On Tree-Connection of Generalized Line Graphs

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Abstract

The line graph $L(G)$ of a nonempty graph G has the set of edges in G as its vertex set where two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. Let $k \geq 2$ be an integer and let G be a graph containing k-paths (paths of order k). The k-path graph $\mathcal{P}_k(G)$ of G has the set of k-paths of G as its vertex set where two distinct vertices of $\mathcal{P}_k(G)$ are adjacent if the corresponding k-paths of G have a $(k-1)$-path in common. Thus, $\mathcal{P}_2(G) = L(G)$ and $\mathcal{P}_3(G) = L(L(G))$. Hence, the k-path graph $\mathcal{P}_k(G)$ of a graph G is a generalization of the line graph $L(G)$. Let G be a connected graph of order $n \geq 3$ and let k be an integer with $2 \leq k \leq n-1$. The graph G is k-tree-connected if for every set S of k distinct vertices of G, there exists a spanning tree T of G whose set of end-vertices is S. Thus, G is 2-tree-connected if and only if G is Hamiltonian-connected. It was conjectured that if T is a tree of sufficiently large order containing no vertices of any of the degrees 2, 3, ..., $k+1$ for an integer $k \geq 2$, then $\mathcal{P}_3(T)$ is k-tree-connected. This conjecture was verified for $k = 2, 3$. In this work, we show that if T is a tree of order at least 6 containing no vertices of degree 2, 3, 4, or 5, then $\mathcal{P}_3(T)$ is 4-tree-connected and so verify the conjecture for the case when $k = 4$.

1 Introduction

There are many graphs associated with a given graph. Such graphs are referred to as "derived graphs". For a given graph G, a *derived graph* of G is a graph obtained from G by a graph operation of some type. The study of the structural properties of derived graphs is a popular area of research in graph theory. One of the most familiar graph operations on a graph is that of the line graph. The *line graph* $L(G)$ of a nonempty graph G has the set of edges in G as its vertex set where two vertices of $L(G)$

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are adjacent if the corresponding edges of G are adjacent. More generally, for a nonempty graph G, we write $L^0(G)$ to denote G and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the *kth iterated line graph* $L^k(G)$ is defined as $L(L^{k-1}(G))$, where $L^{k-1}(G)$ is assumed to be nonempty. In particular, $L(L(G)) = L^2(G)$. Over the years, various generalizations of line graphs have been introduced and studied by many (see [1, 9], for example).

Another more general class of derived graphs was inspired by line graphs. Observe that the vertex set of the line graph $L(G)$ is the set of 2-paths of a graph G (the paths P_2 of order 2) where two vertices of $L(G)$ are adjacent if the corresponding paths of G have a path P_1 in common. This observation leads us to a generalization of line graphs. Let $k \geq 2$ be an integer and let G be a graph containing k-paths. The *k-path graph* $\mathcal{P}_k(G)$ of G has the set of k-paths of G as its vertex set where two distinct vertices of $\mathcal{P}_k(G)$ are adjacent if the corresponding k-paths of G have a $(k-1)$-path in common. Thus, the 2-path graph of a nonempty graph is its line graph. The 3-path graph $\mathcal{P}_3(G)$ of a connected graph G of order at least 3 therefore has the set of 3-paths in G as its vertex set, where two distinct vertices of $\mathcal{P}_3(G)$ are adjacent if the corresponding 3-paths of G have a 2-path (an edge) in common. Since every 3-path in a graph G is both a vertex of $\mathcal{P}_3(G)$ and an edge of $L(G)$ and every 3-path is obtained from a pair of adjacent edges of G, it follows that $\mathcal{P}_3(G) = L^2(G)$. However, if $k \geq 4$ and G is a connected graph having k-paths, then $\mathcal{P}_k(G) \neq L^{k-1}(G)$ in general. For example, if G is the double star of order 5, then $L(G) = K_{1,3} + e$. Thus, $L^2(G) = C_4 + e$ and so $L^3(G) = C_4 + K_1$, which is the wheel of order 5. Since $\mathcal{P}_4(G) = K_2$ it follows that $\mathcal{P}_4(G) \neq L^3(G)$. This concept was introduced by Gary Chartrand and studied in [2, 3, 10], where the primary emphasis in these papers was on 3-path graphs.

A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and a graph having a Hamiltonian cycle is a *Hamiltonian graph*. Harary and Nash-Williams [8] characterized those graphs whose line graph is Hamiltonian. Their characterization primarily involved the existence of a circuit in a graph called a *dominating circuit* in which every edge of the graph is incident with a vertex of the circuit.

Theorem 1.1 [8] *Let G be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if G is the star $K_{1,t}$ for some integer $t \geq 3$ or G contains a dominating circuit.*

While a connected graph G with no vertices of degree 1 or 2 need not have a Hamiltonian line graph, Chartrand and Wall [4] verified that if G is connected graph with $\delta(G) \geq 3$, then $L(G)$ must have a spanning subgraph containing an Eulerian circuit, which is a dominating circuit of $L(G)$ and consequently, gives the following result in terms of 3-path graphs.

Theorem 1.2 [4] *If G is a connected graph with $\delta(G) \geq 3$, then $\mathcal{P}_3(G)$ is Hamiltonian.*

There are graphs possessing a variety of Hamiltonian properties where spanning trees or spanning walks play a major role. For example, a *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph G is *Hamiltonian-connected* if every two vertices of G are connected by a Hamiltonian path. The concept of Hamiltonian-connected graphs can be looked at in a different way. That is, a connected graph G is Hamiltonian-connected if for every two vertices u and v, there exists a spanning tree T of G whose only end-vertices are u and v. This observation gives rise to an extension of Hamiltonian-connected graphs. Let G be a connected graph of order $n \geq 3$ and let k be an integer with $2 \leq k \leq n-1$. The graph G is *k-tree-connected* if for every set S of k distinct vertices of G, there exists a spanning tree T of G whose set of end-vertices is S. Thus, G is 2-tree-connected if and only if G is Hamiltonian-connected. These concepts were studied in [2, 3, 7, 10]. The following conjecture was due to Chartrand (see [10]).

Conjecture 1.3 *If T is a tree of sufficiently large order containing no vertices of any of the degrees $2, 3, \ldots, k+1$ for each integer $k \geq 2$, then $\mathcal{P}_3(T)$ is k-tree-connected.*

By viewing the line graph $L^2(G)$ of the line graph $L(G)$ of a connected graph G in terms of its 3-path graph $\mathcal{P}_3(G)$, we are able to apply techniques involving paths or walks in the graph to establish sufficient conditions for the 3-path graph of a connected graph to possess stronger Hamiltonian properties In particular, the following two results appear in [2, 3], which verify Conjecture 1.3 for $k = 2, 3$.

Theorem 1.4 *If T is a tree of order at least 5 containing no vertices of degree 2 or 3, then $\mathcal{P}_3(T)$ is Hamiltonian-connected and, equivalently, 2-tree-connected.*

Theorem 1.5 *If T is a tree of order at least 6 containing no vertices of degree 2, 3 or 4, then $\mathcal{P}_3(T)$ is 3-tree-connected.*

In this work, we verify Conjecture 1.3 for the case when $k = 4$. That is, we present an extension of Theorem 1.5 to show that if T is a tree of order at least 6 containing no vertices of degree 2, 3, 4, or 5, then $\mathcal{P}_3(T)$ is 4-tree-connected. We refer to the book [5] for graph theory notation and terminology not described in this paper.

2 Main Result

First, we introduce an additional definition. A *Hamiltonian walk* in a connected graph G is a closed walk of minimum length that contains every vertex of G. This concept was introduced by Goodman and Hedetniemi [6] who showed that if G is a connected graph of order n and size m, then the length of Hamiltonian walk W in G is at least n and at most $2m$. Furthermore, every edge of G occurs at most twice in W. The length of W is n i and only if G is Hamiltonian (in which case W is a Hamiltonian cycle) anc the length of W is $2m$ if and only if G is a tree (in which case each edge o G appears exactly twice in W).

Every embedding of a tree T in the plane gives rise to a Hamiltoniar walk in T. For example, let T be the star $K_{1,4}$ of order 5 whose fou edges are labeled a, b, c, d. Figures 1(a) and 1(b) show two different embeddings of T in the plane. By tracing the walk as shown in Figure 1(c using the embedding of T in Figure 1(a), we construct the Hamiltonia walk $W_1 = (w, v, x, v, y, v, z, v, w)$ or, in terms of edges of T, the wal $W_1 = (a, b, b, c, c, d, d, a)$. While every edge of T occurs exactly twice o W_1, the 3-path $(w, v, x) = (a, b) = ab$ occurs once in W_1 but the 3-pat $(w, v, y) = ac$ does not occur at all in W_1. On the other hand, the embedding of T shown in Figure 1(b) gives rise to the Hamiltonian wal $W_2 = (w, v, y, v, x, v, z, v, w) = (a, c, c, b, b, d, d, a)$, which contains the 3 path (w, v, y) but not the 3-path (w, v, x).

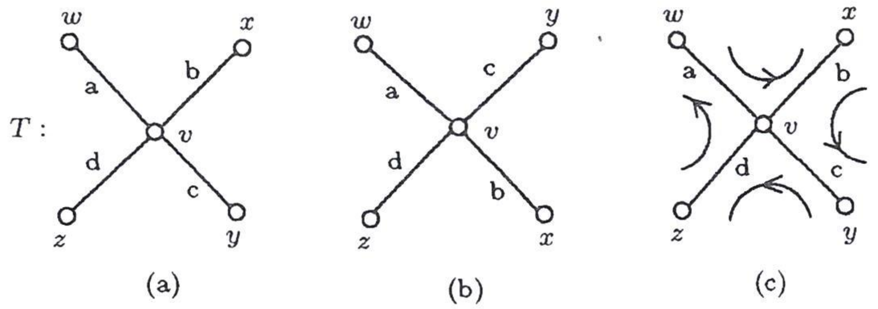


Figure 1: Two embeddings of $K_{1,4}$ in the plane

We are now prepared to present the main result of this work.

Theorem 2.1 *If T is a tree of order at least 7 containing no vertices degree 2, 3, 4, or 5, then $\mathcal{P}_3(T)$ is 4-tree-connected.*

Proof. Let P, Q, R_1 and R_2 be four 3-paths of T. We show that $\mathcal{P}_3(T)$ $\{R_1, R_2\}$ contains a Hamiltonian P-Q path. It suffices to show that the exists an ordering

$$P = A_1, A_2, \ldots, A_p = Q \quad ($$

of those 3-paths A_i $(1 \leq i \leq p)$ of T that do not include R_1 and R_2, beginning with P and ending with Q such that A_i and A_{i+1} have an edge in common for $i = 1, 2, \ldots, p-1$. Indeed, since the interior vertices of R_1 and R_2 must have degree at least 6, there exist at least five of these 3-paths distinct from P, Q and R_2 that contain an edge of R_1 and at least five of these 3-paths distinct from P, Q and R_1 that contain an edge of R_2. Thus, each of R_1 and R_2 shares an edge with A_k for some $k \notin \{1, p\}$. By taking the Hamiltonian $P-Q$ path together with R_1 and R_2 and the edges A_i and A_j in $\mathcal{P}_3(T)$ for $i, j \notin \{1, p\}$ (we may have $A_i = A_j$) joining R_1 and R_2, respectively, a spanning tree of $\mathcal{P}_3(T)$ is formed whose set of end-vertices is $\{P, Q, R_1, R_2\}$.

We consider the following four cases, depending on the location of P and Q in T:

(1) P and Q have an edge in common,

(2) P and Q do not have an edge in common and there exists a path in T containing both P and Q,

(3) P and Q do not have an edge in common and there exists a path in T containing one edge of each of P and Q but there is no path in T containing one of these paths and one edge of the other,

(4) P and Q do not have an edge in common and there is no path in T containing both P and Q but there is a path containing one of P and Q and one edge of the other.

Case 1. *P and Q have an edge in common, say $P = e_1e_2$ and $Q = e_2e_3$.* Thus, either P and Q have the same interior vertex or P and Q have distinct adjacent interior vertices (see Figure 2). We consider these two possibilities.

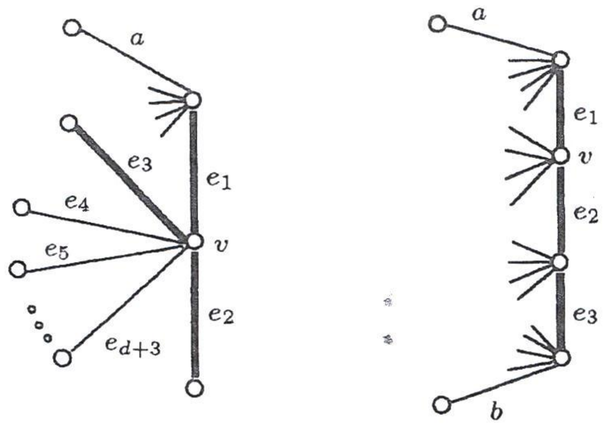
Subcase 1.1. *P and Q have the same interior vertex v.* See Figure 2(a). So, v is incident with all three edges e_1, e_2, e_3.

First, suppose that v is also the interior vertex of R_1 and R_2, say $R_1 = f_1g_1$ and $R_2 = f_2g_2$, where f_1, g_1, f_2, g_2 are four edges incident with v. Then we consider two situations. In each situation, we will provide an ordering of the edges incident with v such that e_1 and e_2 appear consecutively, and the pairs e_2, e_3 and f_1, g_1, f_2, g_2 do not appear consecutively.

First, suppose that R_1 and R_2 have no edge in common. So, we have the following situations.

(*i*) One of f_i and g_i $(i = 1, 2)$ is e_1 or e_2, say $f_1 = e_1$.

(*ii*) Neither f_i nor g_i $(i = 1, 2)$ is e_1 or e_2.



(a) (b)

Figure 2: The 3-paths P and Q have the edge e_2 in common

- ⋆ In situation (i), $f_1 = e_1$, $g_1 \neq e_1$ and $g_2 \notin \{e_1, e_2\}$. Let $d \geq 0$ be th number of edges not in P, Q, R_1, R_2. For $d \geq 1$, let $e_4, e_5, \dots e_{d+3}$ b the distinct edges incident with v that are none of $e_1, e_2, e_3, g_1, f_2, g$

  If $g_1 = e_3$, then $d \geq 2$ and we produce the sequence

$$e_1, e_2, e_{d+3}, \dots, e_6, g_1 = e_3, f_2, e_5, g_2, e_4.$$

  If $g_1 \neq e_3$, then $d \geq 1$ and we produce the sequence

$$e_1, e_2, e_{d+3}, \dots, e_5, g_1, g_2, e_4, e_3.$$

- ⋆ In situation (ii), let $d \geq 0$ be the number of edges not in P, Q, R_1, R For $d \geq 1$, let $e_4, e_5, \dots e_{d+3}$ be the distinct edges incident with that are not e_1, e_2, e_3, f_1, f_2, g_1, g_2. If $f_i = e_3$ or $g_i = e_3$ $(i = 1, 2$ then we may assume that $f_1 = e_3$ (since R_1 and R_2 have no edge common). So $d \geq 1$. We can then order the edges as follows:

$$e_1, e_2, e_{d+3}, \dots, e_6, g_1, g_2, e_5, e_4, f_1 = e_3, f_2.$$

  If none of f_i and g_i $(i = 1, 2)$ is e_3, then $d \geq 0$. Note here that if $d =$ then $\{e_4, e_5, \cdots e_{d+3}\}$ is the empty set. In this case, we produce tl sequence

$$e_1, e_2, e_{d+3}, \dots, e_5, f_1, e_4, f_2, e_3, g_1, g_2.$$

Next, suppose that R_1 and R_2 have an edge in common. So, $R_1 = f_1$ and $R_2 = f_1 g_2$. Then we have the following situations.

(i') One of f_i and g_i ($i = 1, 2$) is e_1 or e_2, say $f_1 = e_1$.

(ii') Neither f_i nor g_i ($i = 1, 2$) is e_1 or e_2.

⋆ In situation (i'), we may assume that $g_i \notin \{e_1, e_2\}$. Let d be the number of edges not in P, Q, R_1, R_2, where then $d \geq 1$. Let $e_4, e_5, \ldots, e_{d+3}$ be the distinct edges incident with v that are not e_1, e_2, e_3, g_1, g_2.

If $g_1 = e_3$ or $g_2 = e_3$, say $g_1 = e_3$, then $d \geq 2$ and we produce the sequence

$$e_1, e_2, e_{d+3}, e_{d+2}, \ldots, e_5, g_2, g_1 = e_3, e_4.$$

If $g_1 \neq e_3$ and $g_2 \neq e_3$, then $d \geq 1$ and we produce the sequence

$$e_1, e_2, e_{d+3}, \ldots, e_5, g_1, g_2, e_4, e_3.$$

⋆ In situation (ii'), Let d be the number of edges not in P, Q, R_1, R_2. For $d \geq 1$, let $e_4, e_5, \ldots, e_{d+3}$ be the distinct edges incident with v that are not any of e_1, e_2, e_3, f_1, f_2, g_1, g_2.

If $f_i = e_3$ or $g_i = e_3$ ($i = 1, 2$), then we may assume that $f_1 = e_3$ or $g_1 = e_3$, say $f_1 = e_3$, and so $d \geq 1$. We can then order the edges as follows:

$$e_1, e_2, e_{d+3}, \ldots, e_6, g_1, g_2, e_5, e_4, f_1 = e_3.$$

If none of f_1, g_1, g_2 is e_3, then $d \geq 0$. Note here that if $d = 0$, then $\{e_4, e_5, \cdots e_{d+3}\}$ is the empty set. In this case, we produce the sequence

$$e_1, e_2, e_{d+3}, \ldots, e_5, f_1 = f_2, e_4, e_3, g_1, g_2.$$

Therefore, in any situation, we can embed T so that the edges incident with v appear counterclockwise in the order given in one of the situations above. Then there exists a Hamiltonian walk W of T and a resulting ordering $\mathcal{S}_1$ of those distinct 3-paths of T belonging to W with the following properties.

⋆ The 3-path P appears in $\mathcal{S}_1$.

⋆ None of Q, R_1 and R_2 appears in $\mathcal{S}_1$.

Then ae_1, e_1e_2, e_2b are three consecutive terms in $\mathcal{S}_1$ for some edges a and b in T. Note here that if a and/or b are incident with v, then we have chosen our ordering so that neither ae_1 nor e_2b is Q, R_1 or R_2. Furthermore, if a or

b is not incident with v, then we may embed T in the plane so that neithe ae_1 nor e_2b is R_i for $i = 1, 2$. Indeed, since the degree of every non-end vertex of T is at least 6, there is some ordering of the edges incident witl this vertex for which the edges that constitute R_1 and R_2 do not appea consecutively.

Consider the vertex v. Let X be the set of 3-paths whose interio vertex is v that do not appear in $\mathcal{S}_1$. By the manner in which we hav chosen the ordering of those edges incident with v, it is clear that $X \neq$ (since $Q, R_i \in X$). Now, let $X' = X - \{R_1, R_2\}$. For each integer i witl $1 \leq i \leq d-3$, let $X_i = \{e_ie_j \in X' : i < j\}$, let s_i be any ordering of th 3-paths in X_i ($i \neq 2$) and let s_2 be any ordering of the 3-paths in X_2 whos first term is e_2e_3.

- ⋆ Insert the 3-paths in X_1 in the order s_1 between ae_1 and e_1e_2.
- ⋆ Insert the 3-paths in X_2 in the order s_2 between e_1e_2 and e_2b.
- ⋆ For each integer i with $3 \leq i \leq d-2$, insert the 3-paths in X_i in th order s_i between consecutive terms containing e_i in $\mathcal{S}_1$.

The resulting sequence $\mathcal{S}_2$ has e_1e_2, e_2e_3 as consecutive terms and contain all 3-paths of $\mathcal{P}_3(T)$ belonging to W as well as those 3-paths having v a their interior vertex except for the 3-paths R_1 and R_2.

For every other vertex u of T having degree $d' \geq 5$, let f_1f_2 be a 3-pat on W having u as its interior vertex, labeling the remaining edges incider with u as $f_3, f_4, \ldots, f_{d'}$ as was done with v. Inserting 3-paths with interic vertex u not in $\mathcal{S}_2$, as we did with the vertex v, produces a sequence $\mathcal{S}$ all 3-paths of T with the desired properties.

Next, suppose that neither R_1 nor R_2 contains an edge incident with Then we embed T in the plane so that R_1 and R_2 do not appear in W an we do not add these two 3-paths to $\mathcal{S}_2$ in the final step of the proof.

Subcase 1.2. *P and Q have adjacent interior vertices.* We may assur that e_1, e_2, e_3 is a 4-path in T, where e_1 and e_2 are incident with the vertex and e_2 and e_3 are incident with the vertex u, as shown in Figure 2(b). R_1 and/or R_2 contains an edge that is incident with either u or v, then w may choose an ordering of the edges incident with one of these vertices suc that the edges of R_1 and R_2 do not appear consecutively. There exists Hamiltonian walk W of T and a resulting ordering $\mathcal{S}_1$ of those 3-paths of on W such that e_1e_2, e_2e_3 are consecutive terms in $\mathcal{S}_1$. There are edges and b in T such that $ae_1, e_1e_2, e_2e_3, e_3b$ are consecutive terms in $\mathcal{S}_1$. Agai we may embed T so that neither ae_1 nor e_3b is R_1 or R_2. For each vert w having degree 6 or more, we insert all 3-paths containing an edge e ar with interior vertex w not already in $\mathcal{S}_1$ into the sequence $\mathcal{S}_1$, except for tl 3-paths R_1 and R_2, between two consecutive terms containing e except e_1

and e_2e_3. For the 3-paths containing e_2 that are not R_1 or R_2, we insert such 3-paths between two other consecutive terms containing e_2. Since the edge e_2 occurs elsewhere in W and in $\mathcal{S}_1$, this can be done. This produces a sequence of all 3-paths of T with the desired properties.

Case 2. P and Q do not have an edge in common and there exists a path Γ in T containing both P and Q. Let $P = ab$ and $Q = cd$. See Figure 3(a).

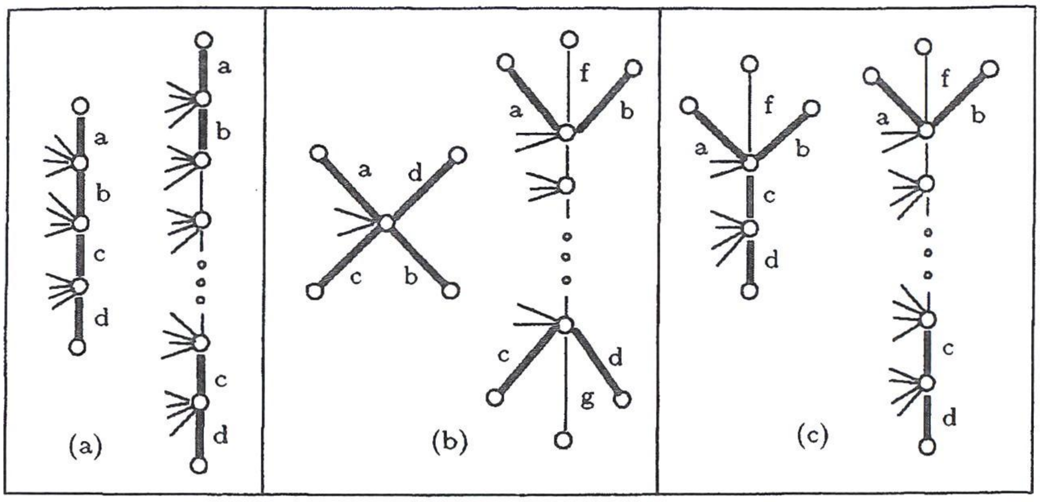


Figure 3: The 3-paths P and Q have no edge in common

By assumption, there exists a path Γ in T containing both P and Q. If none of the edges of R_1 and R_2 appear as 3-paths of Γ, then there is a Hamiltonian walk W of T such that Γ is a path in W such that R_1 and R_2 do not appear in W. Thus, either

$\Gamma : a, b, c, d$ or $\Gamma : a, b, e_1, e_2, \ldots, e_k, c, d$ for some positive integer k.

Let $\mathcal{S}_1$ be a cyclic sequence consisting of those 3-paths of T appearing in the order as they are encountered on W. Thus, either

$$ab, bc, cd \quad \text{or} \quad ab, be_1, e_1e_2, \ldots, e_kc, cd$$

are consecutive terms in $\mathcal{S}_1$.

- ⋆ If $\Gamma : a, b, c, d$, then we delete bc from $\mathcal{S}_1$.
- ⋆ if $\Gamma : a, b, e_1, e_2, \ldots, e_k, c, d$, then we delete the terms be_1, e_1e_2, $\ldots$, e_kc from $\mathcal{S}_1$.

In either situation, a new sequence $\mathcal{S}_2$ is created. Since each edge of T is encountered twice in W, each edge of T occurs in two consecutive terms of $\mathcal{S}_2$. Each 3-path deleted from $\mathcal{S}_1$ and each 3-path in T not in $\mathcal{S}_1$ may now be added in an appropriate position in $\mathcal{S}_2$, with the exception of not adding

R_1 and R_2, creating a new sequence $\mathcal{S} : A_1, A_2, \ldots, A_p$ of all 3-paths of ⁚
such that (1) $P = A_1$ and $Q = A_p$, (2) A_i and A_{i+1} have a single edg
in common for $i = 1, 2, \ldots, p-1$ and (3) R_1 and R_2 are not terms of thi
sequence.

Now, we may assume that R_1 or R_2 (or both) appear in Γ. Thus agair
either $\Gamma : a, b, c, d$ where $R_1 = bc$ or $R_2 = bc$, say $R_1 = bc$ or

$$\Gamma : a, b, e_1, e_2, \ldots, e_k, c, d \text{ for some positive integer } k,$$

where each of R_1 and R_2 (or perhaps one of them) appears as two conse‹
utive terms in this sequence not including a, b or c, d. Let $\mathcal{S}_1$ be a cycl⁚
sequence consisting of those 3-paths of T appearing in the order as the
are encountered on W. Thus, either

$$ab, bc, cd \quad \text{or} \quad ab, be_1, e_1e_2, \ldots, e_kc, cd \text{ are consecutive terms in } \mathcal{S}_1.$$

⋆ If $\Gamma : a, b, c, d$, then we delete bc from $\mathcal{S}_1$.

⋆ If $\Gamma : a, b, e_1, e_2, \ldots, e_k, c, d$, then we delete the terms be_1, e_1e_2, ‥
e_kc from $\mathcal{S}_1$.

In either situation, a new sequence $\mathcal{S}_2$ is created that has the property th⁝
neither of R_1 and R_2 appears in $\mathcal{S}_2$. Since each edge of T is encountere
twice in W, each edge of T occurs in two consecutive terms of $\mathcal{S}_2$. Eac
3-path deleted from $\mathcal{S}_1$ and each 3-path in T not in $\mathcal{S}_1$, except for the tw
3-paths R_1 and R_2, may now be added in an appropriate position in $\mathcal{S}$
creating a new sequence $\mathcal{S} : A_1, A_2, \ldots, A_p$ of all 3-paths of T such that (⁚
$P = A_1$ and $Q = A_p$, (2) A_i and A_{i+1} have a single edge in common f‹
$i = 1, 2, \ldots, p-1$ and (3) R_1 and R_2 are not terms of this sequence. Tl
existence of the sequence $\mathcal{S}$ shows that $\mathcal{P}_3(T)$ contains a Hamiltonian P-
path that avoids R_1 and R_2.

Case 3. *P and Q do not have an edge in common and there is no path*
T containing one of these paths and one edge of the other. Let $P = ab$ ar
$Q = cd$. See Figure 3(b). Necessarily, there exists a path in T containi‹
one edge of P and Q. Let Γ be the path in T connecting the interior vertic
of P and Q. We consider two possibilities, depending on whether Γ is
trivial path.

Subcase 3.1. *Γ is a trivial path.* Thus, P and Q have the same interi
vertex v. See Figure 3(b). Suppose first that $\deg v = 6$. The tree T
embedded in the plane so that the six edges a, b, c, d, e, f incident with
appear as in Figure 4(a).
If R_1 and R_2 also have v as their interior vertex, we may select an ordering
these edges so that none the edges of P, Q, R_1 and R_2 appear consecutive
about v. Possibly as many as three neighbors of v are end-vertices in

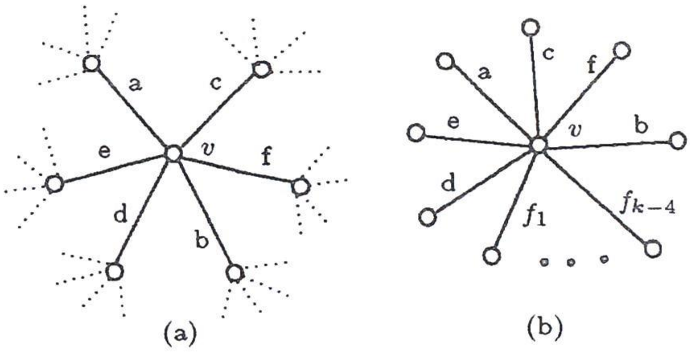


Figure 4: A step in the proof of Subcase 3.1

This embedding gives rise to a Hamiltonian walk W of T and a cyclic sequence $\mathcal{S}_1$ of distinct 3-paths of T lying on W. So, $\mathcal{S}_1$ has the appearance

$$\mathcal{S}_1 : cf, \ldots, fb, \ldots, bd, \ldots, de, \ldots, ea \ldots, xa, ac, cy, \ldots, zc, cf$$

for edges x, y, z in T, where, for example, possibly $x = e$ and/or $y = f$. Among the 3-paths in T not in $\mathcal{S}_1$ are $P = ab$ and $Q = cd$. We now insert the pair $P = ab, Q = cd$ between ac and cy, arriving at

$$\mathcal{S}_2 : P = ab, ac, xa, \ldots, ea, \ldots, de, \ldots bd, \ldots, cf, zc, \ldots, cy, cd = Q.$$

Thus, this noncyclic sequence $\mathcal{S}_2$ of distinct 3-paths begins at P, ends at Q and contains all 3-paths of T on W in addition to P and Q. Furthermore, every two consecutive 3-paths on $\mathcal{S}_2$ have an edge in common. Each 3-path rs in T not in $\mathcal{S}_2$, except for the 3-paths R_1 and R_2, can then be added to $\mathcal{S}_2$, either between two 3-paths containing r or between two 3-paths containing s, to produce a new sequence $\mathcal{S} : A_1, A_2, \ldots, A_p$ of all 3-paths of T such that (1) $P = A_1$ and $Q = A_p$, (2) A_i and A_{i+1} have a single edge in common for $i = 1, 2, \ldots, p-1$ and (3) R_1 and R_2 are not terms of this sequence.

Next, suppose that $\deg v = k \geq 7$, say $f_1, f_2, \ldots, f_{k-6}$ are the remaining $k-6$ edges incident with v. Let T be embedded as in Figure 4(b). We then proceed as above to produce a sequence $\mathcal{S}$ with the desired properties.

Subcase 3.2. Γ is not a trivial path. Let u be the interior vertex of $P = ab$ and v the interior vertex of $Q = cd$. Since the interior vertices of P and Q have degree at least 6, it follows that if R_1 and/or R_2 contains an edge that is incident with one of these interior vertices, then we may select an ordering of these edges such that none of the edges P, Q, R_1 and R_2 appear consecutively about v. Let

$$\Gamma : e_1, e_2, \ldots, e_k,\ k \geq 1, \text{ be the } u - v \text{ path in } T.$$

Since $\deg u \geq 6$ and $\deg v \geq 6$, there exists an edge f incident with different from a, b and e_1 and an edge g incident with v different from $c,$ and e_k. Let T be embedded in the plane, as shown in Figure 3(b). Let V be the resulting Hamiltonian walk of T which contains none of P, Q, R and R_2 but contains the path Γ and let $\mathcal{S}_1$ be the resulting cyclic sequen of 3-paths lying on W. Hence, $\mathcal{S}_1$ has the appearance

$$\mathcal{S}_1 : af, \ldots, fb, bx, \ldots, yb, be_1, e_1e_2, \ldots, e_{k-1}e_k, e_kd, dz, \ldots, wa, af$$

for edges x, y, z and w in T, where possibly $y = f$ and $z = g$. Among tl 3-paths in T not belonging to $\mathcal{S}_1$ are $P = ab$ and $Q = cd$ (see Figure 3(b) Hence, we may insert $P = ab$ between yb and be_1 and $Q = cd$ between e_k and dz. We then delete the 3-paths be_1, e_1e_2, ..., e_kd from $\mathcal{S}_1$, arriving the sequence

$$\mathcal{S}_2 : P = ab, yb, \ldots, bx, fb, \ldots, af, wa, \ldots, dz, cd = Q,$$

consisting of P, Q and all 3-paths of T lying on W, except be_1, e_1e_2, .. e_kd. These 3-paths along with all 3-paths of T not in $\mathcal{S}_2$, except the 3 patl R_1 and R_2, can be appropriately inserted into $\mathcal{S}_2$ to produce a sequence $\mathcal{S}$ $A_1, A_2, \ldots, A_p$ consisting of the distinct 3-paths such that (1) $P = A_1$ ar $Q = A_p$, (2) A_i and A_{i+1} have a single edge in common for $i = 1, 2, \ldots, p-$ and (3) the 3-paths R_1 and R_2 are not terms of this sequence. The existen of the sequence $\mathcal{S}$ shows that $\mathcal{P}_3(T)$ contains a Hamiltonian P-Q path th avoids the 3-paths R_1 and R_2.

Case 4. P and Q do not have an edge in common and there is no pa in T containing both P and Q but there is a path containing one of P and and one edge of the other. Let $P = ab$ and $Q = cd$. See Figure 3(c). V may assume that there exists a path Γ in T containing the 3-path Q a the edge b but not a. Then there is a Hamiltonian walk W of T such th Γ is a path in W. Thus, either

$$\Gamma : b, c, d \quad \text{or} \quad \Gamma : b, e_1, e_2, \ldots, e_k, c, d \text{ for some positive integer } k.$$

Let T be embedded in the plane, as shown in Figure 3(c). Since no vert of T has degree 3, there is an edge f adjacent to a and b but not belongi to Γ that lies between a and b. Consider the following three cases:

1. If one of R_1 or R_2 is af or fb, say $R_1 = af$ or $R_1 = fb$, then sin there are no vertices of degree 5 in T, there is some other edge g th lies either between a and f or between f and b so that the edges R_1 do not appear consecutively.

2. Let $R_1 = af$ and $R_2 = fb$. Then since there are no vertices of degr 5 in T, there exist some other edges g_1 and g_2 such that (a) g_1 l between a and f and (b) g_2 lies between f and b so that the edges R_1 and R_2 do not appear consecutively.

3. If $R_1 = e_1a$ or $R_2 = e_1a$, then we place g between these edges instead.

Let $\mathcal{S}_1$ be the cyclic sequence consisting of those 3-paths of T appearing in the order as they are encountered on W. The 3-path ab therefore does not lie on W. Thus, either

(i) xb, bc, cd are three consecutive terms in $\mathcal{S}_1$ for some edge x of T or

(ii) $xb, be_1, e_1e_2, \ldots, e_kc, cd$ are consecutive terms in $\mathcal{S}_1$ for some edge x of T.

If (i) occurs, then we insert the 3-path ab between xb and bc and delete bc; while if (ii) occurs, we insert ab between xb and be_1 and delete the terms be_1, e_1e_2, $\ldots$, e_kc. In either situation, a new sequence $\mathcal{S}_2$ is created. Since each edge of T is encountered twice in W, each edge of T occurs twice in two consecutive terms of $\mathcal{S}_1$. Each 3-path deleted from $\mathcal{S}_1$ and each 3-path in T not in $\mathcal{S}_1$ may now be added in an appropriate position in $\mathcal{S}_2$ except for the 3-paths R_1 and R_2.

Specifically, if (i) occurs and $R_1, R_2 \neq bc$, then the 3-path bc can now be inserted between two consecutive terms containing c. If (ii) occurs and R_1 and R_2 are not 3-paths of Γ, then the 3-path be_1 can be inserted between two consecutive terms containing e_1, the 3-path e_1e_2 can be inserted between two consecutive terms containing e_2, and so on. This creates a new sequence $\mathcal{S} : A_1, A_2, \ldots, A_p$ of all 3-paths of T that begins at P and ends at Q such that every two consecutive terms of $\mathcal{S}$ have an edge in common and R_1 and R_2 are not terms of $\mathcal{S}$. The existence of the sequence $\mathcal{S}$ shows that $\mathcal{P}_3(T)$ contains a Hamiltonian P-Q path that avoids R_1 and R_2.

In each case, there is a Hamiltonian P-Q path in $\mathcal{P}_3(T)$ that avoids R_1 and R_2 and so $\mathcal{P}_3(T)$ is 4-tree-connected. ■

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