subset C$. This induces lengths $1, 4, 7, 5$, and 6, respectively. Then label F as shown in Figure 16.



(a) P_4 (b) $K_{1,3}$ (c) $P_2 + P_3$ (d) $P_2 + P_2 + P_2$

Figure 16: Labels of F when $|E(H_5)| = 5$

If $|E(H_5)| = 6$ or 7, label a vertex of degree one with ∞ so that the unlabeled rtices of G and the edges incident with them form a graph of type $(1,0,0,0,0)$ us an isolated edge. Apply the labels as shown in Figure 17. If $|E(H_5)| = 7$, $F \cong P_3$ we are done. If $F \cong P_2 + P_2$, let 5 be the vertex incident with ∞ and t $5 \in A$. We have described a 1-rotational ρ-tripartite labeling of the graph $\cong H_5 + F$.

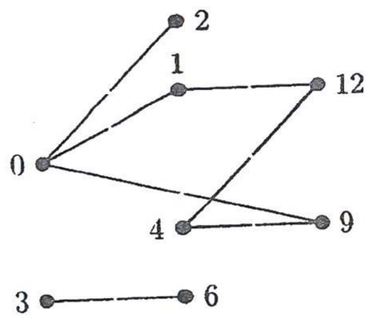


Figure 17: Edges with finite lengths; $|E(H_5)| = 6$ or 7

We have found a 1-rotational ρ-tripartite labeling of every disconnected unicyclic graph with eight edges that contains a C_5. Therefore, the next theorem follows directly from Theorems 3.7 and 6.1.

Theorem 6.2. *Let G be a disconnected unicyclic graph with eight edges that contains a C_5. The graph G decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

7 Graphs containing a heptagon

Let G be a unicyclic graph with eight edges that contains a C_7. Then G is either the disjoint or non-disjoint union of C_7 with an edge.

Theorem 7.1. *Let G be a unicyclic graph with eight edges that contains a C_7. The graph G decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

Proof. Figures 18 and 19 provide ρ-tripartite and 1-rotational ρ-tripartite labelings, respectively, of G. The result follows now from Theorems 3.5 and 3.7. □

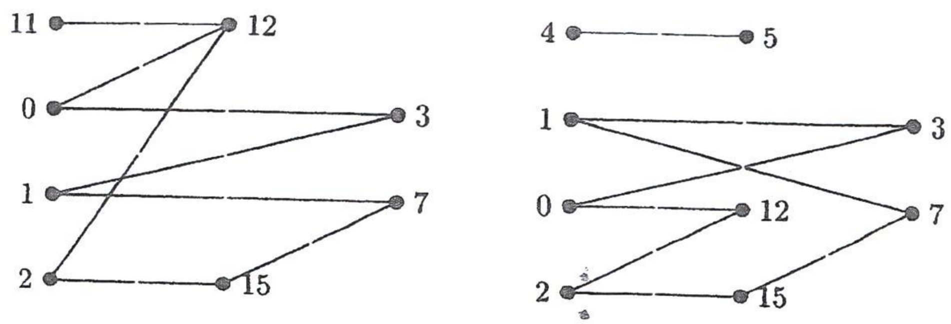


Figure 18: ρ-tripartite labelings of unicyclic graphs with eight edges that contain a heptagon

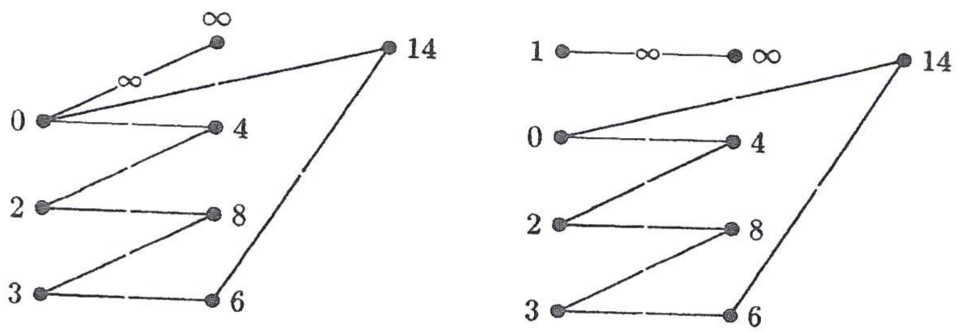


Figure 19: 1-rotational ρ-tripartite labelings of unicyclic graphs with eight edges that contain a heptagon

8 Main result

We have completely classified the integers n such that K_n allows a G-decomposition where G is a unicyclic graph with eight edges. We conclude with our main result.

Theorem 8.1. *Let G be a unicyclic graph with eight edges other than C_8. The graph G decomposes K_n if and only if $n \equiv 0, 1 \pmod{16}$.*

Proof. If G is bipartite, the conclusion follows from Theorems 2.2 and 2.3. If G is not bipartite, Theorems 2.4, 4.2, 5.2, 6.2, and 7.1 give the result. □

9 Acknowledgements

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References

[1] R.C. Bunge, On 1-rotational decompositions of complete graphs into tripartite graphs, *Opuscula Math.*, **39(5)** (2019), 623–643.

[2] R.C. Bunge, A. Chantasartrassmee, S.I. El-Zanati, and C. Vanden Eynden, On cyclic decompositions of complete graphs into tripartite graphs, *J. Gr. Theory*, **72** (2013), 90–111.

[3] D. Donovan, S. I. El-Zanati, C. Vanden Eynden, and S. Sutinuntopas, Labelings of unions of up to four uniform cycles, *Australas. J. Combin.*, **29** (2004), 323–336.

[4] S. I. El-Zanati and C. Vanden Eynden, On Rosa-type labelings and cyclic graph decompositions, *Math. Slovaca*, **59(1)** (2009), 1–18.

[5] S. I. El-Zanati, C. Vanden Eynden, and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.*, **24** (2001), 209–219.

[6] J. Fahnenstiel and D. Froncek, Decomposition of complete graphs into connected bipartite unicyclic graphs with eight edges, *Electron. J. Graph Theory Appl.*, **7(2)** (2019), 235–250.

[7] B. Freyberg and N. Tran, Decomposition of complete graphs into bipartite unicyclic graphs with eight edges, in print.

[8] D. Froncek and O. Kingston, Decomposition of complete graphs into connected unicyclic graphs with eight edges and pentagon, *Indonesian J. Combin.* **3(1)** (2019), 24–33.

[9] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Intl. Symp. Rome 1966), Gordon and Breach, Dunod, Paris, 1967, 349–355.