On 2-fold Graceful Labelings

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Abstract

A long-standing conjecture by Kotzig, Ringel, and Rosa states that every tree admits a graceful labeling. That is, for any tree T with n edges, it is conjectured that there exists a labeling $f\colon V(T) \to \{0, 1, \ldots, n\}$ such that the set of induced edge labels $\big\{|f(u) - f(v)| : \{u, v\} \in E(T)\big\}$ is exactly $\{1, 2, \ldots, n\}$. We extend this concept to allow for multigraphs with edge multiplicity at most 2. A *2-fold graceful labeling* of a graph (or multigraph) G with n edges is a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, n\}$ such that the multiset of induced edge labels is comprised of two copies of each element in $\big\{1, 2, \ldots, \lfloor n/2 \rfloor\big\}$ and, if n is odd, one copy of $\big\{\lceil n/2 \rceil\big\}$. When n is even, this concept is similar to that of 2-equitable labelings which were introduced by Bloom and have been studied for several classes of graphs. We show that caterpillars, cycles of length $n \not\equiv 1 \pmod 4$, and complete bipartite graphs admit 2-fold graceful labelings. We also show that under certain conditions, the join of a tree and an empty graph (i.e., a graph with vertices but no edges) is 2-fold graceful.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \varnothing$). Let $\mathbb{N}$ denote the set of nonnegative integers, $\mathbb{Z}^+$ the set of positive integers, and $\mathbb{Z}_t$ the group of integers modulo t. For a set S and a positive integer λ, let ${}^{\lambda}S$ denote the multiset obtained from S by repeating each element λ times. Thus for example, ${}^{2}[1, 4]$ is the multiset $\{1, 1, 2, 2, 3, 3, 4, 4\}$.

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For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_t) = [0, t-1]$. The *label* of an edge $\{i, j\}$ in K_t is $|i-j|$ while the *length* of $\{i, j\}$ is $\min\{|i-j|, t-|i-j|\}$. Thus if t is odd, then K_t consists of t edges of length ℓ for $\ell \in [1, (t-1)/2]$. If t is even, then K_t consists of t edges of length ℓ for $\ell \in [1, (t-2)/2]$ and $t/2$ edges of length $t/2$.

1.1 Labelings of Simple Graphs

For any graph G, a one-to-one function $f\colon V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [11], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(v) : v \in V(G)\}$. Define a function $\bar{f}\colon E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$ and call this the set of induced edge labels. Consider the following conditions:

($\ell 1$) $f(V(G)) \subseteq [0, 2n]$,

($\ell 2$) $f(V(G)) \subseteq [0, n]$,

($\ell 3$) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

($\ell 4$) $\bar{f}(E(G)) = [1, n]$.

Then a labeling satisfying the conditions:

($\ell 1$) and ($\ell 3$) is called a *ρ-labeling*;

($\ell 1$) and ($\ell 4$) is called a *σ-labeling*;

($\ell 2$) and ($\ell 4$) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. A β-labeling is better known as a *graceful* labeling. Furthermore, if G is bipartite with vertex bipartition $\{A, B\}$ and f is a β-labeling of G where $f(a) \le \lambda < f(b)$ for all $a \in A$ and $b \in B$, then f is called an *α-labeling*, and λ is called the *boundary value* of f. Labelings of the types above are called *Rosa-type* because of Rosa's original article [11] on the topic. (See [6] for a survey of Rosa-type labelings.) A dynamic survey on general graph labelings is maintained by Gallian [7].

1.2 Labelings of Multigraphs

Let G be a multigraph (or simple graph) of size n and with edge multiplicity (at most) 2. A *2-fold graceful labeling* of G is a one-to-one function

$f\colon V(G) \to [0,n]$ such that the set of induced edge labels is

$$\{\bar{f}(e) : e \in E(G)\} = \begin{cases} {}^2[1, n/2] & \text{if } n \text{ is even,} \\ {}^2[1, (n-1)/2] \cup \{(n+1)/2\} & \text{if } n \text{ is odd.} \end{cases}$$

A graph G is said to be *2-fold graceful* if it admits a 2-fold graceful labeling.

A similar concept was introduced by Bloom [4] in 1994 in the context of k-equitable labelings and investigated further by Barrientos, Dejter, and Hevia [3] and by Mitsou [9]. (We note that Mitsou used the term "k-fold graceful", but the definition presented in [9] aligns with the more restrictive definition of "k-equitable".) A labeling of the vertices of a graph G of size kt with vertex labels from $[0, kt]$ is *k-equitable* if the set of induced edge labels has k edges labeled ℓ for each $\ell \in [1,t]$. Thus if $k = 2$, then a 2-equitable labeling is also a 2-fold graceful labeling. However, a graph with a 2-equitable labeling necessarily has even size. The 2-fold graceful concept as defined above allows for an odd number of edges. It can be shown (see [5]) that if G with n edges is 2-fold graceful, then there exists a cyclic G-decomposition of ${}^2K_{n+1}$.

2 Main Results

We investigate 2-fold graceful labelings of several classes of graphs including complete graphs, caterpillars, cycles, and complete bipartite graphs. It is known that caterpillars of even size (see [3]) and cycles of even size (see [12]) are 2-equitable and are hence 2-fold graceful. Odd cycles of length $n \equiv 1 \pmod 4$ cannot be 2-fold graceful. We show that all caterpillars, all complete bipartite graphs, and all cycles of length $n \equiv 3 \pmod 4$ are 2-fold graceful. We also show that under certain conditions, the join of a tree T and $\overline{K}_m$ is 2-fold graceful, where $\overline{K}_m$ is the empty graph on m vertices.

2.1 Complete Graphs

While it easy to see that K_2 and K_3 are 2-fold graceful, we find that these are actually the exception in that all other non-empty complete graphs are not 2-fold graceful.

Theorem 1. *The complete graph K_v is 2-fold graceful if and only if $v \in \{1,2,3\}$.*

Proof. The sufficiency of $v \in \{1,2,3\}$ is clear from labeling the vertex of K_1 with label 0; labeling the vertices of K_2 with labels 0 and 1; and labeling the vertices of K_3 with labels 0, 1, and 2. To show necessity, we let $v \geq 4$ and assume there exists a 2-fold graceful labeling, say f, of K_v. Let

$V(K_v) = \{u_1, u_2, \ldots, u_v\}$ and let $n = v(v-1)/2$, i.e., the size of K_v, which is even for $v \equiv 0, 1 \pmod 4$ and odd otherwise. Without loss of generality, we may assume that the minimum vertex label is $f(u_1) = 0$. Since each vertex is adjacent to u_1, the maximum vertex label is thus no more than $\lceil n/2 \rceil$, i.e., the largest possible edge label. If n is even, then there must be two edges with label $n/2$; however, this is impossible since at most one vertex can have label $n/2$ and all other vertex labels are between 0 and $n/2 - 1$, inclusively.

Next, we consider when n odd. Note that in this case $v \geq 6$, and thus $n \geq 15$. In this case, there is exactly one edge with label $(n+1)/2$ and two edges of each (unique) label in the set $\{(n-1)/2,\ (n-3)/2,\ 1\}$. We may again assume, without loss of generality, that the maximum vertex label is $f(u_2) = (n+1)/2$. (In fact, because K_v is vertex transitive, we continue to assume without loss of generality that any given vertex has a required vertex label.) Now, in order to achieve two edges with label $(n-1)/2$, we must have vertices, say u_3 and u_4, with labels 1 and $(n-1)/2$. Then, the multiset of labels on the edges in the set $\{\{u_1, u_2\},\ \{u_1, u_3\},\ \{u_2, u_4\},\ \{u_2, u_3\},\ \{u_1, u_4\},\ \{u_3, u_4\}\}$ is $\{(n+1)/2,\ 1,\ 1,\ (n-1)/2,\ (n-1)/2,\ (n-3)/2\}$. Note that we still need another edge with label $(n-3)/2$, and the only possible values for $f(u_5)$ are between 1 and $(n-1)/2$, exclusively. This forces $f(u_5)$ to be either $(n-3)/2$ or 2 so that either edge $\{u_1, u_5\}$ or edge $\{u_2, u_5\}$, respectively, has edge label $(n-3)/2$. However, in both situations, we end up with another edge (either $\{u_2, u_5\}$ or $\{u_1, u_5\}$, respectively) with edge label 1. This contradiction concludes the proof. ∎

2.2 Trees and Caterpillars

El-Zanati has conjectured that all trees are 2-fold graceful. We have verified this conjecture for all trees on up to 11 vertices. Because of space constraints, we will not list those labelings here. Next we show that all caterpillars are 2-fold graceful. As stated previously, this result is known for even size caterpillars [3], a special case of the results of this paper.

A tree is a called a *caterpillar* if the subgraph induced by the non-degree 1 vertices is either empty or a path. In the latter case, the induced path is called the *spine* of the caterpillar (where we are allowing a path to be of length 0). We call an α-labeling of a caterpillar *standard* if it is of the form described in [11], which has the following properties:

- 0 and 1 are the vertex labels if the caterpillar is a K_2,
- 0 is the label of an endpoint of the spine otherwise, and
- the largest vertex label is not on the spine.

Hence for a nontrivial caterpillar the largest vertex label is on a degree 1 vertex that is adjacent to the endpoint of the spine with label 0.

Let f be a β-labeling of a graph G with n edges. The labeling f' of G defined by $f'(v) = n - f(v)$ is called the *complementary labeling* of f. Note that a complementary labeling of a β- or α-labeling is necessarily also a β- or α-labeling, respectively. Furthermore, the complementary labeling of a standard α-labeling of a caterpillar has the largest vertex label on an endpoint of the spine.

Theorem 2. *All caterpillars are 2-fold graceful.*

Proof. Let G be a caterpillar of size n. If $n = 1$ (i.e., $G \cong K_2$), then the standard α-labeling of G is also a 2-fold graceful labeling. For the remainder of the proof, we assume $n \geq 2$ (hence, there exists a spine with at least one vertex). Let G_1 and G_2 be edge-disjoint subgraphs of G such that

- G_1 and G_2 are caterpillars,
- $|E(G_1)| = \lceil n/2 \rceil$ and $|E(G_2)| = \lfloor n/2 \rfloor$,
- $V(G_1) \cap V(G_2) = \{v\}$.

Since G_1 and G_2 are edge-disjoint, the common vertex v must be on the spine of G. Furthermore, for each $i \in \{1, 2\}$, vertex v is either an endpoint of the spine of G_i or a degree 1 vertex in G_i adjacent to an endpoint of the spine (if it exists) of G_i.

If v is not an endpoint of the spine of G_1, then let f_1 be a standard α-labeling of G_1 such that $f_1(v) = |E(G_1)|$. Otherwise, let f_1 be the complementary labeling of a standard α-labeling of G_1 such that $f_1(v) = |E(G_1)|$.

On the other hand, if v is not an endpoint of the spine of G_2, then let f_2 be the complementary labeling of a standard α-labeling of G_2 such that $f_2(v) = 0$. Otherwise, let f_2 be a standard α-labeling of G_2 such that $f_2(v) = 0$.

Now, define a labeling $f\colon V(G) \to [0, n]$ by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in V(G_1) \setminus \{v\}, \\ f_2(x) + \lceil n/2 \rceil & \text{if } x \in V(G_2). \end{cases}$$

An example of a caterpillar with the described labeling f can be seen in Figure 1. The minimum and maximum of $f(V(G))$ respectively are

$$\min f_1(V(G_1)) = 0, \quad \max f_2(V(G_2)) + \lceil n/2 \rceil = \lfloor n/2 \rfloor + \lceil n/2 \rceil = n.$$

Note that $f(V(G_1))$ is disjoint from $f(V(G_2))$ except $f(v) = f_2(v) + \lceil n/2 \rceil = 0 + \lceil n/2 \rceil = f_1(v)$.

Finally, the set of induced edge labels are

$$\bar{f}(E(G_1)) = \bar{f}_1(E(G_1)) = \big[1, \lceil n/2 \rceil\big]$$

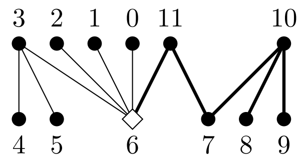

Figure 1: A caterpillar with the 2-fold graceful labeling described in the proof for Theorem 2. Subgraphs G_1 and G_2, as defined in the proof, are shown with thin and thick lines, respectively, and v is the vertex with the square shape.

and

$$\bar{f}(E(G_2)) = \bar{f}_2(E(G_2)) = \big[1, \lfloor n/2 \rfloor\big].$$

Therefore, by definition, f is a 2-fold graceful labeling of G. ■

These results provide further evidence in support of the following conjecture by El-Zanati.

Conjecture 1. *Every tree is 2-fold graceful.*

2.3 Cycles

If every vertex in a graph G has even degree, then it is known that in any Rosa-type labeling of G the number of edges with odd labels must be even. This is known as the *parity condition* (see [6]). Thus cycles of the form C_{4r+1} cannot be graceful or 2-fold graceful. As stated previously, even cycles are 2-equitable [12] and hence 2-fold graceful. It remains to be shown that C_{4r+3} is 2-fold graceful.

First, we note the following result by Abrham and Kotzig [1] regarding α-labelings of paths.

Lemma 3. *Let k be a positive integer and let f be an α-labeling of a path of length n with the endvertices w and z such that $f(w) < f(z)$. Then the following relations hold:*

I. $f(w) + f(z) = k$ *if* $n = 2k$ *and* $f(w) \leq k - 1$,

II. $f(w) + f(z) = 3k$ *if* $n = 2k$ *and* $f(w) \geq k$,

III. $f(z) - f(w) = k$ *if* $n = 2k - 1$.

The removal of any edge in C_{4r+3} yields a path of length $4r + 2$, which is necessarily a caterpillar. Applying the results from the previous theorem and lemma, we can show the following.

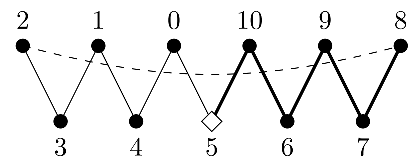

Figure 2: A C_{11} with the 2-fold graceful labeling described in the proof for Theorem 4. Subgraphs G_1 and G_2, as defined in the proof, are shown with thin and thick lines, respectively, v is the vertex with the square shape, and edge $\{w, z\}$ is shown with a dashed line.

Theorem 4. *For all $r \in \mathbb{N}$, the cycle C_{4r+3} is 2-fold graceful.*

Proof. Let r be a nonnegative integer and let G be a path of length $4r + 2$ with end vertices w and z. Since G is a caterpillar, we define G_1, G_2, v, f_1, f_2, and f as in the proof for Theorem 2. Note the following 3 items: (i) v is distinct from both w and z, (ii) both G_1 and G_2 are paths of length $2r + 1$, and (iii) v is an end vertex of both G_1 and G_2. Without loss of generality, we assume that w is an end vertex of G_1, which implies that $f_1(w) < f_1(v) = 2r + 1$ and $f_2(z) > f_2(v) = 0$. Thus, by Lemma 3, we have that $f_1(v) - f_1(w) = r + 1 = f_2(z) - f_2(v)$, which implies that $f_1(w) = f_1(v) - r - 1 = r$ and $f_2(z) = f_2(v) + r + 1 = r + 1$.

Finally, consider the graph G' with vertex set $V(G)$ and edge set $E(G) \cup \big\{\{w, z\}\big\}$. Clearly, G' is a cycle of length $4r+3$ with $f\big(V(G')\big) \subseteq [0, 4r+3]$. An example of C_{11} with the described labeling f can be seen in Figure 2. The set of edge labels of G' induced by f include $\bar{f}(E(G_1)) = [1, 2r + 1]$ and $\bar{f}(E(G_2)) = [1, 2r + 1]$. Furthermore, the induced edge label on $\{w, z\}$ is

$$f(z) - f(w) = \big(f_2(z) + 2r + 1\big) - f_1(w) = (r + 1) + 2r + 1 - (r) = 2r + 2.$$

Therefore, by definition, f is a 2-fold graceful labeling of G'. ■

2.4 Complete Bipartite Graphs

Next we show that complete bipartite graphs are 2-fold graceful.

Theorem 5. *All complete bipartite graphs are 2-fold graceful.*

Proof. Let G be the complete bipartite graph $K_{m,n}$ with vertex bipartition $\{U, V\}$ where $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. If either m or n is 1, then $K_{m,n}$ is a caterpillar, and the result follows from Theorem 2. Thus, we assume both m and n are at least 2.

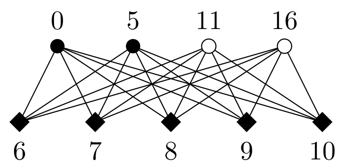

Figure 3: A $K_{4,5}$ with the 2-fold graceful labeling described in Case 1 of the proof for Theorem 5. Vertex sets A_1, A_2, and V, as defined in the proof, are shown with differing vertex shapes: black circles, white circles, and black squares, respectively.

Case 1: m is even.
Let $m = 2k$ where $k \geq 1$, let A_1 denote the vertex set $\{u_1, u_2, \ldots, u_k\}$, and let A_2 denote the vertex set of $\{u_{k+1}, u_{k+2}, \ldots, u_{2k}\}$. Now, define a labeling $f\colon V(G) \to [0, 2kn]$ by

$$f(x) = \begin{cases} (i-1)n & \text{if } x = u_i \in A_1, \\ (i-1)n + 1 & \text{if } x = u_i \in A_2, \\ (k-1)n + i & \text{if } x = v_i \in V. \end{cases}$$

An example of $K_{4,5}$ with the described labeling f can be seen in Figure 3. The minimum value in $f(A_1)$ is 0, and the maximum value is $(k-1)n$. The minimum value in $f(A_2)$ is $kn+1$, and the maximum value is $(2k-1)n+1$. The minimum value of $f(V)$ is $(k-1)n+1$, and the maximum value is kn. Thus, the vertex labels are all unique because

$$f(A_1) < f(V) < f(A_2).$$

Consider the edge labeling induced by f. By the division algorithm, there exist unique integers q and r such that $\ell = qn + r$ with $0 \leq q \leq k-1$ and $1 \leq r \leq n$. Then ℓ is the induced edge label on $\{u_{k-q}, v_r\}$ and $\{u_{k+1+q}, v_{n+1-r}\}$ because

$$\begin{aligned} f(v_r) - f(u_{k-q}) &= \big((k-1)n + r\big) - \big((k-q-1)n\big) \\ &= kn - n + r - kn + qn + n \\ &= qn + r \end{aligned}$$

and

$$\begin{aligned} f(u_{k+1+q}) - f(v_{n+1-r}) &= \big((k+1+q-1)n + 1\big) \\ &\qquad - \big((k-1)n + n + 1 - r\big) \\ &= kn + n + qn - n + 1 - kn + n - n - 1 + r \\ &= qn + r. \end{aligned}$$

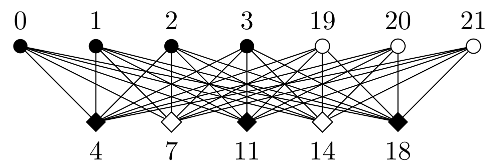

Figure 4: A $K_{7,5}$ with the 2-fold graceful labeling described in Case 2 of the proof for Theorem 5. Vertex sets A_1, A_2, B_1, and B_2, as defined in the proof, are shown with differing vertex shapes: black circles, white circles, black squares, and white squares, respectively.

Therefore, by definition, f is a 2-fold graceful labeling of $K_{2k,n}$.

CASE 2: m and n are odd.
Let $m = 2k+1$ and let $n = 2s+1$ where $k, s \in \mathbb{N}$. We also define the following vertex sets:

$$A_1 = \{u_1, u_2, \ldots, u_{k+1}\}, \qquad B_1 = \{v_1, v_3, v_5, \ldots, v_{2s+1}\},$$
$$A_2 = \{u_{k+2}, u_{k+3}, \ldots, u_{2k+1}\}, \qquad B_2 = \{v_2, v_4, v_6, \ldots, v_{2s}\}.$$

Now, define a labeling $f\colon V(G) \to \big[0, (2k+1)(2s+1)\big]$ by

$$f(x) = \begin{cases} i-1 & \text{if } x = u_i \in A_1, \\ (2k+1)s + i & \text{if } x = u_i \in A_2, \\ \frac{i+1}{2} \cdot (2k+1) - k & \text{if } x = v_i \in B_1, \\ \frac{i}{2} \cdot (2k+1) & \text{if } x = v_i \in B_2. \end{cases}$$

An example of $K_{7,5}$ with the described labeling f can be seen in Figure 4. The minimum value of $f(A_1)$ is 0, and the maximum value is k. The minimum value of $f(A_2)$ is $(2k+1)s + k + 2$, and the maximum value is $(2k+1)s+2k+1$. The minimum value of $f(B_1)$ is $k+1$, and the maximum value is $(2s+2)/2 \cdot (2k+1) - k = (2k+1)s + k + 1$. The minimum value of $f(B_2)$ is $2k+1$, and the maximum value is $(2k+1)s$. Also,

$$f(B_1) = \{t \cdot (2k+1) - k : 0 \leq t \leq s+1\},$$
$$f(B_2) = \{t \cdot (2k+1) \qquad : 0 \leq t \leq s\},$$

and since $t_1 \cdot (2k+1) - k = t_2 \cdot (2k+1)$ has no solution with t_1 and t_2 both integers, $f(B_1)$ and $f(B_2)$ are disjoint. Hence, the minimum value of $f(V)$ is $k+1$ and the maximum value is $(2k+1)s + k + 1$, and the vertex labels are all unique because

$$f(A_1) < f(V) < f(A_2).$$

Next, suppose we look for the largest edge label $\lceil (2k+1)(2s+1)/2 \rceil = (2k+1)s+k+1$. This is the induced edge label on $\{u_1, v_{2s+1}\}$ because

$$\begin{aligned} f(v_{2s+1}) - f(u_1) &= \big((2s+2)/2 \cdot (2k+1) - k\big) - (0) \\ &= (2k+1)s+k+1. \end{aligned}$$

Suppose we look for an edge label among the next largest edge labels: $\ell \in \big[(2k+1)s+1, (2k+1)s+k\big]$. That is, suppose $\ell = (2k+1)s+r$ where $1 \le r \le k$. Then ℓ is the induced edge label on $\{u_{k+2-r}, v_{2s+1}\}$ and $\{u_{k+1+r}, v_1\}$ because

$$\begin{aligned} f(v_{2s+1}) - f(u_{k+2-r}) &= \big((2s+2)/2 \cdot (2k+1) - k\big) - (k+2-r-1) \\ &= (2k+1)s+k+1-k-2+r+1 \\ &= (2k+1)s+r \end{aligned}$$

and

$$\begin{aligned} f(u_{k+1+r}) - f(v_1) &= \big((2k+1)s+k+1+r\big) - \big(1 \cdot (2k+1) - k\big) \\ &= (2k+1)s+k+1+r-2k-1+k \\ &= (2k+1)s+r. \end{aligned}$$

Finally, suppose we look for an edge label $\ell \in \big[1, (2k+1)s\big]$. By the division algorithm, there exist unique integers q and r such that $\ell = (2k+1)q+r$ with $0 \le q \le s-1$ and $1 \le r \le 2k+1$. If $1 \le r < k+1$, then ℓ is the induced edge label on $\{u_{k+2-r}, v_{2q+1}\}$ and $\{u_{k+1+r}, v_{2s+1-2q}\}$ because

$$\begin{aligned} f(v_{2q+1}) - f(u_{k+2-r}) &= \big((2q+2)/2 \cdot (2k+1) - k\big) - (k+2-r-1) \\ &= (2k+1)q+k+1-k-2+r+1 \\ &= (2k+1)q+r \end{aligned}$$

and

$$\begin{aligned} f(u_{k+1+r}) - f(v_{2s+1-2q}) &= \big((2k+1)s+k+1+r\big) \\ &\qquad - \big((2s+2-2q)/2 \cdot (2k+1) - k\big) \\ &= (2k+1)s+k+1+r \\ &\qquad - (2k+1)s+(2k+1)q-k-1 \\ &= (2k+1)q+r. \end{aligned}$$

If $r = k+1$, then ℓ is the induced edge label on $\{u_1, v_{2q+1}\}$ and $\{u_{k+1}, v_{2q+2}\}$ because

$$\begin{aligned} f(v_{2q+1}) - f(u_1) &= \big((2q+2)/2 \cdot (2k+1) - k\big) - (0) \\ &= (2k+1)q+k+1 \\ &= (2k+1)q+r \end{aligned}$$

and

$$\begin{aligned} f(v_{2q+2}) - f(u_{k+1}) &= \big((2q+2)/2 \cdot (2k+1)\big) - (k+1-1) \\ &= (2k+1)q + 2k + 1 - k \\ &= (2k+1)q + r. \end{aligned}$$

If $k+1 < r \leq 2k+1$, then ℓ is the induced edge label on $\{u_{2k+2-r}, v_{2q+2}\}$ and $\{u_r, v_{2s-2q}\}$ because

$$\begin{aligned} f(v_{2q+2}) - f(u_{2k+2-r}) &= \big((2q+2)/2 \cdot (2k+1)\big) - (2k+2-r-1) \\ &= (2k+1)q + 2k + 1 - 2k - 2 + r + 1 \\ &= (2k+1)q + r \end{aligned}$$

and

$$\begin{aligned} f(u_r) - f(v_{2s-2q}) &= \big((2k+1)s + r\big) - \big((2s-2q)/2 \cdot (2k+1)\big) \\ &= (2k+1)s + r - (2k+1)s + (2k+1)q \\ &= (2k+1)q + r. \end{aligned}$$

Hence, the set of induced edge labels contains ${}^2[1, (2k+1)s+k] \cup \big\{(2k+1)s+k+1\big\}$, and by the counting principle, this must be the entire set of edge labels. Therefore, by definition, f is a 2-fold graceful labeling of $K_{2k+1,2s+1}$. ■

2.5 Joins of Trees and Empty Graphs

The *join* of graphs G and H, denoted $G \vee H$, is the graph obtained from the vertex-disjoint union of G and H together with all edges joining the vertices of G with the vertices of H. That is, $G \vee H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$, where $V(G) \cap V(H) = \varnothing$. In [8], Koh, Rogers, and Lim proved that $T \vee \overline{K}_m$ is graceful if T is a graceful tree. We prove an analogous result.

Theorem 6. *Let T be a tree and let m be a positive integer.*

- *If T is graceful and m is odd, then $T \vee \overline{K}_m$ is 2-fold graceful.*
- *If T is 2-fold graceful and m is even, then $T \vee \overline{K}_m$ is 2-fold graceful.*

As a corollary of this result, several classes of graphs are proven to be 2-fold graceful. For example, paths are both graceful and 2-fold graceful, and so the fan graph $P_n \vee \overline{K}_m$ is 2-fold graceful for any positive integers n and m. If the conjectures that every tree is graceful and that every tree is

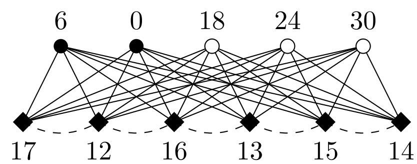

Figure 5: The join of P_6 and $\overline{K}_5$ with the 2-fold graceful labeling described in the proof for the first statement in Theorem 6, where the standard α-labeling is used as the initial graceful labeling of the P_6 shown here with dashed lines for edges. Vertex sets $V(P_6)$, A, and B, as defined in the proof, are shown with differing vertex shapes: black squares, black circles, and white circles, respectively.

2-fold graceful hold, this theorem shows $T \vee \overline{K}_m$ is 2-fold graceful for any tree T and positive integer m.

The result here can be extended to greater generality. If we replace in the above statement the tree T with any graph G such that $|V(G)| - |E(G)| = 1$ (such a graph that is not a tree is necessarily disconnected), then the proof below still applies. In [2], Acharya shows how such a graph can be constructed by adding disconnected vertices to any graceful graph. Using a similar construction and the above theorem implies that, given any graceful or 2-fold graceful graph H, infinitely many 2-fold graceful graphs exists with H as a subgraph.

Proof of Theorem 6, first statement. Suppose m is odd and T is a graceful tree with n vertices. Let $m = 2k + 1$ where $k \geq 0$ and let $G = T \vee \overline{K}_{2k+1}$. Then G has $n - 1 + (2k + 1)n = 2kn + 2n - 1$ edges. Let g be a graceful labeling of T with $\bar{g}$ the edge labeling induced by g. Note in particular that $g(V(T)) = [0, n - 1]$ and that the size of G is odd. If $k = 0$, let $V(\overline{K}_{2k+1}) = \{v_1\} = B$; otherwise, let $\{A, B\}$ be a partition of $V(\overline{K}_{2k+1})$ with $A = \{u_1, u_2, \ldots, u_k\}$ and $B = \{v_1, v_2, \ldots, v_{k+1}\}$. Now, define a labeling $f \colon V(G) \to [0, 2kn + 2n - 1]$ by

$$f(x) = \begin{cases} g(x) + kn & \text{if } x \in V(T), \\ (k - i)n & \text{if } x = u_i \in A, \\ (k + i)n & \text{if } x = v_i \in B. \end{cases}$$

An example of the described labeling f can be seen in Figure 5. The minimum value of $f(V(T))$ is kn, and the maximum value is $n - 1 + kn$. The minimum value of $f(A)$ is 0, and the maximum value is $(k - 1)n$. The minimum value of $f(B)$ is $(k + 1)n$, and the maximum value is $(2k + 1)n$.

Thus, the vertex labels are all unique because

$$f(A) < f(V(T)) < f(B).$$

Consider the edge labeling $\bar{f}$ induced by f. First, for $e = \{x_1, x_2\} \in E(T)$, we have

$$\begin{aligned}\bar{f}(e) &= |f(x_1) - f(x_2)| \\ &= |(g(x_1) + kn) - (g(x_2) + kn)| \\ &= |g(x_1) - g(x_2)| \\ &= \bar{g}(e).\end{aligned}$$

Hence $\bar{f}(E(T)) = \bar{g}(E(T)) = [1, n-1]$. Second, for an edge e incident with $u_i \in A$, we have

$$\begin{aligned}\bar{f}(e) &= f(x) - f(u_i) \\ &= (g(x) + kn) - ((k-i)n) \\ &= g(x) + i \cdot n.\end{aligned}$$

Since $g(V(T)) = [0, n-1]$, the set of labels on all edges incident with $u_i \in A$ is thus $[i \cdot n, (i+1)n - 1]$. Hence, the set of edge labels on all such edges incident with a vertex in A is

$$\bigcup_{i=1}^{k} [i \cdot n, (i+1)n - 1] = [n, kn + n - 1].$$

Third, for an edge e incident with $v_i \in B$, we have

$$\begin{aligned}\bar{f}(e) &= f(v_i) - f(x) \\ &= ((k+i)n) - (g(x) + kn) \\ &= i \cdot n - g(x).\end{aligned}$$

Since $g(V(T)) = [0, n-1]$, the set of labels on all edges incident with $v_i \in B$ is thus $[(i-1)n + 1, i \cdot n]$. Hence, the set of edge labels on all such edges incident with a vertex in B is

$$\bigcup_{i=1}^{k+1} [(i-1)n + 1, i \cdot n] = [1, kn + n].$$

Therefore, the set of all edge labels is ${}^2[1, kn + n - 1] \cup \{kn + n\}$, and f is a 2-fold graceful labeling of G. ■

The proof for the latter statement in Theorem 6 is similar to that of the former, but covered in two cases depending on the parity of the size/order of the tree T.

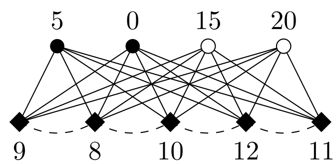

Figure 6: The join of P_5 and $\overline{K}_4$ with the 2-fold graceful labeling described in Case 1 of the proof for the second statement in Theorem 6, where the 2-fold graceful labeling described in the proof for Theorem 2 is used as the initial 2-fold graceful labeling of the P_5 shown here with dashed lines for edges. Vertex sets $V(P_5)$, A, and B, as defined in the proof, are shown with differing vertex shapes: black squares, black circles, and white circles, respectively.

Proof of Theorem 6, second statement. Suppose m is even and T is a 2-fold graceful tree with n vertices. Let $m = 2k$ where $k \geq 1$, let $G = T \vee \overline{K}_{2k}$, and let g be a 2-fold graceful labeling of T with $\bar{g}$ the edge labeling induced by g. We partition $V(\overline{K}_{2k})$ into sets $A = \{u_1, u_2, \ldots, u_k\}$ and $B = \{v_1, v_2, \ldots, v_k\}$.

Case 1: n is odd.

Let $n = 2r+1$ where $r \geq 0$. Then G has $2r+2k(2r+1) = 4kr+2k+2r$ edges. Note in particular that $g(V(T)) = [0, 2r]$ and that the size of G is even. Now, define a labeling $f\colon V(G) \to [0, 4kr + 2k + 2r]$ by

$$f(x) = \begin{cases} g(x) + k(2r+1) - r & \text{if } x \in V(T), \\ (k-i)(2r+1) & \text{if } x = u_i \in A, \\ (k+i)(2r+1) & \text{if } x = v_i \in B. \end{cases}$$

An example of the described labeling f can be seen in Figure 6. The minimum value of $f(V(T))$ is $k(2r+1) - r$, and the maximum value is $k(2r+1) + r$. The minimum value of $f(A)$ is 0, and the maximum value is $(k-1)(2r+1)$. The minimum value of $f(B)$ is $(k+1)(2r+1)$, and the maximum value is $2k(2r+1)$. Thus, the vertex labels are all unique because

$$f(A) < f(V(T)) < f(B).$$

Consider the edge labeling $\bar{f}$ induced by f. First, for $e = \{x_1, x_2\} \in$

$E(T)$, we have

$$\begin{aligned}\bar{f}(e) &= |f(x_1) - f(x_2)| \\ &= |(g(x_1) + k(2r+1) - r) - (g(x_2) + k(2r+1) - r)| \\ &= |g(x_1) - g(x_2)| \\ &= \bar{g}(e).\end{aligned}$$

Hence $\bar{f}(E(T)) = \bar{g}(E(T)) = {}^2[1, r]$. Second, for an edge e incident with $u_i \in A$, we have

$$\begin{aligned}\bar{f}(e) &= f(x) - f(u_i) \\ &= (g(x) + k(2r+1) - r) - ((k-i)(2r+1)) \\ &= g(x) + i(2r+1) - r.\end{aligned}$$

Since $g(V(T)) = [0, 2r]$, the set of labels on all edges incident with $u_i \in A$ is thus $[i(2r+1) - r, i(2r+1) + r] = [(i-1)(2r+1) + r + 1, i(2r+1) + r]$. Hence, the set of edge labels on all such edges incident with a vertex in A is

$$\bigcup_{i=1}^{k} [(i-1)(2r+1) + r + 1, i(2r+1) + r] = [r+1, 2kr + k + r].$$

Third, for an edge e incident with $v_i \in B$, we have

$$\begin{aligned}\bar{f}(e) &= f(v_i) - f(x) \\ &= ((k+i)(2r+1)) - (g(x) + k(2r+1) - r) \\ &= i(2r+1) + r - g(x).\end{aligned}$$

Since $g(V(T)) = [0, 2r]$, the set of labels on all edges incident with $v_i \in B$ is thus $[i(2r+1) - r, i(2r+1) + r] = [(i-1)(2r+1) + r + 1, i(2r+1) + r]$. Hence, the set of edge labels on all such edges incident with a vertex in B is

$$\bigcup_{i=1}^{k} [(i-1)(2r+1) + r + 1, i(2r+1) + r] = [r+1, 2kr + k + r].$$

Therefore, the set of all edge labels is ${}^2[1, 2kr + k + r]$, and f is a 2-fold graceful labeling of G.

Case 2: n is even.
Let $n = 2r$ where $r \geq 1$. Then G has $2r - 1 + 2k(2r) = 4kr + 2r - 1$ edges. Note in particular that $g(V(T)) = [0, 2r-1]$ and that the size of G is odd.

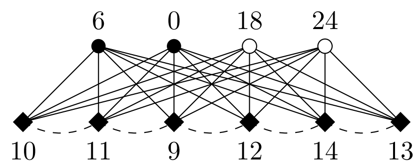

Figure 7: The join of P_6 and $\overline{K}_4$ with the 2-fold graceful labeling described in Case 2 of the proof for the second statement in Theorem 6, where the 2-fold graceful labeling described in the proof for Theorem 2 is used as the initial 2-fold graceful labeling of the P_5 shown here with dashed lines for edges. Vertex sets $V(P_5)$, A, and B, as defined in the proof, are shown with differing vertex shapes: black squares, black circles, and white circles, respectively.

Now, define a labeling $f\colon V(G) \to [0, 4kr + 2r - 1]$ by

$$f(x) = \begin{cases} g(x) + 2kr - r & \text{if } x \in V(T), \\ (k-i) \cdot 2r & \text{if } x = u_i \in A, \\ (k+i) \cdot 2r & \text{if } x = v_i \in B. \end{cases}$$

An example of the described labeling f can be seen in Figure 7. The minimum value of $f(V(T))$ is $2kr - r$, and the maximum value is $2kr + r - 1$. The minimum value of $f(A)$ is 0, and the maximum value is $2kr - 2r$. The minimum value of $f(B)$ is $2kr + 2r$, and the maximum value is $4kr$. Thus, the vertex labels are all unique because

$$f(A) < f(V(T)) < f(B).$$

Consider the edge labeling $\bar{f}$ induced by f. First, for $e = \{x_1, x_2\} \in E(T)$, we have

$$\begin{aligned} \bar{f}(e) &= |f(x_1) - f(x_2)| \\ &= |(g(x_1) + 2kr - r) - (g(x_2) + 2kr - r)| \\ &= |g(x_1) - g(x_2)| \\ &= \bar{g}(e). \end{aligned}$$

Hence $\bar{f}(E(T)) = \bar{g}(E(T)) = {}^2[1, r-1] \cup \{r\}$. Second, for an edge e incident with $u_i \in A$, we have

$$\begin{aligned} \bar{f}(e) &= f(x) - f(u_i) \\ &= (g(x) + 2kr - r) - ((k-i) \cdot 2r) \\ &= g(x) + i \cdot 2r - r. \end{aligned}$$

Since $g(V(T)) = [0, 2r-1]$, the set of labels on all edges incident with $u_i \in A$ is thus $[i \cdot 2r - r, (i+1) \cdot 2r - r - 1]$. Hence, the set of edge labels on all such edges incident with a vertex in A is

$$\bigcup_{i=1}^{k} [i \cdot 2r - r, (i+1) \cdot 2r - r - 1] = [r, 2kr + r - 1].$$

Third, for an edge e incident with $v_i \in B$, we have

$$\begin{aligned} \bar{f}(e) &= f(v_i) - f(x) \\ &= ((k+i) \cdot 2r) - (g(x) + 2kr - r) \\ &= i \cdot 2r + r - g(x). \end{aligned}$$

Since $g(V(T)) = [0, 2r-1]$, the set of labels on all edges incident with $v_i \in B$ is thus $[(i-1) \cdot 2r + r + 1, i \cdot 2r + r]$. Hence, the set of edge labels on all such edges incident with a vertex in B is

$$\bigcup_{i=1}^{k} [(i-1) \cdot 2r + r + 1, i \cdot 2r + r] = [r+1, 2kr + r].$$

Therefore, the set of all edge labels is ${}^2[1, 2kr + r - 1] \cup \{2kr + r\}$, and f is a 2-fold graceful labeling of G. ■

3 Labelings of Cubic Multigraphs

Next we investigate 2-fold graceful labelings of cubic graphs and cubic multigraphs, which necessarily must have edge multiplicity at most 2 in order to be 2-fold graceful. We specifically focus on the cubic graphs and multigraphs of order at most 8. There are 9 such simple graphs, only one of which is disconnected, and 22 multigraphs, only two of which are disconnected. We show that every cubic graph and multigraph of order at most 8 is 2-fold graceful except for K_4. We list all such multigraphs along with 2-fold graceful labelings in Tables 1–4, and we employ a naming scheme inspired by that found in [10] for the connected cubic graphs.

Acknowledgments

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Table 1: Connected cubic simple graphs of order at most 8 shown with a 2-fold graceful labeling where possible.

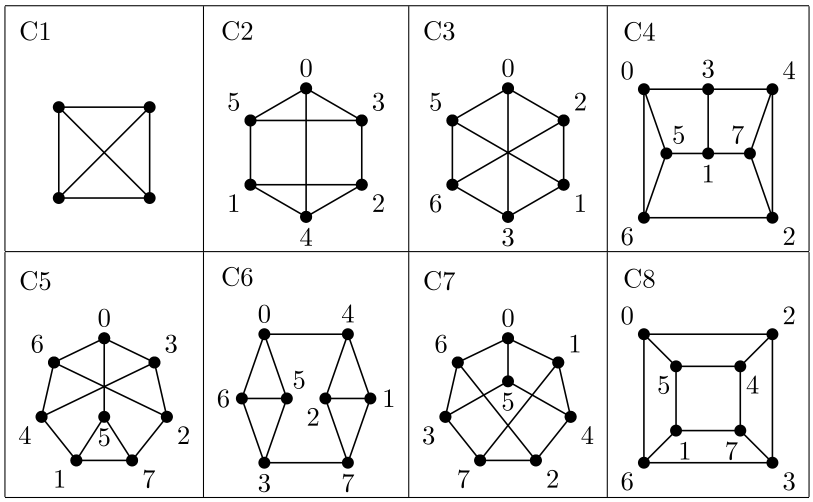


Table 2: Connected cubic multigraphs of order at most 6 shown with a 2-fold graceful labeling.

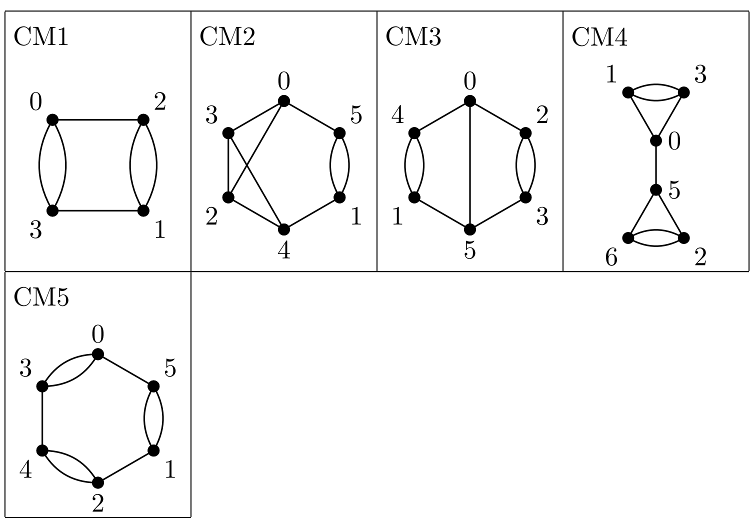


Table 3: Connected cubic multigraphs of order 8 shown with a 2-fold graceful labeling.

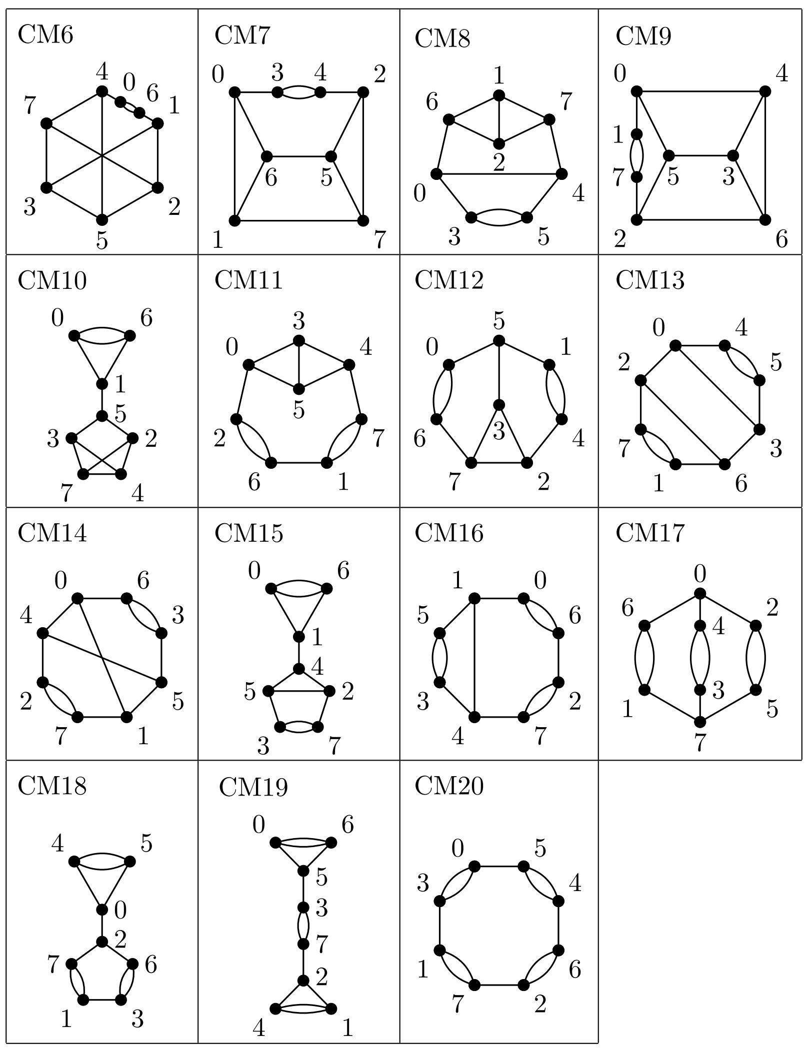
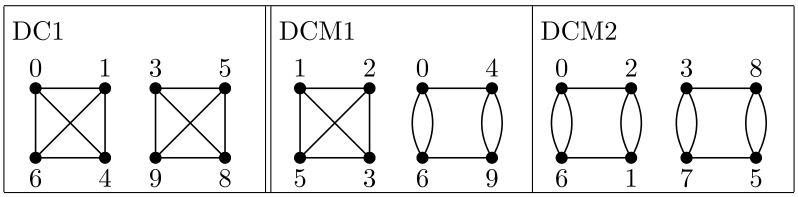


Table 4: All disconnected cubic graphs and multigraphs of order 8 shown with a 2-fold graceful labeling.



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