Rényi Ordering of Tournaments

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Abstract

We develop an ordering function on the class of tournament digraphs (complete antisymmetric digraphs) that is based on the Rényi α-entropy. This ordering function partitions tournaments on n vertices into equivalence classes that are approximately sorted from transitive (the arc relation is transitive — the score sequence is $(0, 1, 2, \ldots, n-1)$) to regular (score sequence $(\frac{n-1}{2}, \ldots, \frac{n-1}{2})$). But the diversity among regular tournaments (there are for example 1123 regular tournaments on 11 vertices, and 1,495,297 regular tournaments on 13 vertices up to isomorphism) is captured to an extent.

1 Introduction

Many entropy functionals have been developed for use on graphs to the ends of a variety of purposes including assessing network complexity [8], approximating similarity [11], characterizing graphs [3], as tools for data analytics involving clustering [1], and for their theoretical value in and of themselves [2]. Possibly all the entropy functionals are developed with the goal of capturing what the classical Shannon entropy for information theory does for probability distributions; in particular, for a discrete probability distribution $P = (p_1, \ldots, p_n)$, the Shannon entropy of P is

$$S(P) = -\sum_i p_i \log_2(p_i).$$

If a graph's Laplacian is normalized in some way so that its trace is equal to 1, then, being symmetric and positive semidefintie, the Laplacian's spec-

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trum is thought of as a probability distribution and the graph's ent: is computed via $-\sum \lambda \log_2 \lambda$, where the sum is over the spectrum of graph's Laplacian (normalized in some way as mentioned). In this case entropy is called the von Neumann entropy, after the way von Neum used it in [10].

Here we explore using Rényi α-entropy to compare tournaments. so-called Rényi α-entropy was developed by Rényi in [9] to identify axioms to which entropy fuinctionals should adhere and to develop a r general functional than Shannon's. Traditionally, the Rényi α-entrop defined using the log function as follows:

$$H_\alpha(P) = \frac{1}{1-\alpha} \log_2 \sum_{i=1}^{n} p_i^\alpha,$$

where $P = (p_1, \ldots, p_n)$ is a discrete probability distribution. For our poses, we suppress the log in order to apply this functional to a com "probability distribution," if you will, and we take the Rényi α-*ent to be

$$H_\alpha^*(P) = -\sum_{i=1}^{n} p_i^\alpha.$$

Note that if $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$, then $H_\alpha^*(P$ $H_\alpha^*(Q)$ iff $H_\alpha(P) = H_\alpha(Q)$. Thus we lose no information by using the r simplistic *entropy. Let T be a tournament with vertex set $V = \{v_1, \ldots,$ arc set $A(T)$, and score sequence $(s_1, \ldots, s_n)$, where the vertices are lab so that s_i is the score of v_i ($s_i = |\{x \in V(T) : v_i \to x \text{ in } T\}|$). We define normalized Laplacian matrix $\bar{L}(T) = (a)_{ij}$ by $a_{ii} = s_i \binom{n}{2}^{-1}$ for $1 \leq i$ and $a_{ij} = -\binom{n}{2}^{-1}$ if $ij \in A(T)$ and 0 otherwise. We pretend the com spectrum of $\bar{L}(T)$ is a probability distribution and define the Rény *entropy of T by

$$H_\alpha^*(T) = -\sum_{\lambda \in \text{spec}(\bar{L}(T))} \lambda^\alpha \in \mathbb{R}$$

for $\alpha > 1$, where spec(A) denotes the spectrum of matrix A. If T is a t nament whose vertex set has size n we may refer to T as an n-*tournam*

We note that our work here is an improvement on the work done Landau in [4] in which a tournament is assessed based on how close it i the what he called the "*hierarchy*" (the transitive tournament). Land assessment mechanism is sensitive to the score sequence of the tournan only; our ordering is sensitive to other structural properties not captu by the score sequence. For example, the regular n-tournaments (tournaments in which all vertices have score $\frac{n-1}{2}$) can be distinguished up to isomorphism to an extent.

We observe that $\bar{L}(T) = \binom{n}{2}^{-1}(D - A)$, where D is the diagonal matrix $\operatorname{diag}(s_1, \dots, s_n)$ and A is the adjacency matrix of T. Recall that raising a matrix to a power raises each of its eigenvalues to that power, and multiplying it by a scalar multiplies the eigenvalues by that scalar. Also recall that for any matrix M, $\operatorname{tr} M$, called the trace of M, is both the sum of its diagonal entries and the sum of its eigenvalues.

Pulling this together, we see that

$$\begin{aligned} H^*_\alpha(T) &= -\sum_{\lambda \in \operatorname{spec}(\bar{L}(T))} \lambda^\alpha \\ &= -\binom{n}{2}^{-\alpha} \operatorname{tr}\left[(D - A)^\alpha\right]. \end{aligned}$$

It directly follows that

$$H^*_2 = -\frac{1}{\binom{n}{2}^2} \sum_{i=1}^{n} s_i^2$$

$$\text{and} \qquad H^*_3 = -\frac{1}{\binom{n}{2}^3} \left(\sum_{i=1}^{n} s_i^3 - 3\left(\binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2} \right) \right).$$

Therefore, if two tournaments have the same score sequence, then they must have the same Rényi 2-*entropy and Rényi 3-*entropy*. However, the converse is not true. For example, Landau's conditions ([5]) guarantee the existence of tournaments with score sequences $(1, 1, 2, 3, 4, 5, 5)$ and $(0, 3, 3, 3, 3, 3, 6)$. Such tournaments each have $H^*_2 = -81/21^2$ and $H^*_3 = -336/21^3$.

Also, it's not true that if two tournaments have distinct Rényi 2-*entropy, then they have distinct Rényi 3-*entropy. Take for examples tournaments with score sequences $(0, 1, 3, 4, 4, 4, 5)$ and $(1, 2, 2, 3, 3, 4, 6)$. The first has $H^*_2 = -79/21^2$, while the second has $H^*_2 = -83/21^2$, with $H^*_3 = -333/21^3$ for both.

Establishing a tournament ordering is further complicated by examples in which $H^*_2(T) < H^*_2(T')$ and $H^*_3(T) > H^*_3(T')$. This is the case with tournaments T and T' with score sequences $(0, 1, 4, 4, 4, 4, 4)$ and $(2, 2, 2, 2, 3, 4, 6)$, respectively. Here, $H^*_2(T) = -81/21^2 < -77/21^2 = H^*_2(T')$ and $H^*_3(T) = -306/21^3 > -318/21^3 = H^*_3(T')$.

To address these issues, we define the *Rényi ordering* of tournai on n vertices using the following structure. Tournaments T and T' ar to be in the same *Rényi α-class* if for every integer β with $2 \leq \beta$ we have $H^*_\beta(T) = H^*_\beta(T')$. If T and T' are in the same Rényi α-clas $H^*_{\alpha+1}(T) > H^*_{\alpha+1}(T')$, then T precedes T' in the Rényi order. For any α, the Rényi α-classes partition the set of tournaments on n vertices the Rényi $(\alpha+1)$-classes refine that partition. For tournaments on and 9 vertices, this refinement is summarized in the following table.

	7 vertices	8 vertices	9 verti
Tournaments, up to isomorphism	456	6880	1915
Score sequences	59	167	672
Rényi 2-classes	15	21	31
Rényi 3-classes	56	145	355
Rényi 4-classes	165	778	387
Rényi 5-classes	270	2152	2107
Rényi 6-classes	334	4176	9316
Rényi 7-classes	334	4664	1429
Rényi 8-classes	334	4664	1606

2 Results

We see from the table that the Rényi 7-classes don't partition the 7-tournaments any more than the 6-classes do. We will show that large enough, the α-classes partition the set of tournaments at the spec level.

Lemma 1. *Let S be the multiset of the n complex roots of the po mial*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \prod_{s \in S}(x - s).$$

Then for $\alpha \in \mathbb{N}$, $p_\alpha = \sum_{s \in S} s^\alpha$ *satisfies the recursion*

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \\ \vdots \\ p_k \\ \vdots \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 & & 0 \\ p_1 & 2 & 0 & \cdots & 0 \\ p_2 & p_1 & 3 & & 0 \\ & \vdots & & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & n \\ & \vdots & & & \vdots \\ p_{k-1} & p_{k-2} & p_{k-3} & \cdots & p_{k-n} \\ & \vdots & & & \vdots \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{bmatrix}.$$

Proof. We define

$$e_k = \sum_{A \in \binom{S}{k}} \prod_{x \in A} x$$

for $k \in \mathbb{N}$. By expanding and reordering terms, we proceed by induction to express p_α recursively. To begin,

$$\begin{aligned} p_\alpha &= \sum_{x \in S} x^\alpha \\ &= \left(\sum_{x \in S} x \right) \left(\sum_{x \in S} x^{\alpha - 1} \right) - \sum_{A \in \binom{S}{2}} \left(\left(\prod_{x \in A} x \right) \sum_{x \in A} x^{\alpha - 2} \right) \\ &= e_1 p_{\alpha - 1} - \sum_{A \in \binom{S}{2}} \left(\left(\prod_{x \in A} x \right) \sum_{x \in A} x^{\alpha - 2} \right). \end{aligned}$$

Now suppose that

$$p_\alpha = \sum_{i=1}^{k} (-1)^{i+1} e_i p_{\alpha - i} + (-1)^k \sum_{A \in \binom{S}{k+1}} \left(\left(\prod_{x \in A} x \right) \sum_{x \in A} x^{\alpha - (k+1)} \right)$$

for some k with $1 \leq k < \alpha - 1$. Then

$$p_\alpha = \sum_{i=1}^{k}(-1)^{i+1}e_i p_{\alpha-i} + (-1)^k \sum_{A\in\binom{S}{k+1}} \left(\left(\prod_{x\in A} x\right) \sum_{x\in A} x^{\alpha-(k+1}\right.$$

$$= \sum_{i=1}^{k}(-1)^{i+1}e_i p_{\alpha-i} + (-1)^k \quad \times$$

$$\sum_{A\in\binom{S}{k+1}} \left(\left(\prod_{x\in A} x\right) \left(p_{\alpha-(k+1)} - \sum_{x\in S\setminus A} x^{\alpha-(k+1)}\right)\right)$$

$$= \sum_{i=1}^{k}(-1)^{i+1}e_i p_{\alpha-i} + (-1)^k e_{k+1} p_{\alpha-(k+1)} + (-1)^{k+1} \quad \times$$

$$\sum_{A\in\binom{S}{k+1}} \left(\left(\prod_{x\in A} x\right) \sum_{x\in S\setminus A} x^{\alpha-(k+1)}\right).$$

The second term is of the form of the first sum with $i = k+1$ summands of the last term are products of $k+2$ distinct element Thus we can simplify this to

$$\sum_{i=1}^{k+1}(-1)^{i+1}e_i p_{\alpha-i} + (-1)^{k+1} \sum_{A\in\binom{S}{k+2}} \left(\left(\prod_{x\in A} x\right) \sum_{x\in A} x^{\alpha-(k+2)}\right.$$

to complete the inductive step. Therefore, taking $k = \alpha - 1$, we get

$$p_\alpha = \begin{cases} \sum_{i=1}^{\alpha-1}(-1)^{i+1}e_i p_{\alpha-i} + (-1)^{\alpha-1}\alpha e_\alpha & \text{for } \alpha \leq n \\ \sum_{i=1}^{n}(-1)^{i+1}e_i p_{\alpha-i} & \text{for } \alpha > n \end{cases}.$$

Clearly, $(-1)^k e_k = a_{n-k}$ for each $k \in \{1, \ldots, n\}$. Therefore, p_α simplified to

$$\begin{cases} -\sum_{i=1}^{\alpha-1} a_{n-i} p_{\alpha-i} - \alpha a_{n-\alpha} & \text{for } \alpha \leq n \\ -\sum_{i=1}^{n} a_{n-i} p_{\alpha-i} & \text{for } \alpha > n \end{cases}.$$

Lemma 2. *For every ordered n-tuple $(p_1, p_2, \ldots, p_n) \in \mathbb{C}^n$, there a unique multiset S of order n containing complex numbers suc*

$$\sum_{x\in S} x^\alpha = p_\alpha$$

for each $\alpha \in \{1, 2, \ldots, n\}$.

Proof. Let A be the invertible matrix

$$\begin{bmatrix} 1 & 0 & 0 & & 0 \\ p_1 & 2 & 0 & \cdots & 0 \\ p_2 & p_1 & 3 & & 0 \\ & \vdots & & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & n \end{bmatrix},$$

and let $(a_{n-1}, a_{n-2}, \ldots, a_0)^t$ be the unique solution to

$$-A\vec{x} = (p_1, p_2, \ldots, p_n)^t.$$

If S is the multiset containing the complex roots of $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then by Lemma 1, $\sum_{x\in S} x^\alpha$ satisfies the same recursion as p_α for $1 \le \alpha \le n$, so they are equal.

Now we show uniqueness. If R is a multiset of order n containing complex numbers such that

$$\sum_{x\in S} x^\alpha = p_\alpha$$

for each $\alpha \in \{1, 2, \ldots, n\}$, then if we write

$$\prod_{r\in R}(x - r) = x^n + b_{n-1}x^{n-1} + \cdots + b_0,$$

we have $(p_1, \ldots, p_n)^t = -A(b_{n-1}, \ldots, b_0)^t$ by Lemma 1. Therefore,

$$(b_{n-1}, \ldots, b_0)^t = (a_{n-1}, \ldots, a_0)^t,$$

so $S = R$. □

Theorem 3. *Tournaments* T *and* T' *on* n *vertices are in the same Rényi* $(n-1)$*-class iff* $L(T)$ *and* $L(T')$ *have the same spectrum.*

Proof. The reverse direction is clear. For the forward direction, note that $L(T)$ and $L(T')$ are singular since their rows sum to 0. Therefore, $0 \in \operatorname{spec}(L(T))$ and $0 \in \operatorname{spec}(L(T'))$. Define S and R to be the multisets of order $n-1$ obtained from deleting one 0 from each of $\operatorname{spec}(L(T))$ and $\operatorname{spec}(L(T'))$, respectively. Since T and T' are in the same Rényi $(n-1)$-class, we have

$$\sum_{\lambda\in S}\lambda^\alpha = \sum_{\lambda\in \operatorname{spec}(L(T))}\lambda^\alpha = \sum_{\lambda\in \operatorname{spec}(L(T'))}\lambda^\alpha = \sum_{\lambda\in R}\lambda^\alpha$$

for each $\alpha \in \{1, 2, \ldots, n-1\}$. By Lemma 2, $S = R$, so $\operatorname{spec}(L(T)) = \operatorname{spec}(L(T'))$. □

Corollary 3.1. *The Rényi $(n-1)$-classes create a maximal pa of the set of tournaments on n vertices, that is, for $\alpha \geq n$, every α-class is equal to a Rényi $(n-1)$-class.*

Theorem 4. *The transitive tournament minimizes Rényi 2-e and 3-entropy.*

Proof. Consider the following algorithm. Let $\vec{s}_0 = (s_1, \ldots, s_n)$ be th sequence of some tournament with $\vec{s}_1 \leq s_2 \cdots \leq s_n$. For $i \geq 1$, ob from $\vec{s}_{i-1}$ by identifying the first repeated score and subtracting the first value in the repeated group and adding 1 to the last value repeated group. Clearly, each step of the algorithm preserves the pr that $s_1 \leq s_2 \leq \cdots \leq s_n$. The algorithm terminates when there repeated values in $\vec{s}_i$, which is only possible when $\vec{s}_i$ is the score se for the transitive tournament on n vertices.

We first show that all sequences given are valid tournament sc quences. Suppose that $\vec{s}_{i-1} = (s_1, s_2, \ldots, s_n)$ is the score sequence c tournament T, and let $s_j = s_k$ be the first and last values in the fi peated group of values in $\vec{s}_{i-1}$. Obtain T' from T by relabeling v j and k (if necessary) so that $j \to k$ in T'. Next, obtain T'' from reversing the arc between j and k so that $k \to j$ in T''. Then T'' ha sequence $\vec{s}_i = (s_1, \ldots, s_j - 1, \ldots, s_k + 1, \ldots, s_n)$. Therefore, by indu each $\vec{s}_i$ given by the algorithm is a valid tournament score sequence

Since

$$(x+1)^2 + (x-1)^2 = 2x^2 + 2 \qquad \text{for all } x \in \mathbb{Z},$$

each step increases the value of p_2 by 2. Therefore, since the algorith minates with the transitive tournament, and since $\vec{s}_0$ was chosen arbi the transitive tournament maximizes p_2 and minimizes H_2^*.

Also, using the fact that

$$H_3^*(\text{spec}(\bar{L}(T))) = -\frac{1}{\binom{n}{2}^3}\left(\sum_{i=1}^{n}\left(s_i^3 + 3\binom{s_i}{2}\right) - 3\binom{n}{3}\right),$$

H_3^* is minimized when $\sum_{i=1}^{n}\left(s_i^3 + 3\binom{s_i}{2}\right) = \sum_{i=1}^{n}\left(s_i^3 + 3(s_i)(s_i - 1\right.$ maximized. It suffices to show that this value also increases with eac of the algorithm. Simply note that the function $f(x) = x^3 + 3x(x$ satisfies $f''(x) > 0$ for $x > 0$. Therefore, $f(x-1) + f(x+1) > 2f$ $x \geq 1$.

Indeed, the value of p_3 increases with each step. Therefore, the transitive tournament maximizes p_3 and minimizes H_3^*.

A straightforward calculation shows that p_2 and p_3 are strictly positive for any tournament besides the 3-cycle, so it follows that the transitive tournament minimizes the traditional 2- and 3-entropies as well for $n \geq 4$. □

Corollary 4.1. *Any transitive tournament comes last in its respective Rényi order.*

Recall that a regular n-tournament is one with all scores equal to $\frac{n-1}{2}$; necessarily n is odd. For even n, *semi-regular n-tournament* is a tournament on n vertices with half of the scores equal to $\frac{n}{2}$ and the other half equal to $\frac{n}{2} - 1$.

Theorem 5. *The first tournament in any Rényi order is regular or semi-regular.*

Proof. We need only show that H_2^* is maximized for regular and semi-regular tournaments. Let T be a tournament on n vertices. Then

$$\begin{aligned}
H_2^*(T) &= -\binom{n}{2}^{-2} \sum_{i=1}^{n} s_i^2 \\
&= -\binom{n}{2}^{-2} \sum_{i=1}^{n} \left((n-1)s_i - \frac{(n-1)^2}{4} + \left(s_i - \frac{n-1}{2}\right)^2 \right) \\
&= -\binom{n}{2}^{-2} \left((n-1)\binom{n}{2} - \frac{n(n-1)^2}{4} + \sum_{i=1}^{n} \left(s_i - \frac{n-1}{2}\right)^2 \right) \\
&= -\frac{1}{n} - \binom{n}{2}^{-2} \sum_{i=1}^{n} \left(s_i - \frac{n-1}{2}\right)^2 .
\end{aligned}$$

If n is odd, it's now clear that $H_2^*(T)$ achieves its maximum value when $s_i = \frac{n-1}{2}$ for each i, and if n is even, it's clear that it achieves its maximum when $\left|s_i - \frac{n-1}{2}\right| = \frac{1}{2}$ for all i. But these are complete characterizations of regular and semi-regular tournaments. □

We now move on to give an exact count of the number of Rényi 2-classes for arbitrary n.

Lemma 6. *Let* T *be a tournament on* n *vertices. Then*

$$p_2 = \sum_{\lambda \in \mathrm{spec}(L(T))} \lambda^2 \text{ is even if and only if } \binom{n}{2} \text{ is even..}$$

Proof. Note that

$$\sum_{\lambda \in \mathrm{spec}(L(T))} \lambda^2 = \mathrm{tr}(L(T)^2) = \sum_{i=1}^{n} s_i^2,$$

where $\vec{s} = (s_1, s_2, \ldots, s_n)$ is the score sequence of T. Since the sum scores in any score sequence is always $\binom{n}{2}$, there are an even num odd s_i's when $\binom{n}{2}$ is even and an odd number of odd s_i's when $\binom{n}{2}$ i Therefore, p_2 is even iff $\binom{n}{2}$ is even.

Theorem 7. *The number of Rényi 2-classes of tournaments on* ı *tices is*

$$\begin{cases} \frac{1}{4}\binom{n+1}{3} + 1 & \text{if } n \text{ is odd,} \\ 2\binom{\frac{n}{2}+1}{3} + 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. We use the algorithm in the proof of Theorem 4 to produce all of $p_2 = \sum_{i=1}^{n} s_i^2$ with $\vec{s}_0$ being a regular or semi-regular score seq that is, $\vec{s}_0$ is $(\frac{n-1}{2}, \ldots, \frac{n-1}{2})$ or $(\frac{n}{2} - 1, \ldots, \frac{n}{2} - 1, \frac{n}{2}, \ldots, \frac{n}{2})$.

We have shown that regular and semi-regular tournaments mi p_2, and that transitive tournaments maximize p_2. Also recall th algorithm increases the value of p_2 by 2 each step until it terminate the transitive tournament. From the Lemma, we know that eith values of p_2 are all even or all odd, so the algorithm generates the sequence of some tournament in each Rényi 2-class.

We can count the number of classes by counting the odd or even nu between minimal and maximal values of p_2. The transitive tourn gives

$$\begin{aligned} p_{2,\max} &= \sum_{i=0}^{n-1} i^2 \\ &= \frac{n(n-\frac{1}{2})(n-1)}{3}. \end{aligned}$$

If n is odd, a regular tournament gives

$$p_{2,\min} = n\left(\frac{n-1}{2}\right)^2 = \frac{n(n-1)^2}{4}$$

The number of Rényi 2-classes for odd n would then be

$$\begin{aligned}\frac{1}{2}(p_{2,\max} - p_{2,\min}) + 1 &= \frac{1}{2}\left(\frac{n(n-\frac{1}{2})(n-1)}{3} - \frac{n(n-1)^2}{4}\right) + 1\\ &= \frac{n(n-1)}{24}\left(4\left(n-\frac{1}{2}\right) - 3(n-1)\right) + 1\\ &= \frac{n(n-1)(n+1)}{24} + 1\\ &= \frac{1}{4}\binom{n+1}{3} + 1.\end{aligned}$$

If n is even, a semi-regular tournament has $\frac{n}{2}$ vertices with score $\frac{n}{2}-1$ and $\frac{n}{2}$ vertices with score $\frac{n}{2}$, so

$$\begin{aligned}p_{2,\min} &= \frac{n}{2}\left(\frac{n}{2}-1\right)^2 + \frac{n}{2}\left(\frac{n}{2}\right)^2\\ &= \frac{n\left((n-2)^2+n^2\right)}{8}\\ &= \frac{n(n^2-2n+2)}{4}.\end{aligned}$$

Therefore, the number of Rényi 2-classes for even n is

$$\begin{aligned}\frac{1}{2}(p_{2,\max} - p_{2,\min}) + 1 &= \frac{1}{2}\left(\frac{n(n-\frac{1}{2})(n-1)}{3} - \frac{n(n^2-2n+2)}{4}\right) + 1\\ &= \frac{n}{24}\left(4\left(n-\frac{1}{2}\right)(n-1) - 3(n^2-2n+2)\right) + 1\\ &= \frac{n(n^2-4)}{24} + 1\\ &= \frac{\frac{n}{2}(\frac{n}{2}-1)(\frac{n}{2}+1)}{3} + 1\\ &= 2\binom{\frac{n}{2}+1}{3} + 1.\end{aligned}$$

□

Finally, we answer the question of which tournaments come firs Rényi order for certain values of n. For a vertex $v \in V(T)$, let denote the outset of v; that is the set $\{x \in V(T) : v \to x \in A(T)\}$. A tournament on $n = 4k+3$ vertices is called *doubly regular* if for any $x, y \in V(T)$, we have $|N^+(x) \cap N^+(y)| = k$. Similarly, a regular tour on $n = 4k+1$ vertices is called *quasi doubly regular* if for any $x, y \in V(T)$, we have $|N^+(x) \cap N^+(y)| = k - 1$ or $|N^+(x) \cap N^+(y)|$

We will show that $H_4^*(T)$ is maximum on $\mathcal{R}_n$ if and only if T doubly regular or doubly regular, when $n = 4k+1$ or $n = 4k+3$, tively. We'll use the following lemma which counts the number of in a tournament.

Let T be an n-tournament and define:

- $c_3(T)$ to be the number of 3-cycles;
- $c_4(T)$ to be the number of 4-cycles;
- $t_4(T)$ to be the number of subtournaments of T isomorphic transitive 4-tournament.

We note that the following lemma addresses a problem similar in [6] (their Proposition 1.1).

Lemma 8. *Let T be an n-tournament on n vertices, and c_3, c t_4 defined as above; then*

$$c_4(T) = t_4(T) - \frac{n-3}{4}\left(\binom{n}{3} - 4c_3(T)\right).$$

Proof. Consider the four 4-tournaments up to isomorphism:

1. TS_4: The strong 4-tournament;
2. TT_4: The transitive 4-tournament;
3. TO_4: The tournament with score sequence (1, 1, 1, 3);
4. TK_4: The tournament with score sequence (0, 2, 2, 2).

It is quickly verified that

$$c_3(C_4) = 2, \quad c_3(T_4) = 0, \quad c_3(TK_4) = 1, \quad c_3(TO_4) = 1.$$

Now let T be any n-tournament. Since each 3-cycle belongs to exactly $n-3$ subtournaments of T on 4 vertices, we have

$$(n-3)c_3(T) = 2c_4(T) + to_4(T) + tk_4(T), \tag{1}$$

where $to_4(T)$ and $tk_4(T)$ are the number of TO_4's and TK_4's in T. Furthermore, the total number of subtournaments of T on 4 vertices is equal to

$$c_4(T) + t_4(T) + to_4(T) + tk_4(T) = \binom{n}{4}. \tag{2}$$

Combining equations (2) and (3), we obtain

$$\begin{aligned} c_4(T) &= t_4(T) - \binom{n}{4} + (n-3)c_3(T) \\ &= t_4(T) - \frac{n-3}{4}\left(\binom{n}{3} - 4c_3(T)\right). \end{aligned}$$

□

Lemma 9. *For regular tournaments, $H_4^*(T)$ is maximized where $t_4(T)$ is minimized, and vice-versa.*

Proof. Let $T = (V, A)$ be a regular tournament on $n = 2m+1$ vertices. First note that for $\alpha \in \mathbb{Z}$ with $\alpha \geq 2$, we have $H_\alpha^*(T) = -\mathrm{tr}(\bar{L}(T)^\alpha)$. Furthermore, since T is regular, we have

$$\bar{L}(T) = \frac{1}{\binom{n}{2}}(mI - M).$$

Therefore, by the linearity of the trace and using Lemma 8, we can express H_4^* in terms of $t_4(T)$, noting that TS_4 is the only tournament on 4 vertices with a walk of length 4 from a vertex to itself.

$$\begin{aligned} H_4^*(T) &= -\binom{n}{2}^{-4} \mathrm{tr}\left(m^4 I - 4m^2 M + 6m^2 M^2 - 4mM^3 + M^4\right) \\ &= -\binom{n}{2}^{-4} \left(m^4 n - 12mc_3(T) + 4c_4(T)\right) \\ &= -\binom{n}{2}^{-4}\left(m^4 n - 12mc_3(T) + 4t_4(T) - (n-3)\left(\binom{n}{3} - 4c_3(T)\right)\right). \end{aligned}$$

Note that n, m and c_3 are all constant for regular tournaments on n vertices. □

Theorem 10. *If n is odd, then a tournament T on n vertic comes first in its respective Rényi order must be doubly reg quasi doubly regular. Furthermore, all doubly regular tourname cospectral and therefore come first in the Rényi order.*

Proof. Since H_2 is maximized by regular tournaments for odd n, w that the first tournament in the Rényi order must be regular, so le a regular tournament on $n = 2m + 1$ vertices. Now we look to mi $t_4(T)$. Consider a vertex $x \in V(T)$ and the corresponding subtourn T' on the m vertices in $N^+(x)$. The number of transitive triples i given by

$$
\begin{aligned}
t_3(T') &= \sum_{y \in N^+(x)} \binom{|N^+(x) \cap N^+(y)|}{2} \\
&= \frac{1}{2} \sum_{y \in N^+(x)} \left(|N^+(x) \cap N^+(y)| - \frac{m-1}{2} \right)^2 \\
&\quad + \frac{(m-2)}{2} \sum_{y \in N^+(x)} |N^+(x) \cap N^+(y)| \; - \; \frac{1}{2} \sum_{y \in N^+(x)} \left(\frac{m-}{2} \right. \\
&= \frac{1}{2} \left(\sum_{y \in N^+(x)} \left(|N^+(x) \cap N^+(y)| - \frac{m-1}{2} \right)^2 + \right. \\
&\qquad\qquad (m-2)\binom{m}{2} - m \left(\frac{m-1}{2} \right)
\end{aligned}
$$

If $n \equiv 3 \pmod 4$ and $n = 4k + 3$, then

$$
\begin{aligned}
t_3(T') &\geq \frac{1}{2} \left((m-2)\binom{m}{2} - m \left(\frac{m-1}{2} \right)^2 \right) \\
&= \frac{m}{2} \left((2k-1)k - k^2 \right) \\
&= m \frac{2k^2 - k - k^2}{2} \\
&= m \binom{k}{2},
\end{aligned}
$$

with equality if and only if $|N^+(x) \cap N^+(y)| = \frac{m-1}{2} = k$ for each $y \in N$ Now, since $t_3(T')$ is also the number of T_4s in T in which x is the s it follows that

$$t_4(T) \geq nm\binom{k}{2},$$

with equality if and only if T is doubly regular.

If $n \equiv 1 \pmod 4$ and $n = 4k+1$, then

$$\begin{aligned}
t_3(T') &\geq \frac{1}{2}\left(m\left(\frac{1}{2}\right)^2 + (m-2)\binom{m}{2} - m\left(\frac{m-1}{2}\right)^2\right) \\
&= \frac{m}{2}\left(\frac{1}{4} + (2k-2)\frac{2k-1}{2} - \left(k-\frac{1}{2}\right)^2\right) \\
&= k\left((k-1)(2k-1) - k^2 + k\right) \\
&= k(k-1)^2,
\end{aligned}$$

with equality if and only if $\left|N^+(x) \cap N^+(y)\right| - \frac{m-1}{2}\Big| = \frac{1}{2}$ for each $y \in N^+(x)$. Therefore,

$$t_4(T) \geq nk(k-1)^2,$$

with equality if and only if T is quasi doubly regular.

Now we have that H_4^* is maximized for (quasi) doubly regular tournaments, which means that the first tournament in the Rényi order in these cases must be one of these types. We now show that all doubly regular tournaments are cospectral, or that Rényi α-*entropy can't distinguish them.

Let A be the adjacency matrix of a doubly-regular tournament T on $n = 2m+1 = 4k+3$ vertices. Then $AA^t = mI + k(J-I)$ and $A + A^t = J - I$, where I is the identity matrix and J is the all-ones matrix. Then

$$\begin{aligned}
(A-\lambda I)(A-\lambda I)^t &= AA^t - \lambda(A+A^t) + \lambda^2 I \\
&= mI + k(J-I) - \lambda(J-I) + \lambda^2 I \\
&= (k-\lambda)J + (m-k+\lambda+\lambda^2)I.
\end{aligned}$$

Since $\mathrm{spec}(J) = \{n, 0^{(n-1)}\}$, we have

$$\mathrm{spec}((A-\lambda I)(A-\lambda I)^t) = \{n(k-\lambda)+m-k+\lambda+\lambda^2, (m-k+\lambda+\lambda^2)^{(n-1)}\}.$$

Therefore,

$$\begin{aligned}|A-\lambda I|^2 &= |(A-\lambda I)(A-\lambda I)^t| \\ &= (n(k-\lambda)+m-k+\lambda+\lambda^2)(m-k+\lambda+\lambda^2)^{n-1} \\ &= (m^2-2m\lambda+\lambda^2)(k+1+\lambda+\lambda^2)^{n-1} \\ &= \left((m-\lambda)(k+1+\lambda+\lambda^2)^m\right)^2.\end{aligned}$$

Therefore, since n is odd,

$$|A-\lambda I| = (m-\lambda)(k+1+\lambda+\lambda^2)^m$$

and

$$\operatorname{spec}(A) = \left\{m, \left(-\frac{1}{2} \pm \frac{i\sqrt{n}}{2}\right)^{(m)}\right\}.$$

Finally, if $\bar{L}$ is the normalized Laplacian matrix of T, then $\bar{L}$ and . related by

$$\bar{L} = \frac{2}{n(n-1)}(mI - A),$$

so

$$\begin{aligned}\operatorname{spec}(\bar{L}) &= \left\{\frac{2}{n(n-1)}(m-m), \frac{2}{n(n-1)}\left(m+\frac{1}{2}\pm\frac{i\sqrt{n}}{2}\right)^{(m)}\right\} \\ &= \left\{0, \frac{1}{n-1}\left(1\pm\frac{i}{\sqrt{n}}\right)^{(m)}\right\}.\end{aligned}$$

3 Concluding Remarks

We conclude with a few conjectures.

Conjecture 1. *If tournaments T and T' have the same spectrum, they have the same score sequence.*

We note that there is reason to doubt this conjecture. Merris [7 structed two cospectral graphs on 11 vertices with different degr quences. Perhaps there is an analogous result for tournaments. Our r for the conjecture is simply because we have not found any counter ples.

There in fact exist tournaments with different score sequences and which belong to the same Rényi α-class for $\alpha \leq 5$. Below are the adjacency matrices of two 8-tournaments with score sequences $(2,2,2,3,4,5,5,5)$ and $(1,3,3,3,4,4,4,6)$ that are in the same Rényi 5-class.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Conjecture 2. *The ratio of the number of Rényi 3-classes of tournaments on n vertices to the number of score sequences of tournaments on n vertices is greater than $1/2$ for all $n \geq 2$.*

Conjecture 3. *The ratio of the number of Rényi $(n-1)$-classes of tournaments on n vertices to the number of tournaments on n vertices up to isomorphism is greater than $1/2$ for all $n \geq 2$.*

References

[1] L. Bai, E. R. Hancock, L. Han, and P. Ren. Graph clustering using graph entropy complexity traces. In *Proceedings of the 21st International Conference on Pattern Recognition (ICPR2012)*, pages 2881–2884. IEEE, 2012.

[2] M. Dairyko, L. Hogben, J. C.-H. Lin, J. Lockhart, D. Roberson, S. Severini, and M. Young. Note on von Neumann and Rényi entropies of a graph. *Linear Algebra and its Applications*, 521:240–253, 2017.

[3] L. Han, F. Escolano, E. R. Hancock, and R. C. Wilson. Graph characterizations from von Neumann entropy. *Pattern Recognition Letters*, 33(15):1958–1967, 2012.

[4] H. G. Landau. On dominance relations and the structure of animal societies: I. effect of inherent characteristics. *The bulletin of mathematical biophysics*, 13(1):1–19, 1951.

[5] H. G. Landau. On dominance relations and the structure of an societies: Iii the condition for a score structure. *Bulletin of Ma matical Biology*, 15(2):143–148, 1953.

[6] Nati Linial and Avraham Morgenstern. On the number of 4-cycl a tournament. *Journal of Graph Theory*, 83(3):266–276, 2016.

[7] R. Merris. Laplacian graph eigenvectors. *Linear algebra an applications*, 278(1-3):221–236, 1998.

[8] F. Passerini and S. Severini. Quantifying complexity in networks von Neumann entropy. *Int. J. Agent Technol. Syst*, 1:58, 2009.

[9] A. Rényi. On measures of entropy and information. Technical re HUNGARIAN ACADEMY OF SCIENCES, Budapest Hungary, 1

[10] J. von Neumann. *Mathematical Foundations of Quantum Mec ics: New Edition*. Princeton university press, 2018.

[11] C. Ye, R. C. Wilson, and E. R. Hancock. A Jensen-Shannon diverg kernel for directed graphs. In *Joint IAPR International Works on Statistical Techniques in Pattern Recognition (SPR) and St tural and Syntactic Pattern Recognition (SSPR)*, pages 196– Springer, 2016.