$L(h, k)$ labelings of $K_n - M$ and $K_n - P_m$ for all values of h and k*

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Abstract

An $L(h, k)$ labeling of a graph G is an integer labeling of the vertices where the labels of adjacent vertices differ by at least h, and the labels of vertices that are at distance two from each other differ by at least k. The *span of an* $L(h, k)$ *labeling* f on a graph G is the largest label minus the smallest label under f. The $L(h, k)$ *span of a graph* G, denoted $\lambda_{h,k}(G)$, is the minimum span of all $L(h, k)$ labelings of G.

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We study $L(h,k)$ labelings of some dense graphs obtained by deleting either a maximum matching or the edges of an arbitrary path from a complete graph. For all non-negative integer values of h and k, we establish the $L(h,k)$ spans of these graphs.

1 Introduction

The classic vertex labeling problem imposes a condition on labels of adjacent vertices. However, motivated by channel assignment problems and other applications, numerous generalizations and modifications have been introduced over the years [11]. One such generalization, called an $L(h,k)$ labeling, imposes conditions on labels of adjacent vertices, as well as labels of vertices that are at distance two.

The channel assignment problem, first introduced by Hale and later modified by Roberts [4, 12], is a long standing problem that describes the assignment of frequencies to transmitters based on their distances to neighboring and nearby nodes in the same network to avoid interference. The $L(h,k)$ labeling problem, introduced by Griggs and Yeh for $h=2$ and $k=1$ was motivated by the channel assignment problem [7].

An $L(h,k)$ labeling of a graph G is an integer labeling of the vertices where adjacent vertices differ in label by at least h, and vertices that are at distance two from each other differ in label by at least k. That is, an *$L(h,k)$ labeling* of G is a vertex labeling $f: V(G) \to \{0\} \cup \mathbb{Z}^+$ such that

- $|f(u) - f(v)| \geq h$ if $d(u,v) = 1$
- $|f(u) - f(v)| \geq k$ if $d(u,v) = 2$.

Note that by $d(u,v)$ we mean the distance between vertices u and v, which is the number of edges in a shortest path between u and v.The *span of an $L(h,k)$ labeling* f of a graph G is the largest label minus the smallest label. By convention, the smallest label used is 0, as all the labels could be shifted by the same value and make the minimum label 0, while maintaining a valid $L(h,k)$ labeling with the same span. So throughout this paper, we assume that the smallest label in a $L(h,k)$ labeling is 0 and hence the span of an $L(h,k)$ labeling f is max $f(u)$ for all $u \in V(G)$. The *$L(h,k)$ span of a graph G*, denoted $\lambda_{h,k}(G)$, is the minimum span of all $L(h,k)$ labelings of G. The decision versions of $L(0,1)$, $L(1,1)$ and $L(2,1)$ span problems are shown to be NP-complete, and the decision versions of $L(h,k)$ span problems for $h \geq k$ are conjectured to be NP-complete [1, 3, 5, 7, 8, 9, 10, 13].

A large number of classes of graphs have been investigated for their (h, k) spans. While most of these efforts have assumed that $h \geq k$, some lasses of graphs have been studied for $L(h, k)$ colorability for all non-egative values of h and k. See [2] for a detailed survey. Since the vertices f a complete graph K_n are pairwise adjacent, we have,

)bservation 1. $\lambda_{h,k}(K_n) = h(n-1)$.

In this paper, we investigate the $L(h, k)$ span of some dense subgraphs of he complete graph, obtained by removing either a maximum matching or he edges of an arbitrary path from a complete graph. For all non-negative alues of h and k, we establish the $L(h, k)$ spans of these dense graphs.

2 Complete graph minus a maximum matching

n this section we investigate the $L(h, k)$ span of the graph obtained by leleting the edges of a maximum matching from a complete graph K_n or all values of h, k and n. Let $K_n = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$. Consider the maximum matching $M = \{v_i v_{i+1} \mid i \text{ is odd and } 1 \leq i < n\}$. We let $K_n' = K_n - M$.

Observation 2. *For all $h \geq 0$ and $k \geq 0$, $\lambda_{h,k}(K_1') = \lambda_{h,k}(K_2') = 0$.*

Theorem 3. *For $h \leq k$ and $n > 2$,*

$$\lambda_{h,k}(K_n') = \begin{cases} (n-1)h, & \text{if } k \leq \lceil \frac{n}{2} \rceil h \\ \left(\lfloor \frac{n}{2} \rfloor - 1 \right) h + k, & \text{if } k > \lceil \frac{n}{2} \rceil h. \end{cases}$$

Proof. Let $k \leq \lceil \frac{n}{2} \rceil h$. Since $h \leq k$ and K_n' is connected, any $L(h, k)$ labeling of K_n' is an $L(h, k)$ labeling of K_n and thus the span of any $L(h, k)$ labeling of K_n' must be at least $\lambda_{h,k}(K_n) = (n-1)h$. Thus, $\lambda_{h,k}(K_n') \geq (n-1)h$.

Consider the labeling f defined as follows:

$$f(v_i) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor h, & \text{if } i \text{ odd} \\ \lfloor \frac{n-1}{2} \rfloor h + \frac{hi}{2}, & \text{if } i \text{ even.} \end{cases}$$

Figure 1 shows this labeling scheme for K_7' when $h = 2$ and $k = 4$.

Let v_i and v_j be adjacent vertices. If both i and j have the same parity, then $|f(v_i)-f(v_j)| \geq h$. Without loss of generality, assume that i is odd and j is even. Then $f(v_j) \geq f(v_2) \geq f(v_r) + h$, where r is odd and $1 \leq r \leq n$.

Thus, $|f(v_j) - f(v_i)| > h$.

Let v_i and v_j be vertices such that $d(v_i, v_j) = 2$. This implies that, without loss of generality, i is odd and $j = i+1$. Then $|f(v_i) - f(v_j)| = f(v_j) - f(v_i) = \lfloor \frac{n-1}{2} \rfloor h + h = \lceil \frac{n}{2} \rceil h \geq k$. Therefore f is an $L(h,k)$ labeling. Note that the span of f is $(n-1)h$. Thus $\lambda_{h,k}(K'_n) = (n-1)h$ when $k \leq \lceil \frac{n}{2} \rceil h$.

Suppose $k > \lceil \frac{n}{2} \rceil h$. Let g be an $L(h,k)$ labeling of K'_n. Let v be a vertex such that $g(v)$ is the largest label. Then there exists a subgraph $H \cong K_{\lfloor \frac{n}{2} \rfloor}$ of $K'_n - \{v\}$ such that every vertex in H has a distance two neighbor in $K'_n - H$. This means $g(v_i) \geq h\left(\lfloor \frac{n}{2} \rfloor - 1\right)$ for some $v_i \in H$, and $|g(v_i) - g(v_j)| \geq k + h\left(\lfloor \frac{n}{2} \rfloor - 1\right)$ where $v_iv_j \notin E(K'_n)$. Thus, the span of any $L(h,k)$ labeling of K'_n is at least $h\left(\lfloor \frac{n}{2} \rfloor - 1\right) + k$.

Consider the labeling f defined as follows:

$$f(v_i) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor h, & \text{if } i \text{ odd} \\ f(v_{i-1}) + k, & \text{if } i \text{ even.} \end{cases}$$

An example of such an f for K'_7 when $h = 4$ and $k = 17$ is shown in Figure 2.

Let v_i and v_j be two adjacent vertices, and without loss of generality assume that $j > i$. If both i and j have the same parity, then $|f(v_i) - f(v_j)| \geq h$. Without loss of generality, assume that i is odd and j is even. Then $f(v_j) \geq f(v_2) = k \geq \lceil \frac{n}{2} \rceil h + 1 \geq f(v_r) + h + 1$, where r is odd and $1 \leq r \leq n$.

Thus $|f(v_j) - f(v_i)| > h$.

Let v_i and v_j be vertices such that $d(v_i, v_j) = 2$. Without loss of generality, assume that i is odd and $j = i+1$. Then $|f(v_i) - f(v_j)| = k$, since $f(v_i) = f(v_{i-1}) + k$. Therefore f is an $L(h,k)$ labeling. Note that the span of f is $h\left(\lfloor \frac{n}{2} \rfloor - 1\right) + k$. Thus, $\lambda_{h,k}(K'_n) = \left(\lfloor \frac{n}{2} \rfloor - 1\right) h + k$ when $k > \lceil \frac{n}{2} \rceil h$. □

Lemma 4. [6] For $h > k$, $\lambda_{h,k}(P_3) = h + k$.

Theorem 5. *For $h > k$, $\lambda_{h,k}(K'_{2m}) = m(k+h) - h$ where $m > 1$.*

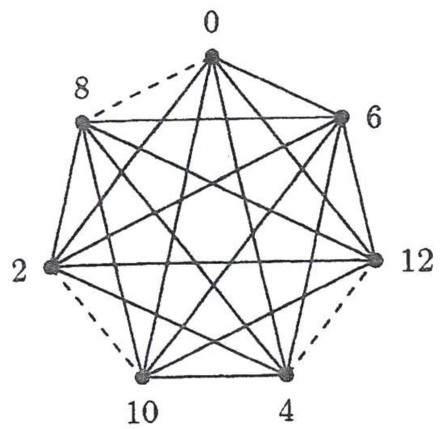


Figure 1: $L(2,4)$ labeling of K'_7 as explained in the proof of Theorem 3.

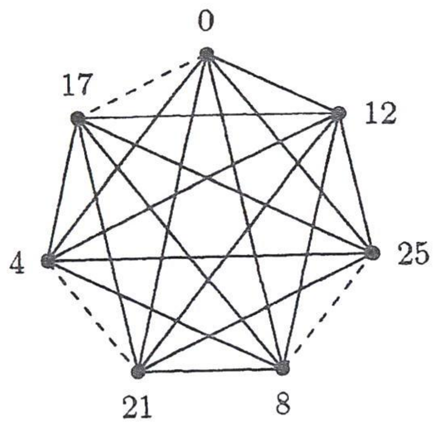


Figure 2: $L(4,17)$ labeling of K'_7 as explained in the proof of Theorem 3.

Proof. First, we will show that $\lambda_{h,k}(K'_{2m}) \geq m(k+h) - h$. To do this assume that this is not true and let f be a labeling of K'_{2m} which produces a minimum counter example, that is there is no counter example for smaller choices of m. Denote the span of the labeling f by λ_f, where $\lambda_f < m(k+h) - h$. Since $K'_4 \cong C_4$, and by [6], $\lambda_{h,k}(K'_4) = \lambda_{h,k}(C_4) = 2(k+h) - h$ for $h > k$, we can assume that $m > 2$.

Let u be the vertex with the largest label, λ_f, v be the vertex with the second largest label, possibly still λ_f, and w be the vertex with the third largest label. Note that the third largest label cannot be λ_f as the independence number of $K'_{2n} = 2$.

Now the subgraph H induced by $\{u, v, w\}$ is either a K_3 or a P_3. If H is a K_3, then $f(w) \leq \lambda_f - 2h \leq \lambda_f - h - k$. If H is a P_3, then, by Lemma 4, $f(w) \leq \lambda_f - h - k$. Now, the labeling f restricted to $K'_{2m} - \{u, v\}$ is an $L(h,k)$ labeling of a $K'_{2(m-1)}$ with span less than or equal to $\lambda_f - h - k < m(k+h) - h - k - h = (m-1)(k+h) - h$ a contradiction to m being the smallest value that produces of a counter example. Thus, $\lambda_{h,k}(K'_{2m}) \geq m(k+h) - h$ when $m > 1$.

Consider the labeling f defined as follows. $f(v_1) = 0$ and

$$f(v_i) = \begin{cases} f(v_{i-1}) + k, & \text{if } i \text{ even} \\ f(v_{i-1}) + h, & \text{if } i \text{ odd.} \end{cases}$$

Let v_i and v_j be adjacent vertices. Without loss of generality assume $i < j$. Note that since v_i and v_j are adjacent, $j \neq i+1$ if i is odd. Then $|f(v_i) - f(v_j)| \geq h$. Let v_i, v_j be vertices such that $d(v_i, v_j) = 2$. Then i must be odd and $j = i+1$. This gives that $|f(v_i) - f(v_j)| = k$.

Therefore f is an $L(h,k)$ labeling. Note that the span of f is $m(k+h)-h$. Thus $\lambda_{h,k}(K'_{2m}) \leq m(k+h) - h$ when $h > k$. □

An example of an optimal $L(5,3)$ labeling of K'_6 is shown in Figure 3.

The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in G_1, v \in G_2\}$.

Lemma 6. *Suppose $h \geq k$ and $diam(G) \leq 2$. Let $H = G \vee K_1$. Then $\lambda_{h,k}(G) \geq \lambda_{h,k}(G) + h$.*

Proof. Let $V(K_1) = \{v\}$. Let f be an optimal $L(h,k)$ labeling of H. Consider the labeling g of G as follows.

$$g(v_i) = \begin{cases} f(v_i), & \text{if } f(v_i) < f(v) \\ f(v_i) - h, & \text{if } f(v_i) \geq f(v). \end{cases}$$

We will show that g is an $L(h,k)$ labeling of G. Let $x, y \in V(G)$. Since $diam(G) \leq 2$, $d_G(x,y) = d_H(x,y)$. If $f(x) \leq f(v)$ and $f(y) \leq f(v)$, or if $f(x) \geq f(v)$ and $f(y) \geq f(v)$ then $|g(x) - g(y)| = |f(x) - f(y)|$. So, without loss of generality, assume that $f(x) < f(v)$ and $f(y) \geq f(v)$. Since v is adjacent to both x and y, $f(v) - f(x) \geq h$ and $f(y) - f(v) \geq h$, and thus $f(y) - f(x) \geq 2h$. Thus $g(y) - g(x) = f(y) - h - f(x) \geq h \geq k$. Therefore, g is an $L(h,k)$ labeling of G. Note that $\lambda_g = \lambda_f - h$. Thus $\lambda_{h,k}(H) = \lambda_f = \lambda_g + h \geq \lambda_{h,k}(G) + h$. □

Theorem 7. *For $h > k$, $\lambda_{h,k}(K'_{2m+1}) = m(k+h)$ where $m > 0$.*

Proof. $K'_{2m+1} = K'_{2m} \vee \{v_{2m+1}\}$, and by Lemma 6, $\lambda_{h,k}(K'_{2m+1}) \geq \lambda_{h,k}(K'_{2m}) + h$. Now by Theorem 5, $\lambda_{h,k}(K'_{2m+1}) \geq m(k+h) - h + h = m(k+h)$.

Consider the same labeling f used in the proof of Theorem 5. f is an $L(h,k)$ labeling of K'_{2m+1}. Note that the span of f is $m(k+h)$. Thus, $\lambda_{h,k}(K'_{2m+1}) \leq m(k+h)$. □

An example of an optimal $L(5,3)$ labeling of K'_7 is shown in Figure 4.

3 Complete graph minus a path

In this section, we establish the $L(h,k)$ span of the graph obtained by removing the edges of an arbitrary path from a complete graph, K_n, for all

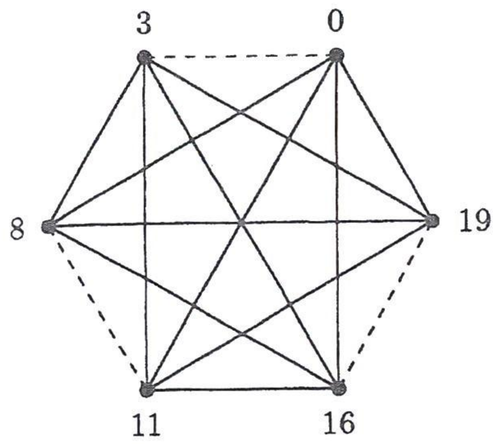


Figure 3: $L(5,3)$ labeling of K_6'

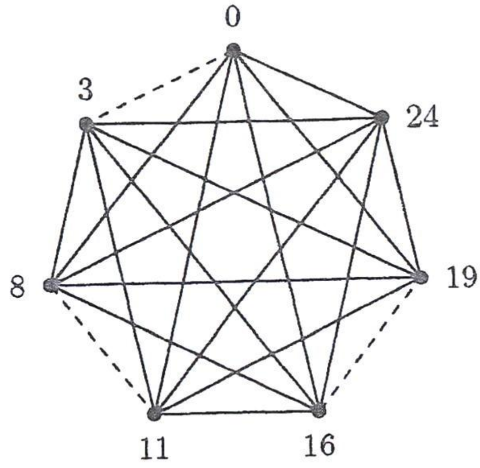


Figure 4: $L(5,3)$ labeling of K_7'

values of h, k and n. Let $K_n - P_m = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$ and P_m consists of vertices $\{v_1, v_2, \dots, v_m\}$ and edges $v_i v_{i+1}$ for all $1 \le i < m$.

Theorem 8. *$\lambda_{h,k}(K_n - P_m) = h(n-1)$ when $h \le k \le (n - \lceil \frac{m}{2} \rceil)\, h$ and $n > 3$.*

Proof. Since $k \ge h$, any $L(h,k)$ labeling of $K_n - P_m$ is an $L(h,k)$ labeling of K_n. Thus $\lambda_{h,k}(K_n - P_m) \ge \lambda_{h,k}(K_n) = h(n-1)$.

Consider the labeling f where $f(v_2) = 0$ and

$$f(v_i) = \begin{cases} f(v_{i-2}) + h, & \text{if } i \text{ even and } 4 \le i \le m \\ f(v_{i-2}) + h, & \text{if } i = m+1 \text{ and } m+1 \text{ is even} \\ f(v_{i-1}) + h, & \text{if } i = m+1 \text{ and } m+1 \text{ is odd} \\ f(v_{i-1}) + h, & \text{if } m+1 < i \le n \\ f(v_n) + h, & \text{if } i = 1 \\ f(v_{i-2}) + h, & \text{if } i \text{ odd and } 3 \le i \le m. \end{cases}$$

An example of such an $L(3,11)$ labeling of $K_7 - P_5$ is shown in Figure 5.

The largest label under f is $f(v_r) = (n-1)h$ where r is the largest odd integer such that $1 \le r \le m$. Thus the span of f is $(n-1)h$.

Since labels used are distinct multiples of h, we have $|f(v_i) - f(v_j)| \ge h$ for any two adjacent vertices v_i and v_j. Suppose $d(v_i, v_j) = 2$. Then $1 \le i, j \le m$ and $|i - j| = 1$. Without loss of generality, assume that i is odd, and let $i = 2r + 1$. Then $j = 2r$, or $j = 2r + 2$.

$$\begin{aligned} |f(v_i) - f(v_j)| &= f(v_i) - f(v_j) \\ &\geq f(v_{2r+1}) - f(v_{2r+2}), \text{ because } f(v_{2r+2}) = h + f(v_{2r}) \\ &= \left(\left(\lfloor \tfrac{m}{2} \rfloor - 1\right) h + (n-m)h + h + rh\right) - rh \\ &= n - \lceil \tfrac{m}{2} \rceil \\ &\geq k. \end{aligned}$$

Therefore, f is an $L(h, k)$ labeling with span $h(n-1)$. Thus $\lambda_{h,k}(K_n - P_m) = h(n-1)$.

□

Theorem 9. *$\lambda_{h,k}(K_n - P_m) = \left(\lceil \frac{m}{2} \rceil - 1\right) h + k$ when $k \geq \left(n - \lceil \frac{m}{2} \rceil\right) h$ and $n > 3$.*

Proof. Consider the labeling f defined as follows, where $f(v_2) = 0$ and

$$f(v_i) = \begin{cases} k, & \text{if } i = 1 \\ f(v_{i-2}) + h, & \text{if } i \text{ odd and } 1 < i \leq m+1 \\ f(v_{i-2}) + h, & \text{if } i \text{ even and } 2 < i \leq m \\ f(v_{m-1}) + h, & \text{if is } i \text{ even and } i = m+1 \\ f(v_{i-1}) + h, & \text{if } m+1 < i \leq n. \end{cases}$$

An example of such an $L(3, 13)$ labeling of $K_7 - P_5$ is shown in Figure 6.

Let v_i, v_j be adjacent vertices. To show that $|f(v_i) - f(v_j)| \geq h$, it suffices to show that $|f(v_1) - f(v_j)| \geq h$, as $f(v_1)$ is the only value in the recursive definition that does not increase by h from the previous step. If $f(v_j) > f(v_1)$, then j is odd and $j \leq m+1$. So let $j = 2l+1$. Then $|f(v_{2l+1}) - f(v_1)| = k + (l-1)h - k = (l-1)h \geq h$. Now if $f(v_j) < f(v_1)$, then

$$\begin{aligned} |f(v_1) - f(v_j)| &= f(v_1) - f(v_j) \\ &\geq f(v_1) - f(v_n) \\ &= k - \left(\left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right) h + (n-m)h\right) \\ &= k - \left(\left(n - \left\lceil \frac{m}{2} \right\rceil\right) h\right) + h \\ &\geq h. \end{aligned}$$

Let v_i, v_j be vertices such that $d(v_i, v_j) = 2$. Without loss of generality let $i = 2l$ and $j = 2l+1$ for some $j \leq m$. Then $|f(v_{2l}) - f(v_{2l+1})| = f(v_{2l+1}) - f(v_{2l}) = k + (l-1)h - (l-1)h = k$.

Thus $\lambda_{h,k}(K_n - P_m) \leq \left(\lceil \frac{m}{2} \rceil - 1\right) h + k$.

Let $G_1 \cong K_{\lceil m/2 \rceil}$ with vertex set $\{v_1, v_3, v_5, ..., v_t\}$ where $t = m$ or $m-1$ and let $G_2 \cong K_{\lfloor m/2 \rfloor}$ with vertex set $\{v_2, v_4, ..., v_{t'}\}$ where $t' = m-1$ or m.

If $\min\{f(v_i), f(v_{i+2})\} < f(v_{i+1}) < \max\{f(v_i), f(v_{i+2})\}$ for any $1 \leq i \leq m-2$, then, we have

$$\begin{aligned} \max\{f(v_i), f(v_{i+2})\} &\geq 2k + \min\{f(v_i), f(v_{i+2})\}, \\ &\quad \text{since } d(v_i, v_{i+1}) = d(v_{i+2}, v_{i+1}) = 2 \\ &\geq k + \left(n - \lceil \tfrac{m}{2} \rceil\right) h, \text{ because } k \geq \left(n - \lceil \tfrac{m}{2} \rceil\right) h \\ &\geq \left(\lceil \tfrac{m}{2} \rceil - 1\right) h + k. \end{aligned}$$

Suppose $\max\{f(v_i), f(v_{i+2})\} < f(v_{i+1})$ or $f(v_{i+1}) < \min\{f(v_i), f(v_{i+2})\}$ for all $1 \leq i \leq m-2$. Then either $f(v) < f(u)$ for all $v \in V(G_1)$ and $u \in V(G_2)$, or $f(v) > f(u)$ for all $v \in V(G_1)$ and $u \in V(G_2)$. In the first case, the span of f must be at least $\left(\lceil \frac{m}{2} \rceil - 1\right) h + k$. In the second case, the smallest label for a vertex in G_1 must be at least k, and thus there must be a label in G_1 that is at least $\left(\lceil \frac{m}{2} \rceil - 1\right) h + k$.

Thus in any case, span of f is at least $\left(\lceil \frac{m}{2} \rceil - 1\right) h + k$.

Therefore $\lambda_{h,k}(K_n - P_m) = \left(\lceil \frac{m}{2} \rceil - 1\right) h + k$. □

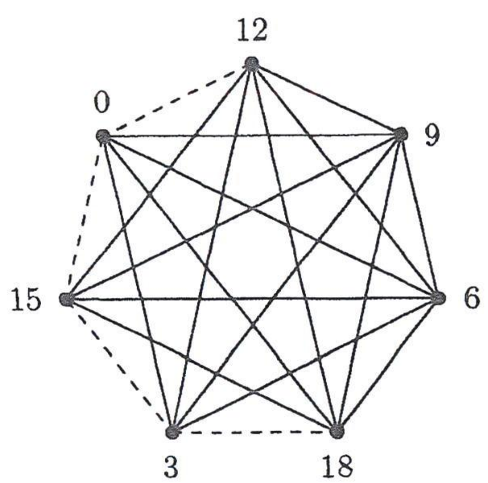


Figure 5: $L(3, 11)$ labeling of $K_7 - P_5$

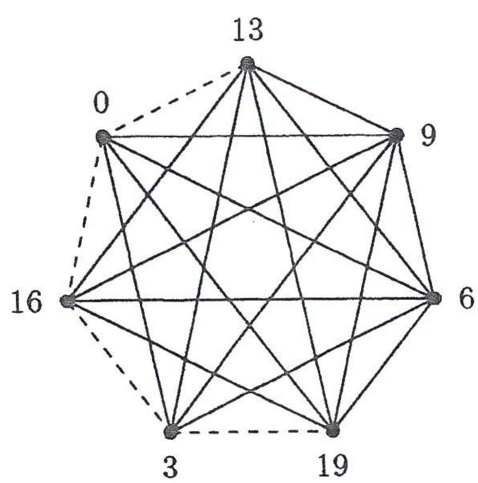


Figure 6: $L(3, 13)$ labeling of $K_7 - P_5$

Lemma 10. *$\lambda_{h,k}(K_n - P_2) = h(n-2) + k$ for $k < h$.*

Proof. By Lemma 4, $\lambda_{h,k}(K_3 - P_2) = \lambda_{h,k}(P_3) = h + k$. Assume that $\lambda_{h,k}(K_{n-1} - P_2) = (n-3)h + k$. Now, since $K_n - P_2 = (K_{n-1} - P_2) \vee K_1$, and $diam(K_{n-1} - P_2) = 2$, by Lemma 6, $\lambda_{h,k}(K_n - P_2) \geq h(n-2) + k$.

Consider the labeling f defined as follows, with $f(v_1) = 0$ and $f(v_i) = k + h(i-2)$ for all $2 \leq i \leq n$. Let v_i, v_j be adjacent vertices. Without loss of generality, let $i > j$. If $j > 2$, then $|f(v_i) - f(v_j)| = (i-2)h + k - ((j-2)h + k)) \geq h$. If $j = 1$ or $j = 2$, then $|f(v_i) - f(v_j)| \geq f(v_3) - f(v_2) = h$. If v_i, v_j are vertices where $d(v_i, v_j) = 2$, then $i = 1, j = 2$ or $i = 2, j = 1$, and $|f(v_i) - f(v_j)| = k$. Thus, $\lambda_{h,k}(K_n - P_2) \leq h(n-2) + k$. □

Theorem 11. *$\lambda_{h,k}(K_n - P_m) = (m-1)k + (n-m)h$ for $k < h < 2k$ and $n > 3$.*

Proof. We first show that $\lambda_{h,k}(K_n - P_m) \geq (m-1)k + (n-m)h$. Note that we know this holds for the case of $m = 2$ from Lemma 10. Now assume that $\lambda_{h,k}(K_n - P_m) < (m-1)k + (n-m)h$, and $m > 2$ is the smallest such m. Now consider the $L(h,k)$ span of $K_n - P_m$. Let g be a labeling of $H = K_n - P_{m-1}$ defined as follows:

$$g(v_i) = \begin{cases} f(v_i), & \text{if } f(v_i) < f(v_m) \\ f(v_i) + (h-k), & \text{if } f(v_i) \geq f(v_m) \end{cases}$$

Let v_i, v_j be adjacent vertices in H. If $f(v_i), f(v_j) < f(v_m)$, or if $f(v_i), f(v_j) \geq f(v_m)$, then $|g(v_i) - g(v_j)| = |f(v_j) - f(v_i)| \geq h$. Without loss of generality, let $f(v_i) < f(v_m) \leq f(v_j)$. If $i = m-1$ and $j = m$, then $d_G(v_i, v_j) = 2$ and $|g(v_i) - g(v_j)| = f(v_j) + (h-k) - f(v_i) = (f(v_j) - f(v_i)) + h - k \geq k + h - k \geq h$. If $i \neq m-1$, or $j \neq m$, then $d_G(v_i, v_j) = 1$ and $|g(v_i) - g(v_j)| = f(v_j) + (h-k) - f(v_i) = (f(v_j) - f(v_i)) + h - k \geq 2h - k > h$.

Now, let v_i, v_j be vertices such that $d_H(v_i, v_j) = 2$. Note that in this case, $i, j \neq m$. If $f(v_i), f(v_j) < f(v_m)$, or if $f(v_i), f(v_j) \geq f(v_m)$, then $|g(v_i) - g(v_j)| = |f(v_i) - f(v_j)| \geq k$. Without loss of generality, let $f(v_i) < f(v_m) \leq f(v_j)$. Since, $j \neq m$, we have $|g(v_i) - g(v_j)| = f(v_j) + (h-k) - f(v_i) \geq f(v_j) - f(v_i) + (h-k) \geq k + (h-k) = h > k$.

Thus, $\lambda_{h,k}(K_n - P_m) \geq (m-1)k + (n-m)h$.

Consider the labeling f of $K_n - P_m$ recursively defined as follows, where $f(v_1) = 0$ and

$$f(v_i) = \begin{cases} f(v_{i-1}) + k, & \text{if } i \leq m \\ f(v_{i-1}) + h, & \text{if } i > m. \end{cases}$$

Note that the span of f is $(m-1)k+(n-m)h$. Let v_i, v_j be adjacent vertices. If $i \geq m$ or $j \geq m$, then $\left|f(v_i)-f(v_j)\right| \geq h$. Without loss of generality, let $m \geq i > j$. Then i and j must differ by at least two to be adjacent, so $\left|f(v_i)-f(v_j)\right| = (i-1)k-(j-1)k \geq 2k > h$.

Let v_i, v_j be vertices such that $d(v_i, v_j) = 2$. Without loss of generality, $i = j+1$, so $\left|f(v_i)-f(v_j)\right| = k$. Therefore, f is an $L(h,k)$ labeling with span $(m-1)k+(n-m)h$ and hence $\lambda_{h,k}(K_n - P_m) \leq (m-1)k+(n-m)h$. □

Theorem 12. *For $h \geq 2k$ and $n > 3$,*

$$\lambda_{h,k}(K_n - P_m) = \begin{cases} \left(n - \frac{m+1}{2}\right)h, & \text{if } m \text{ is odd} \\ \left(n - \frac{m}{2} - 1\right)h + k & \text{if } m \text{ is even.} \end{cases}$$

Proof. Suppose m is odd. Then $v_1, v_3, \ldots, v_m, v_{m+1}, v_{m+2}, \ldots, v_n$ is a complete graph of order $n - \frac{m-1}{2}$, which means the $\lambda_{h,k}(K_n - P_m) \geq \left(n - \frac{m-1}{2} - 1\right)h = \left(n - \frac{m+1}{2}\right)h$.

Suppose m is even, and let f be an $L(h,k)$ labeling of $K_n - P_m$ such that $|f| \leq \left(n - \frac{m}{2} - 1\right)h + k - 1$. Let g be a labeling of $H = K_n - P_{m-1}$, where $m-1$ is odd, defined as follows:

$$g(v_i) = \begin{cases} f(v_i), & \text{if } f(v_i) < f(v_m) \\ f(v_i) + (h-k), & \text{if } f(v_i) \geq f(v_m) \end{cases}$$

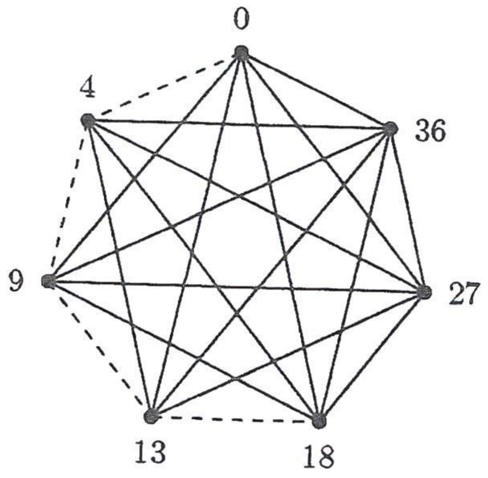
Using similar arguments as in the case of the proof of Theorem 11, g is an $L(h,k)$ labeling of $K_n - P_{m-1}$ with $|g| = |f| + h - k \leq \left(n - \frac{m}{2} - 1\right)h + k - 1 + h - k \leq \left(n - \frac{(m-1)+1}{2}\right)h - 1$, a contradiction. Thus $\lambda_{h,k}(K_n - P_m) \geq \left(n - \frac{m}{2} - 1\right)h + k$ when m is even.

Now, consider the labeling f defined as follows:

$$f(v_i) = \begin{cases} \left(\frac{i-1}{2}\right)h, & \text{if } i \text{ is odd and } i \leq m \\ f(v_{i-1}) + k, & \text{if } i \text{ is even and } i \leq m \\ f(v_{i-1}) + h, & \text{if } i > m. \end{cases}$$

Note that $f(v_j) \geq h + f(v_i)$ for $j \geq i+2$ or if $m \leq i < j \leq n$. Therefore $|f(v_i) - f(v_j)| \geq h$ for adjacent vertices v_i and v_j. Suppose $d(v_i, v_j) = 2$. Then $|i-j| = 1$, and if $j = i+1$ then $|f(v_i) - f(v_j)| = k$ and if $i = j+1$ then $|f(v_i) - f(v_j)| = h - k \geq 2k - k \geq k$. Thus f is an $L(h,k)$ labeling of $K_n - P_m$. The span of f is $\left(\frac{m-1}{2}\right)h + (n-m)h = \left(n - \frac{m+1}{2}\right)h$ if m is odd, and $\left(\frac{m}{2} - 1\right)h + k + (n-m)h = \left(n - \frac{m}{2} - 1\right)h + k$ if m is even. □

An example of an optimal $L(9,4)$ labeling of $K_7 - P_5$ shown in Figure 7 and an $L(9,4)$ labeling of $K_7 - P_6$ are shown in Figure 8.



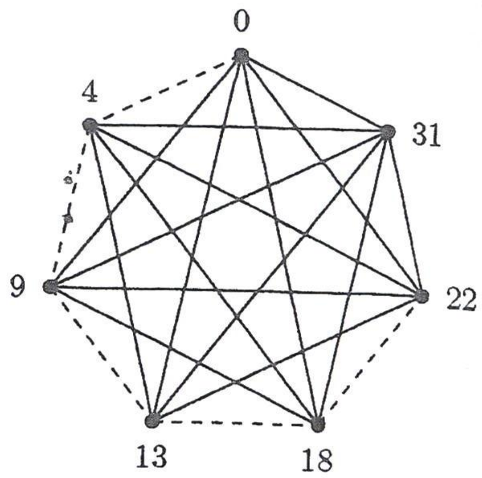


Figure 7: $L(9,4)$ labeling of $K_7 - P_5$.Figure 8: $L(9,4)$ labeling of $K_7 - P_6$.

4 Conclusion

In this paper we established the $L(h,k)$ span of all subgraphs of complete graphs obtained by either removing a maximum matching, or by removing the edges of an arbitrary path, for all non-negative integer values of h and k. In both of these classes of graphs, and especially in the case of $K_n - P_m$, we carefully partitioned the values of h and k to establish the $L(h,k)$ spans of graphs as the spans turned out to be significantly different for different ranges of values of h and k.

These results lead us to believe that finding the $L(h,k)$ span of most other subgraph classes of complete graphs might be challenging for all non-negative values of h and k. However, it would be interesting to study the $L(h,k)$ labelings of more subgraphs of complete graphs for at least some values of h and k. The graph obtained by removing the edges of a regular subgraph from a complete graph may be of particular interest.

In [7], the authors showed that $L(2r,r)$ labelings, where r is a positive real number, are equivalent to $L(2,1)$ labelings for a given graph. Since then, various classes of graphs have been studied for their $L(r,1)$ span where r is a positive real number. One interesting direction would be to study the $L(r,1)$ span of the classes of graphs we considered, for any positive real number r.

References

[1] A. A. Bertossi and M. A. Bonuccelli. Code assignment for hidden terminal interference avoidance in multihop packet radio networks. *IEEE/ACM Transactions on Networking (TON)*, 3(4):441–449, 1995.

[2] T. Calamoneri. The $L(h,k)$-labelling problem: an updated survey and annotated bibliography. *The Computer Journal*, 54(8):1344–1371, 2011.

[3] T. Fiala, Jiřıand Kloks and J. Kratochvıl. Fixed-parameter complexity of λ-labelings. *Discrete Applied Mathematics*, 113(1):59–72, 2001.

[4] P. C. Fishburn and F. S. Roberts. Frequency assignment: theory and applications. *Proceedings of IEEE*, 68(12):1497–1514, 1980.

[5] D. Fotakis, S. E. Nikoletseas, V. G. Papadopoulou, and P. G. Spirakis. Radiocoloring in planar graphs: Complexity and approximations. *Theoretical Computer Science*, 340(3):514–538, 2005.

[6] J. P. Georges and D. W. Mauro. Generalized vertex labelings with a condition at distance two. *Congressus Numerantium*, pages 141–160, 1995.

[7] J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM J. Discrete Math.*, 5(4):586–595, 1992.

[8] P. Heggernes and J. A. Telle. Partitioning graphs into generalized dominating sets. *Nord. J. Comput.*, 5(2):128–142, 1998.

[9] E. L. Lloyd and S. Ramanathan. On the complexity of distance-2 coloring. In *Computing and Information, 1992. Proceedings. ICCI'92., Fourth International Conference on*, pages 71–74. IEEE, 1992.

[10] S. T. McCormick. Optimal approximation of sparse hessians and its equivalence to a graph coloring problem. *Mathematical Programming*, 26(2):153–171, 1983.

[11] F. S. Roberts. From garbage to rainbows: generalizations of graph coloring and their applications. In *Graph theory, combinatorics, and applications. Vol. 2 (Kalamazoo, MI, 1988)*, Wiley-Intersci. Publ., pages 1031–1052. Wiley, New York, 1991.

[12] F. S. Roberts. T-colorings of graphs: recent results and open problems. *Discrete Math.*, 93(2-3):229–245, 1991.

[13] A. Sen and M. L. Huson. A new model for scheduling packet radio networks. *Wireless Networks*, 3(1):71–82, 1997.