Some Generalizations and Limit Theorems on Cancellable Numbers

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Abstract

A cancellable number (CN) is a fraction in which a decimal digit can be removed ("canceled") in the numerator and denominator without changing the value of the number; examples include $\frac{64}{16}$ where the 6 can be canceled and $\frac{98}{49}$ where the 9 can be canceled. We present a few limit theorems and provide several generalizations.

1 Introduction

In 1933, Morley [3] posed the problem of illegitimate cancellation, identifying $\frac{16}{64} = \frac{\not{6}4}{1\not{6}}$, $\frac{65}{26} = \frac{\not{6}5}{2\not{6}}$, $\frac{95}{19} = \frac{\not{9}5}{1\not{9}}$ and $\frac{98}{49} = \frac{\not{9}8}{4\not{9}}$ as the only proper fractions with denominators less than one hundred which could be reduced by the indicated illegal cancellation. B·L· Schwartz [4] provided the solutions for three digit denominators in 1961. A cancellable number (CN) is a fraction in which a decimal digit can be removed ("canceled") in the numerator and denominator without changing the value of the number; examples include $\frac{64}{16}$ where the 6 can be canceled and $\frac{98}{49}$ where the 9 can be canceled. In [1], we studied cancellable numbers that had a cancellable nine, a cancellable zero, a sequence of adjacent cancellable zeros, sequences of adjacent cancellable zeros and nines of the same length. In [2], we introducedP_H-CN numbers for $\frac{L}{M}$ to be a representative for $\frac{L}{M}$ such that H can be canceled $1, 2, 3, \ldots, P$ times, each cancellation producing a representative of $\frac{L}{M}$. In [2] we showed the canceling line never has a positive slope. It was also shown that in base 10, zero and nine are preferential digits for cancellations. In base K, zero and $k-1$ are preferential.

2 A Counting process

Many results were established in our previous two works and that is why we strongly recommend reading [1] and [2] first. For each rational number $\frac{L}{M}$, there is an infinite set of CN representations; yet clearly, not every representative of $\frac{L}{M}$ is cancellable. All sets of cardinality $\aleph_0$ are equivalent so some other device must be found in order to determine just how the CN's are spread out among all the representations of a rational. We begin with the following definition. Let $Z(L, M, n, H, k, l)$ denote the number of CN representations of $\frac{L}{M}$ such that there are n digits in the denominator (priori to cancellation), H, the coefficient of 10^k in the numerator is cancellable with H, the coefficient of 10^l in the denominator. In this section we limit our investigation to the case, $k = l$. We start with the following cancellation:

$$\frac{L}{M} = \frac{a_m 10^m + \ldots + a_{k+1}10^{k+1} + H10^k + a_{k-1}10^{k-1} + \ldots + a_0}{b_n 10^n + \ldots + b_{k+1}10^{k+1} + H10^k + b_{k-1}10^{k-1} + \ldots + b_0}$$

$$= \frac{a_m 10^{m-1} + \ldots + a_{k+1}10^k + a_{k-1}10^{k-1} + \ldots + a_0}{b_n 10^{n-1} + \ldots + b_{k+1}10^{k+1} + b_{k-1}10^{k-1} + \ldots + b_0}$$

Certain sets are always considered together and consistent with our work in [2]. We make the following observations:

$$10^{k+1}A_1 = a_m 10^m + \ldots + a_{k+1}10^{k+1}, \qquad 10^{k+1}B_1 = b_n 10^n + \ldots + b_{k+1}10^{k+1}$$

$$A_2 = a_{k-1}10^{k-1} + \ldots + a_0, \qquad B_2 = b_{k-1}10^{k-1} + \ldots + b_0$$

$$\frac{L}{M} = \frac{10^{k+1}A_1 + H10^k + A_2}{10^{k+1}B_1 + H10^k + B_2} = \frac{10^k A_1 + A_2}{10^k B_1 + B_2}$$

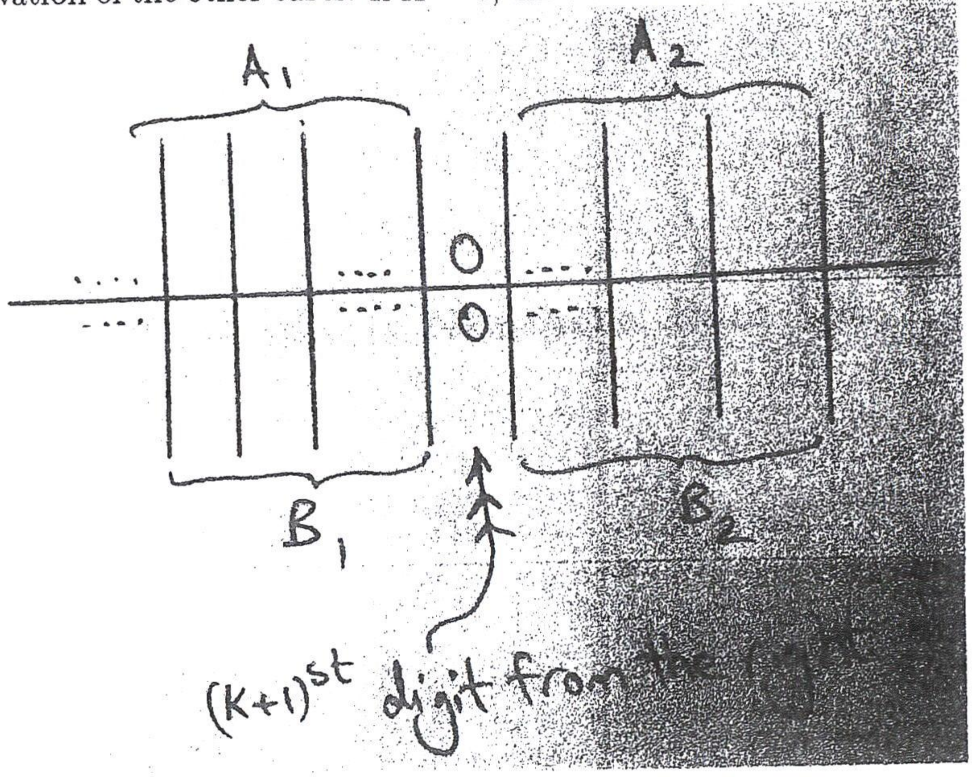
$$\frac{L}{M} = \frac{9A_1 + H}{9B_1 + H}, \qquad \frac{L}{M} = \frac{H10^k - 9A_2}{H10^k - 9B_2}$$

Equivalently, we arrive at the following results:

$$A_1 = \frac{L}{9M}(9B_1 + H) - \frac{H}{9}$$

$$A_2 = \frac{H10^k}{9} + \frac{L}{9M}(9B_2 - H10^k)$$

Recalling that the number of integers contained in $U < x \le V$ is $[V] - [U]$, where [V] is the greatest integer function of V, it is now possible to derive expressions for $Z(L, M, n, H, k, k)$. In addition to the assumption $L \ge M$, it will be assumed that $(L, M) = 1$. The Z function for $H = 0$ and $H = 9$ can be acquired with relative simplicity and these are noted prior to the derivation of the other cases. If $H = 0$, the schematic for a CN is as follows:



From the above set of equations, we have $A_1 = \frac{LB_1}{M}$. Since $(L, M) = 1$ and A_1 is integral, B_1 is a multiple of M and A_1 is a multiple of L. So $B_1 = NM$ and $A_1 = NL$. We define S_1 and S_2 as permissible values for B_1 and A_1, respectively. B_1 has $n - (k+1)$ digits. We make the following observations:

$$\frac{1}{M}(10^{n-(k+2)} - 1) < S_1 \le \frac{1}{M}(10^{n-(k+1)} - 1)$$

$$S_1 = \left\{ \left[\frac{1}{M}(10^{n-(k+1)} - 1)\right], \left[\frac{1}{M}(10^{n-(k+1)} - 1)\right] - 1, \ldots, \left[\frac{1}{M}(10^{n-(k+2)} - 1)\right] + 1 \right\}$$

The above defines values of S_1 that satisfy the constraints on B_1. Similarly, A_1 has at least $n - (k+1)$ digits, hence $S_2 > \frac{1}{L}(10^{n-(k+2)} - 1)$.

$$S_2 = \left\{ \left[\frac{1}{L}(10^{n-(k+2)} - 1) \right] + 1, \left[\frac{1}{L}(10^{n-(k+2)} - 1) \right] + 2, \ldots \right\}$$

The above defines values of S_2 that satisfy the constraints on A_1. Clearly, $S = S_1 \cap S_2 = S_1$ and the number of elements of S_1 is as follows:

$$O(S) = \left[\frac{1}{M}(10^{n-(k+1)} - 1) \right] - \left[\frac{1}{M}(10^{n-(k+2)} - 1) \right]$$

Quite analogously, $A_2 = \frac{LB_2}{M}$, where B_2 and A_2 are again multiples of M and L, respectively. A_2 and B_2 have k or fewer digits. If T is the set of permissible numbers, the following results are obtained:

$$T = \left\{ \left[\frac{1}{M}(10^k - 1) \right], \ldots, 0 \right\} \cap \left\{ \frac{1}{L}(10^k - 1), \ldots, 0 \right\} =$$

$$\left\{ \frac{1}{L}(10^k - 1), \ldots, 0 \right\}$$

and $O(T) = \left[\frac{1}{L}(10^k - 1) \right] - \left[-\frac{1}{L} \right] = \left[\frac{1}{L}(10^k - 1) \right] + 1$

We now arrive at our first result.

Theorem 2.1: The number of CN representatives $Z(L, M, n, 0, k, k)$ is:

$$\left(\left[\frac{1}{M}(10^{n-(k+1)} - 1) \right] - \left[\frac{1}{M}(10^{n-(k+2)} - 1) \right] \right) \left(\left[\frac{1}{L}(10^k - 1) \right] + 1 \right).$$

As an example, consider $Z(23, 2, 4, 0, 2, 2) =$

$$\left(\left[\tfrac{1}{2}(10^{4-3} - 1) \right] - \left[\tfrac{1}{2}(10^{4-4} - 1) \right] \right) \left(\left[\tfrac{1}{23}(10^2 - 1) \right] + 1 \right) = 20$$

The values of A_1 and B_1 follow from equations for S_1 and S_2, while the values of A_2 and B_2 follow from equations for T. The 20 CN representations given by these equations are:

$$\frac{23000}{2000}, \frac{23023}{2002}, \frac{23046}{2004}, \frac{23069}{2006}, \frac{23092}{2008}$$

$$\frac{46000}{4000}, \frac{46023}{4002}, \frac{46046}{4004}, \frac{46069}{4006}, \frac{46092}{4008}$$

$$\frac{69000}{6000}, \frac{69023}{6002}, \frac{69046}{6004}, \frac{69069}{6006}, \frac{69092}{6008}$$

$$\frac{92000}{8000}, \frac{92023}{8002}, \frac{92046}{8004}, \frac{92069}{8006}, \frac{92092}{8008}$$

Similarly, for $H = 9$ and $k = l$, we can derive the following.

Theorem 2.2: The number of CN representatives $Z(L, M, n, 9, k, k)$ is

$$\left(\left[\frac{1}{M}(10^{n-(k+1)})\right] - \left[\frac{1}{M}(10^{n-(k+2)})\right]\right)\left(\left[-1-(\frac{-(1+10^k)}{L})\right]\right).$$

The sets $\{A_1\}$, $\{B_1\}$, $\{A_2\}$, and $\{B_2\}$ fall out of the derivation of the Z function for $H = 9$ just as for $H = 0$.

For arbitrary L and M, one can always find CN representations of $\frac{L}{M}$ where either a zero or nine (or both) can be canceled. Such generality for L and M does not apply if other values of H are to be canceled. This lack of generality creates considerable difficulty in the derivation of the Z function for H's other than zero and nine. In order to determine $Z(L, M, n, H, k, k)$, $H = 1, 2, 3, 4, 5, 6, 7, 8, 9$; the defining equations of A_1 and A_2 are initially subject only to the constraint that the A's and B's shall be integers and later the constraint on magnitude is imposed. A_1 and B_1 will be considered first. From using previous equations in [1] and solving for B_1 rather than A_1, we get

$$B_1 = L^{\Phi(M)-1}(\frac{M-L}{9})H(mod\ \ M)$$

$$A_1 = \frac{1}{M}\left\{L^{\Phi(M)}(\frac{M-L}{9})Hmod\ \ M) - (\frac{M-L}{9})H\right\}$$

The constraints on the magnitude of B_1 and A_1 are $10^{n-(k+2)} - 1 < B_1 \leq 10^{n-(k+1)} - 1$ and $A_1 > b_1$. These inequalities lead us to the following:

$$\max\ B_1 = \max\left\{L^{\Phi(M)-1}(\tfrac{M-L}{9})H(mod\ \ M)\right\} \leq 10^{n-(k+1)} - 1$$

$$\min\ B_1 = \min\left\{L^{\Phi(M)-1}(\tfrac{M-L}{9})H(mod\ \ M)\right\} > 10^{n-(k+2)} - 1$$

$$\max\ A_1 = \tfrac{1}{M}\left\{L \cdot maxB_1 - (\tfrac{M-L}{9})H\right\}$$

min $A_1 = \frac{1}{M}\left\{L \cdot minB_1 - (\frac{M-L}{9})H\right\}$

nilarly, we obtain these equalities:

$$B_2 = L^{\Phi(M)-1}(\frac{L-M}{9})10^k H(mod\ \ M)$$

$$A_2 = \frac{1}{M}\left\{L^{\Phi(M)(\frac{L-M}{9})10^k H} mod\ \ M) - 10^k H(\frac{L-M}{9})\right\}$$

r A_2 and B_2, the conditions on magnitude are $-1 < A_2 \leq 10^k - 1$ and $< B_2 \leq 10^k - 1$. Accordingly,

max B_2=max $\left\{L^{\Phi(M)-1}(\frac{L-M}{9})10^k H(mod\ \ M)\right\} \leq 10^k - 1$

min B_2=min $\left\{L^{\Phi(M)-1}(\frac{L-M}{9})10^k H(mod\ \ M)\right\} > -1$

max $A_2 = \frac{1}{M}\left\{L \cdot maxB_2 - 10^k H(\frac{L-M}{9})\right\}$

min $A_2 = \frac{1}{M}\left\{L \cdot minB_2 - 10^k H(\frac{L-M}{9})\right\}$

ieorem 2.3: The number of CN representatives $Z(L, M, n, H, k, k)$ is

$$\left(\frac{maxB_1 - minB_1}{M} + 1\right)\left(\frac{maxB_2 - minB_2}{M} + 1\right)$$

ere max B_1, min B_1, max B_2 and min B_2 are defined by equations ove.

an example, consider $Z(28, 19, 4, 2, 2, 2)$ with $\Phi(19) = 18$, we obtain

max $B_1 = max\left\{28^{17}(-2)(mod\ \ 19)\right\} \leq 9 = 4$

min $B_1 = min\left\{28^{17}(-2)(mod\ \ 19)\right\} > 0 = 4$

max $A_1 = \frac{1}{19}\left\{28(4) + 2\right\} = 6$

min $A_1 = \frac{1}{19}\left\{28(4) + 2\right\} = 6$

$$\max B_2 \leq max\left\{ 28^{17}(200)(mod\ \ 19) \right\} \leq 99 \leq 94$$

$$\min B_2 \geq min\left\{ 28^{17}(200)(mod\ \ 19) \right\} > -1 = 18$$

$$\max A_2 \leq \tfrac{1}{19}\left\{ 28(94) - 200 \right\} \leq 128$$

$$\min A_2 \geq \tfrac{1}{19}\left\{ 28(18) - 200 \right\} = 16$$

Since 128 has too many digits, max A_2=128-2(28)=72. Accordingly, max B_2= 94-2(19)=56. Hence, $Z(28, 19, 4, 2, 2, 2) = (\frac{4-4}{19} + 1) \cdot (\frac{56-18}{19} + 1) = 3$. The required representations of $\frac{28}{19}$ are: $\frac{6216}{4218}$, $\frac{6244}{4237}$, and $\frac{6272}{4256}$. It is of interest to note that the first representation is the same as that given in [2].
As an example, we try to find $Z(38, 17, 5, 6, 2, 2)$. Notice that $\Phi(17) = 16$.

$$\max B_1 = max\left\{ 38^{15}(-14)(mod\ \ 17) \right\} \leq 99 = 90$$

$$\min B_1 = min\left\{ 38^{15}(-14)(mod\ \ 17) \right\} > 9 = 22$$

$$\max A_1 = \tfrac{1}{17}\left\{ 38(90) + 14 \right\} = 202$$

$$\min A_1 = \tfrac{1}{17}\left\{ 38(22) + 14 \right\} = 50$$

$$\max B_2 \leq max\left\{ 38^{15}(1400)(mod\ \ 17) \right\} \leq 99 \leq 95$$

$$\min B_2 \geq min\left\{ 38^{15}(1400)(mod\ \ 17) \right\} > -1 = 10$$

$$\max A_2 \leq \tfrac{1}{17}\left\{ 38(95) - 1400 \right\} \leq 130 = 92$$

$$\min A_2 \geq \tfrac{1}{17}\left\{ 38(10) - 1400 \right\} \geq -60 = 16$$

Here, maxA_2 = 92, minA_2 = 16, maxB_2 = 78, and minB_2 = 44. Accordingly, $Z(38, 17, 5, 6, 2, 2) = 5(3) = 15$. The required representation of $\frac{38}{17}$ are indicated by the schemes:

$$\frac{50616}{22644}, \frac{50654}{22661}, \frac{50692}{22678}$$
$$\frac{88616}{39644}, \frac{88654}{39661}, \frac{88692}{22678}$$

. . . .

$$\frac{202616}{90644}, \frac{202654}{90661}, \frac{202692}{90678}$$

Another Attempt

all L and M, we can find a CN with vertical cancellable zero and a CN
ı vertical cancellable nine. It is not unreasonable to guess that the ratio
ıe respective Z functions might be near unity. We have immediately:

$$K = \frac{Z(L, M, n, 0, k, k)}{Z(L, M, n, 9, k, k)} =$$

$$\frac{\left(\left[\frac{1}{M}(10^{n-(k+1)} - 1)\right] - \left[\frac{1}{M}(10^{n-(k+2)} - 1)\right]\right)\left(\left[\frac{1}{L}(10^k - 1)\right] + 1\right)}{\left(\left[\frac{1}{M}(10^{n-(k+1)})\right] - \left[\frac{1}{M}(10^{n-(k+2)})\right]\right)\left(-\left[\frac{1}{L}(1 + 10^k)\right] - 1\right)}$$

ːorem 3.1: The following limit is established:

$$\lim_{n\to\infty} K = \lim_{n\to\infty}\left(\frac{Z(L, M, n, 0, k, k)}{Z(L, M, n, 9, k, k)}\right) = -\frac{1 + \left[\frac{1}{L}(10^k - 1)\right]}{1 + \left[-\frac{1}{L}(10^k + 1)\right]}.$$

ʳ let $\frac{10^k}{L} = S + \frac{R}{L}$. We distinguish 2 cases.
Case I: $R \neq 0$ where $\lim_{n\to\infty} K = 1 + \frac{1}{S}$
Case II: $R = 0$ where $\lim_{n\to\infty} K = 1$
ither case, $\lim_{n\to\infty} K$ is approximately unity for $10^k >> L$. The most
ptional case is for $L > 10^k$ and $R \neq 0$ when the limit becomes infinite.
ın example, with $L = 13$ and $K = 1$, we have

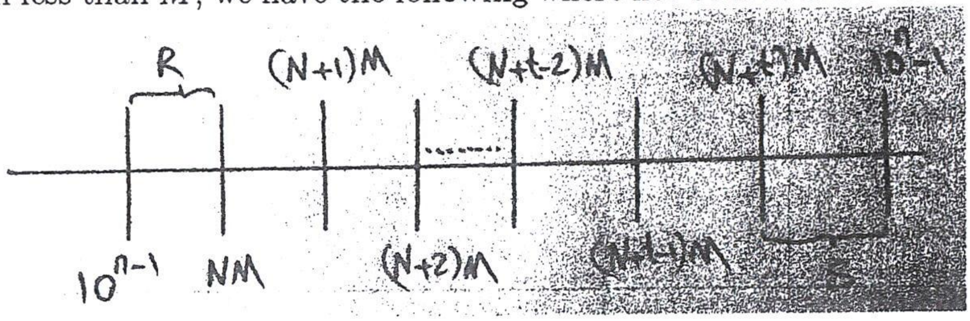
$$\lim_{n\to\infty} \frac{Z(L, Q, n, 0, 1, 1)}{Z(L, Q, n, 9, 1, 1)} = -\frac{1 + [\frac{9}{13}]}{1 + [-\frac{11}{13}]} = \infty$$

e 1_0-CN 's for 13 with $k = 1$ are $\frac{1300}{100}$, $\frac{131300}{10100}$, etc. If $M = 2$, we have
, $\frac{131300}{20200}$, etc. On the other hand, if there is a 1_9-CN for $\frac{13}{M}$, there must
ıne for $\frac{13}{1}$. Such a CN must have the form $\frac{13}{1} = \frac{100u+90+y}{100x+90+y} = \frac{u+v}{x+y}$,
vhich there is no Diophantine solution such that $0 < v < 10$. Similar
rems may be derived for ratios of Z functions with other H's.

Some Limit Theorems

. far more interesting problem is to see what proportions of representations f a rational fraction are cancellable and subject to certain constraints. 'nce again, the method will be illustrated for the arithmetically simple ase of cancellable zeros. The method will consist of the following steps: ind bounds on $D(1, M, n)$, the number of representatives of $\frac{L}{M}$ with n igits in the denominator. Then compute $\lim_{n\to\infty} \frac{Z(L,M,n,0,k,k)}{D(L,M,n)}$.

'here are $9\times 10^{n-1}$ n-digit numbers, namely: $10^n - 1, \ldots, 10^{n-1}$. The first nd last members of this set have no common factors, so not both $10^n - 1$ nd 10^{n-1} are divisible by M unless $M = 1$. If $M \neq 1$, and with R and S ach less than M, we have the following where not both R and S are zeros.



'he number of intervals is

$$\frac{10^n - 1 - S - 10^{n-1} - R}{M}.$$

'he maximum number of intervals is found if R or S is zero and S or R quals unity; the minimum is found when both R and S equal $M - 1$. The ifference, for $M \neq 1$ is $\frac{1}{M}(2M - 3) = 2 - \frac{3}{M}$. If $M = 1$, this difference ; clearly zero. Since $D(L, M, n)$ is one more than the number of intervals, 'e can conclude the following:

$$\frac{9\times 10^{n-1} - M + 1}{M} \leq D(L, M, n) \leq \begin{cases} \frac{9\times 10^{n-1} - M - 2}{M} & M \neq 1 \\ 9\times 10^{n-1} & M = 1 \end{cases}$$

ı order to effectively work with the ratio of the D and Z functions, it is ecessary to make some exchanges for the bracket functions. We use the bvious inequalities, $[\frac{a}{b}] > \frac{a}{b} - 1$ and $[\frac{a}{b}] < \frac{a}{b} + 1$. Accordingly,

$$\frac{Z(L, M, n, 0, k, k)}{D(L, M, n)} \leq$$

$$\left(\left[\frac{1}{M}(10^{n-(k+1)} - 1)\right] - \left[\frac{1}{M}(10^{n-(k+2)} - 1)\right]\right)\times$$