Some Systems of Simultaneous Linear Recurrences and Their Applications to Computing of Graph Expression Lengths

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Abstract

The paper proposes techniques which provide closed-form solutions for special simultaneous systems of two and three linear recurrences. These systems are characterized by particular restrictions on their coefficients. We discuss the application of these systems to some algorithmic problems associated with relationship between algebraic expressions and graphs. Using decomposition methods described in the paper we generate the simultaneous recurrences for graph expression lengths and solve them with the proposed approach.

1 Introduction

A *recurrence relation* (*recurrence*) for the sequence a_0, a_1, a_2, ... is an equation that expresses the term a_n of a sequence as a function of certain preceding terms a_i, $i < n$, for each $n \geq n_0$.

A recurrence with *finite history* depends on a fixed number of earlier values, while an equation that depends on all preceding values has a *full* *history*. The recurrence with finite history is of *k-th order* if a_n can be expressed in terms of $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$, i.e.

$$a_n = f\left(a_{n-1}, a_{n-2}, \ldots, a_{n-k}\right), \qquad n \geq k.$$

Initial conditions for this recurrence specify particular values (*initial values*) of $a_0, a_1, \ldots, a_{k-1}$.

The recurrence is *linear* if it expresses a_n as a linear function of preceding terms. Otherwise the recurrence is *nonlinear*. A linear recurrence of k-th order is an equation of the form

$$a_n = C_{n-1}a_{n-1} + C_{n-2}a_{n-2} + \ldots + C_0 a_0 + \alpha(n), \qquad n \geq 1. \tag{1}$$

The function $\alpha(n)$ is the *particularity function*. If $\alpha(n) = 0$ the recurrence (1) is called *homogeneous*. Otherwise, it is *nonhomogeneous*. Each coefficient C_i may be either a *constant coefficient*, or a function on n, i.e., a *variable coefficient*.

The goal of numerous investigations devoted to recurrence relations is to find *closed-form* (explicit) solutions for these equations, i.e., to express a_n directly in n. Specifically, generic recurrences and methods for their solutions are discussed in [9], [15], [16], [21], [23], [24], [32].

Linear recurrences of k-th order with constant coefficients are the most common examples of recurrence relations. Many of them are solved by *methods* of *characteristic equations* (*roots*) and *generating functions*. Also, other methods for solving special equations of this type are considered in [9], [23], [24], [32]. A *matrix method* to solve the recurrences of order $k \geq 3$, when use of the traditional methods is rather difficult, is provided in [22].

The usual methods can be used elegantly when constant coefficients are of special form. Linear recurrences with coefficients in arithmetic and geometric progression are solved in [7].

Linear nonhomogeneous recurrences with constant coefficients and particularity functions of special forms are studied in [36] and [38].

Linear recurrences with variable, mainly, polynomial coefficients are surveyed in [5], [6], [15], [16], [30], [31]. Specifically, the approach presented in [6] is to transform an equation to a previously solved equation.

Some linear recurrences with full history are analyzed in [15]. A special linear recurrence with full history that arises in a number of applications, is solved in [35].

As noted in [40], sometimes recurrences working in tandem are more effective than a single recurrence. Ways for solving simultaneous systems of two linear recurrences are discussed in [16], [24], [40]. With simultaneous recurrences, one uses a substitution from one recurrence to reduce the number of different sequences occurring in other recurrences. The objective is to reduce the solution of the initial system to the solution of one or more independent recurrences. In the general case, for two or more simultaneous recurrences, it is possible to divide the system into individual recurrences using the Hamilton-Cayley theorem [4]. The obtained recurrences each of which has a single unknown will be of a higher order than the initial ones. They are to be solved by the standard methods for regular recurrences.

The purpose of this paper is to solve a following simultaneous system of three linear nonhomogeneous recurrences of first order with constant coefficients:

$$\begin{cases} a_n = \alpha_{11}a_{n-1} + \alpha_{12}b_{n-1} + \alpha_{13}c_{n-1} + \alpha_1 \\ b_n = \alpha_{21}a_{n-1} + \alpha_{22}b_{n-1} + \alpha_{23}c_{n-1} + \alpha_2 \\ c_n = \alpha_{31}a_{n-1} + \alpha_{32}b_{n-1} + \alpha_{33}c_{n-1} + \alpha_3, \end{cases} \tag{2}$$

when

$$\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{21} + \alpha_{22} + \alpha_{23} = \alpha_{31} + \alpha_{32} + \alpha_{33}$$

and a sequence b_n is an *affine combination* (linear combination with the sum of weights equal to 1) of sequences a_n and c_n (a_0, b_0, and c_0 are initial values of a, b, and c, respectively). The particularity functions (α_1, α_2, α_3) in all recurrences of (2) are constants (*absolute terms*).

The application of this system to some algorithmic theory problems is discussed in Section 3.

2 Graphs and graph expressions

A *graph* G consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge corresponds to a pair (v, w) of vertices. If the edges are ordered pairs of vertices (i.e., the pair (v, w) is different from the pair (w, v)), then we call the graph *directed* or *digraph*; otherwise, we call it *undirected*. If (v, w) is an edge in a digraph, we say that (v, w) *leaves* vertex v and *enters* vertex w. A vertex in a digraph is a *source* if no edges enter it, and a *sink* if no edges leave it. A *path* from vertex v_0 to vertex v_k in a graph G is a sequence of its vertices $[v_0, v_1, v_2, \ldots, v_{k-1}, v_k]$ such that $(v_{i-1}, v_i) \in E(G)$ for $1 \le i \le k$. G is an *acyclic graph* if there is no closed path $[v_0, v_1, v_2, \ldots, v_k, v_0]$ in G. A two-terminal directed acyclic graph (*st-dag*) has only one source s and only one sink t. In an st-dag, every vertex lies on some path from s to t.

A graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. A graph G is *homeomorphic* to a graph G' (a *homeomorph* of G') if G can be obtained by subdividing edges of G' with new vertices. Two graphs G and G' are *isomorphic* if there exists a bijection $f : V(G) \to V(G')$ such that $(v, w) \in E(G)$ if and only if $(f(v), f(w)) \in E(G')$.

The *transpose* [8] of a digraph G is another digraph on the same set of vertices with all of the edges reversed compared to the orientation of the corresponding edges in G. In graph theory, a *skew-symmetric graph* is a directed graph that is isomorphic to its own transpose graph.

Given a graph G, an *edge labeling* is a function $E(G) \longrightarrow R$, where R is a ring equipped with two binary operations $+$ (addition or disjoint union) and $\cdot$ (multiplication or concatenation, also denoted by juxtaposition when no ambiguity arises). In what follows, elements of R are called *labels*, and a *labeled graph* refers to an edge-labeled graph with all labels distinct.

Each path between the source and the sink (a *spanning path*) in an st-dag can be represented by a product of all edge labels of the path. We define the sum of edge label products corresponding to all possible spanning paths of an st-dag G as the *canonical expression* of G. The label order in every product (from the left to the right) is identical to the order of corresponding edges in the path (from the source to the sink). An algebraic expression is called a *graph expression* (a *factoring of an st-dag* in [2]) if it is algebraically equivalent to the canonical expression of an st-dag. A graph expression consists of labels, the two ring operators $+$ and $\cdot$, and parentheses. For example, clearly, the algebraic expression $ab+bc$ is not a graph expression.

We define the total number of labels in an algebraic expression as its *complexity*. An *optimal representation of the algebraic expression F* is an expression of minimum complexity algebraically equivalent to F. Graph expressions with a minimum (or, at least, a polynomial) complexity may be considered as a key to generating efficient algorithms on distributed systems. Therefore, our intention is to simplify a graph expression to its optimal representation or, at least, to the expression with polynomial complexity in relation to the graph's size.

A *series-parallel graph* is defined recursively as follows:

(i) A single edge (u, v) is a series-parallel graph with source u and sink v.

(ii) If G_1 and G_2 are series-parallel graphs, so is the graph obtained by either of the following operations:

(a) *Parallel composition*: identify the source of G_1 with the source of G_2 and the sink of G_1 with the sink of G_2.

(b) *Series composition*: identify the sink of G_1 with the source of G_2.

A series-parallel graph expression has a representation in which each abel appears only once [2], [19] (a *read-once formula* [12] in which Boolean operations are replaced by their arithmetic counterparts). This representation is optimal for the series-parallel graph expression. For example, he canonical expression of the series-parallel graph presented in Fig. 1 is $abd+abe+acd+ace+fe+fd$ and it can be reduced to $(a(b+c)+f)(d+e)$.

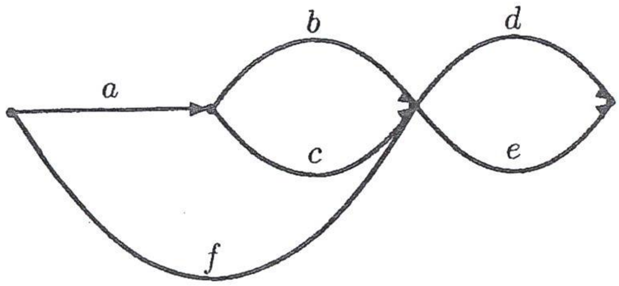

Figure 1: A series-parallel graph.

A *Fibonacci graph* [14] which gives a generic example of non-series-parallel graphs has vertices $\{1,2,3,\ldots,n\}$ and edges $\{(v,v+1) \mid v = 1,2,\ldots,n-1\} \cup \{(v,v+2) \mid v=1,2,\ldots,n-2\}$. As shown in [3], an st-dag is series-parallel if and only if it does not contain a subgraph which is a homeomorph of the *forbidden subgraph* positioned between vertices 1 and 4 of the Fibonacci graph illustrated in Fig. 2. Possible optimal representations of its expression are $a_1(a_2a_3+b_2)+b_1a_3$ or $(a_1a_2+b_1)a_3+a_1b_2$. For this reason, an expression of a non-series-parallel st-dag can not be represented as a read-once formula. However, generating the optimum factored form for expressions which cannot be reduced to read-once formulae is **NP**-complete [39].

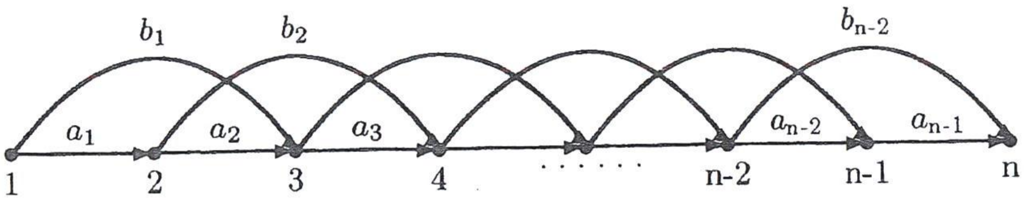

Figure 2: A Fibonacci graph.

Problems related to computations on labeled graphs have applications in various areas. Specifically, flow [37], scheduling [11], reliability [33], economical [29] problems which are either intractable or have complicated solutions in the general case are solvable for series-parallel graphs.

Interrelations between graphs and expressions are discussed in [1], [2], [10], [13], [17], [18], [19], [25],[26], [27], [28], [29], [34], and other works. In a number of papers, in particular, in [13], [25], the algorithms developed in order to obtain good factored forms are presented. In this paper we describe *decomposition methods* for generating compact graph expressions.

3 Generating graph expressions by decomposition methods

These methods are based on recursive revealing subgraphs of approximately equal sizes in a graph of a regular structure. The resulting expression is produced by a special composition of subexpressions describing these subgraphs. Subgraphs revealed in all recursive steps are divided into the same number of subgraphs of proportionally decreasing sizes. The existence of a decomposition method for a graph G is a sufficient condition for the existence of a polynomial-size expression for G. The expression's complexity depends, in particular, on the number of revealed subgraphs in each recursive step of the decomposition procedure.

In [19] we apply a decomposition method to a *Fibonacci graph*. Denote by $F(p,q)$ a subexpression related to its subgraph (which is a Fibonacci graph as well) having a source p and a sink q. If $q-p \geq 2$, then we choose any *decomposition vertex* i $(p+1 \leq i \leq q-1)$ in a subgraph, and, in effect, split it at this vertex (Fig. 3). Any path from vertex p to vertex q passes through vertex i or avoids it via edge b_{i-1}. Therefore, in the general case a current subgraph is decomposed into four new subgraphs and

$$F(p,q) \leftarrow F(p,i)F(i,q) + F(p,i-1)b_{i-1}F(i+1,q).$$

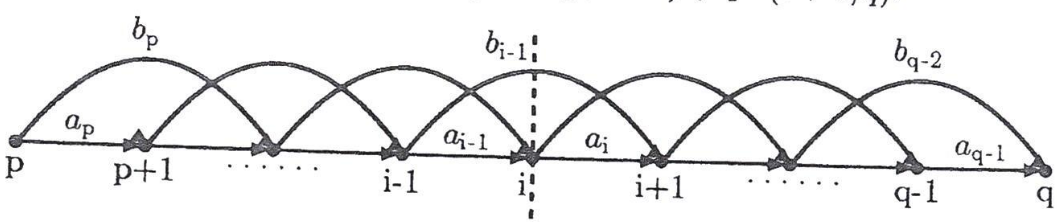

Figure 3: Decomposition of a Fibonacci subgraph at vertex i.

As shown in [19], the shortest expression obtained by the decomposition method is achieved for i chosen as $\lfloor \frac{q+p}{2} \rfloor$ or $\lceil \frac{q+p}{2} \rceil$ in each recursive step. By the *master theorem* [8], the total number of labels $T(m)$ in this expression for an m-vertex Fibonacci graph is $O\left(m^2\right)$. For m that is a power of two ($m = 2^r$ for some positive integer $r \geq 2$),

$$T(m) = 2T\left(\frac{m}{2}\right) + T\left(\frac{m}{2} - 1\right) + T\left(\frac{m}{2} + 1\right) + 1$$

that is presented in closed form as

$$T(m) = \frac{1}{3}\left(\frac{19}{16}m^2 - 1\right).$$

Consider a more complicated graph called a *full square rhomboid* [18]. We split every non-trivial subgraph through two decomposition vertices with the same absolute ordinal numbers which are chosen in the middle of the *upper* and the *lower vertex rows* in the graph (see the example in Fig. 4). Any path from the source to the sink of the graph in Fig. 4 passes either through one of the decomposition vertices or through edge b_3. The graph is decomposed into six subgraphs two of which are also full square rhomboids (FSR) and four ones are so called *single-leaf full square rhomboids* (FSR_1). Each FSR_1 is decomposed into six new subgraphs in the similar way (Fig. 5(a)). They are one FSR, three FSR_1 and two *dipterous full square rhomboids* (FSR_2). Decomposition of possible varieties of FSR_2 (see the example in Fig. 5(b)) gives two FSR_1 and four FSR_2 subgraphs.

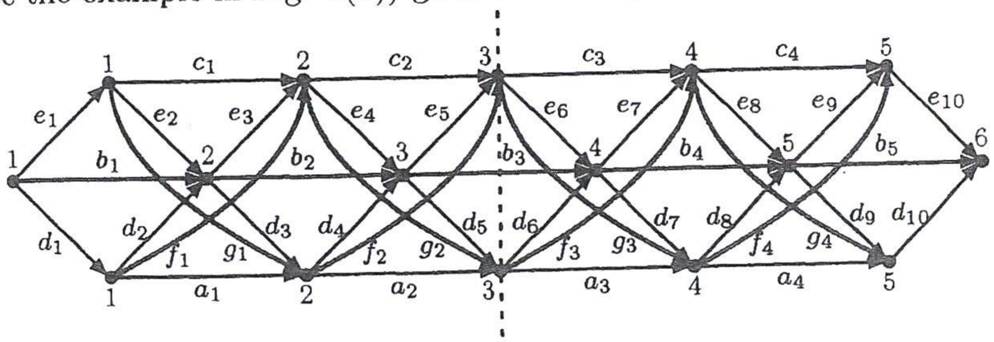

Figure 4: Decomposition of a full square rhomboid.

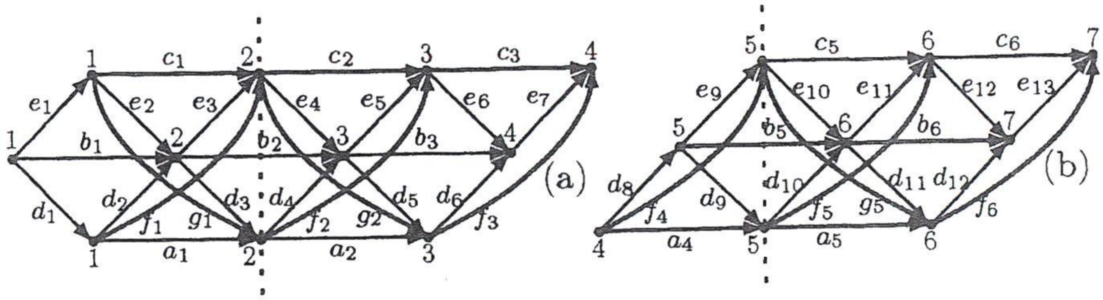

Figure 5: Decomposition of single-leaf and dipterous full square rhomboids.

We use a number of vertices of the *central vertex row* in the graph as a parameter which characterizes the size of the graph. Thus the total number of labels $T(m)$ in the expression of a full square rhomboid of size

m for $m = 2^r$ $(r \geq 1)$ is defined recursively as follows:

$$\begin{cases} T(m) = 2T\left(\frac{m}{2}\right) + 4T_1\left(\frac{m}{2}\right) + 1 \\ T_1(m) = T\left(\frac{m}{2}\right) + 3T_1\left(\frac{m}{2}\right) + 2T_2\left(\frac{m}{2}\right) + 1 \\ T_2(m) = 2T_1\left(\frac{m}{2}\right) + 4T_2\left(\frac{m}{2}\right) + 1\,, \end{cases} \quad (3)$$

where $T_1(m)$ and $T_2(m)$ are the total numbers of labels in expressions of FSR_1 and FSR_2, respectively, of size m and $T(1) = 0$, $T_1(1) = 1$ $T_2(1) = 3$.

One can see that the sum of the coefficients in each of three simultaneous recurrences (3) equals 6 and $T_1(m) = \frac{1}{2}T(m) + \frac{1}{2}T_2(m)$ for $m \geq 2$. Therefore, system (3) may be presented in general terms as (2).

The similar graph called a *square rhomboid* is considered in [18] and [20]. This graph can be decomposed through two decomposition vertices into six subgraphs in the same way as a full square rhomboid. However, numerically it is more efficient to split this graph through one decomposition vertex located in the middle of the central vertex row [20] (Fig. 6). Any path from the source to the sink of the graph in Fig. 6 passes either through the decomposition vertex or through edge c_3 or through edge a_3. As a result, six analogous subgraphs of three kinds (*square rhomboid, single-leaf square rhomboid* and *dipterous square rhomboid*) appear in each recursive step. Since at this time, the splitting vertex belongs to the row whose size determines the size of the graph, the system for the expression's complexity looks as follows for $m = 2^r$ $(r \geq 3)$:

$$\begin{cases} T(m) = T\left(\frac{m}{2}+1\right) + T\left(\frac{m}{2}\right) + 2T_1\left(\frac{m}{2}\right) + 2T_1\left(\frac{m}{2}-1\right) + 2 \\ T_1(m) = T\left(\frac{m}{2}+1\right) + 3T_1\left(\frac{m}{2}\right) + 2T_2\left(\frac{m}{2}-1\right) + 2 \\ T_2(m) = T_1\left(\frac{m}{2}+1\right) + T_1\left(\frac{m}{2}\right) + 2T_2\left(\frac{m}{2}\right) + 2T_2\left(\frac{m}{2}-1\right) + 2\,. \end{cases} \quad (4)$$

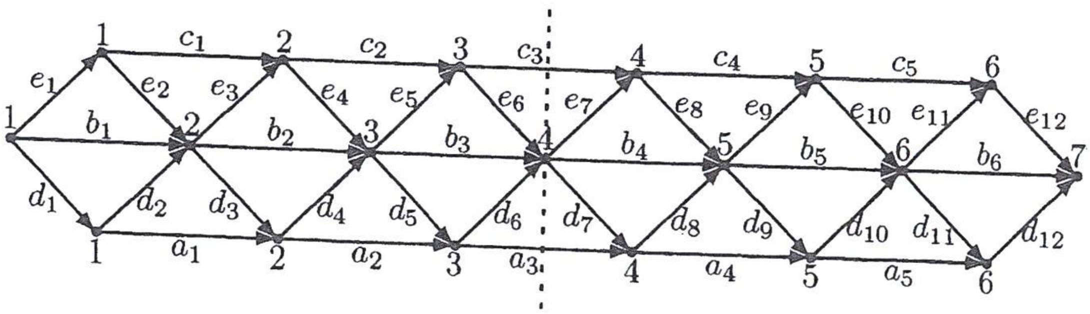

Figure 6: Decomposition of a square rhomboid.

System (4) can not be presented as (2). However, by a number of intermediate findings [20] which allow to express characteristics with arguments $\frac{m}{2} - 1$ and $\frac{m}{2} + 1$ in characteristics with argument $\frac{m}{2}$ and using initial values of $T(m)$, $T_1(m)$, $T_2(m)$, simultaneous recurrences (4) transform to the following ones:

$$\begin{cases} T(m) = \frac{37}{6}T\left(\frac{m}{2}\right) - \frac{1}{6}T_1\left(\frac{m}{2}\right) + 2 \\ T_1(m) = \frac{15}{2}T\left(\frac{m}{2}\right) - \frac{7}{2}T_1\left(\frac{m}{2}\right) + 2T_2\left(\frac{m}{2}\right) + 2 \\ T_2(m) = \frac{16}{3}T\left(\frac{m}{2}\right) - \frac{10}{3}T_1\left(\frac{m}{2}\right) + 4T_2\left(\frac{m}{2}\right) + 2\,, \end{cases} \tag{5}$$

where $m = 2^r$ $(r \geq 3)$, $T(4) = 41$, $T_1(4) = 47$, $T_2(4) = 60$, and $T_1(m) = \frac{13}{19}T(m) + \frac{6}{19}T_2(m)$ for $m \geq 4$. One can see that the sum of the coefficients in each of three simultaneous recurrences (5) equals 6 and, therefore, (5) is an example of a general system (2).

The considered examples show that graphs may be decomposed in different ways. We define graph vertices whose number determines the size of the graph as *basic vertices* (for instance, vertices of the central row in a square and a full square rhomboids or all vertices in a Fibonacci graph). In the case of a full square rhomboid we have a decomposition into subgraphs with disjoint sets of basic vertices *(disjoint decomposition)*. It is possible that a graph of even size, m is decomposed only into subgraphs of sizes $\frac{m}{2}$. An *overlapping decomposition* takes place when subgraphs revealed from the left and from the right of the location of the split have common basic vertices. Specifically, this decomposition is applied to Fibonacci graphs and square rhomboids. An overlapping decomposition implemented on a full square rhomboid divides the graph into ten subgraphs [18] and thus it is not efficient in this case.

Both disjoint and overlapping decompositions can occur under various scenarios.

In the simple case, all subgraphs revealed in the course of decomposition may be of the same kind as the initial graph (for example, in a Fibonacci graph). The complexity of the graph expression is expressed by a single recurrence in this case.

In another graphs only some of subgraphs emerged in the result of decomposition are exactly of the same structure as the initial one. Others are supplemented at one of the end by elements which were in the middle of the split graph. The structure of these *one-sided* subgraphs (single-leaf square and full square rhomboids in the examples above) does not change in the middle and, hence, they are decomposed in the same way. This gives, together with subgraphs of the initial structure and one-sided subgraphs, *two-sided* subgraphs (dipterous square and full square rhomboids in the examples above) supplemented at both ends by the elements of inner structure. The two-sided subgraphs are decomposed likewise and their splitting yields new one-sided and two-sided subgraphs. Hence, subgraphs with ends of two kinds arise throughout decomposition and three simultaneous recurrences appear in this case.

Also, subgraphs with ends of more than two kinds may be formed. This yields more complex systems of recurrences.

Thus complexities of graph expressions derived by decomposition methods are represented with simultaneous systems of linear recurrences. Coefficients in each of the recurrences in a system are equal to the numbers of respective subgraphs. Since subgraphs of all kinds are decomposed into the same numbers of new subgraphs, sums of coefficients in all recurrences of the system are equal. An absolute term in each recurrence equals the number of edges which connect revealed subgraphs (*connecting edges*).

Theorem 1 *Given a skew-symmetric st-dag G of size $m = 2^r$ ($r \geq r_0$, where r_0 is any natural number) suppose an expression of G is derived by disjoint decomposition so that G is decomposed only into subgraphs of sizes $\frac{m}{2}$ and subgraphs with ends of two kinds arise throughout decomposition. Then in the general case the total number of labels $T(m)$ in the expression of G of size m is defined recursively by the following simultaneous system of three linear recurrences with constant coefficients and absolute terms:*

$$\begin{cases} T(m) = \alpha_{11} T\left(\frac{m}{2}\right) + \alpha_{12} T_1\left(\frac{m}{2}\right) + \alpha \\ T_1(m) = \alpha_{21} T\left(\frac{m}{2}\right) + \alpha_{22} T_1\left(\frac{m}{2}\right) + \alpha_{23} T_2\left(\frac{m}{2}\right) + \alpha \\ T_2(m) = \alpha_{32} T_1\left(\frac{m}{2}\right) + \alpha_{33} T_2\left(\frac{m}{2}\right) + \alpha \, , \end{cases} \tag{6}$$

where $T_1(m)$ and $T_2(m)$ are the total numbers of labels in expressions of one-sided and two-sided subgraphs, respectively, of size m revealed in the course of decomposition. At that,

$$\alpha_{11} + \alpha_{12} = \alpha_{21} + \alpha_{22} + \alpha_{23} = \alpha_{32} + \alpha_{33} \tag{7}$$

and

$$T_1(m) = \frac{1}{2} T(m) + \frac{1}{2} T_2(m) . \tag{8}$$

Proof. Since G is skew-symmetric then the number of subgraphs with the end of a given kind adjacent to the location of the split from the left is equal to the number of subgraphs with the end of the same kind adjacent to the location of the split from the right (*symmetric-ends property*).

Because m is a power of two, a size, m' of each revealed subgraph in even and each of the subgraphs is decomposed in its turn only into subgraphs of sizes $\frac{m'}{2}$, etc. The initial graph is decomposed only into subgraphs of the same kind and one-sided subgraphs. A two-sided subgraph is decomposed only into one-sided and two-sided subgraphs. For this reason coefficient α_{13} of $T_2\left(\frac{m}{2}\right)$ in the first recurrence and coefficient α_{31} of $T\left(\frac{m}{2}\right)$ in the third recurrence of system (6) are zeros. Therefore, since sums of coefficients in all recurrences are equal we obtain equation (7). Subgraphs of all kinds are decomposed in the same way, specifically, with the same number of connecting edges. Thus absolute terms in all equations are equal.

Proof of (8) is based on the fact that decomposition rules do not depend on the ends of the graph and thus are the same for subgraphs of all kinds. The initial graph (and subgraph of the same kind) is decomposed into α_{11} subgraphs of the same kind and α_{12} one-sided subgraphs. A two-sided subgraph is decomposed into α_{32} one-sided subgraphs and α_{33} two-sided subgraphs. Hence, $\alpha_{11} = \alpha_{32}$ and $\alpha_{12} = \alpha_{33}$. We denote a subgraph of the same kind as G by [], one-sided subgraphs with the ends supplemented by additional elements adjacent to the location of the split from the left and from the right by [} and {], respectively, and a two-sided subgraph by {}. Using these denotations and based on the symmetric-ends property we illustrate possible decompositions corresponding to the first and the second (without loss of generality) recurrences of (6) as

$$[] \rightarrow \begin{pmatrix} \frac{\alpha_{11}}{2}[] & \frac{\alpha_{11}}{2}[] \\ \frac{\alpha_{12}}{2}[\} & \frac{\alpha_{12}}{2}\{] \end{pmatrix}$$

and

$$\{] \rightarrow \begin{pmatrix} \frac{\alpha_{11}}{2}\{] & \frac{\alpha_{11}}{2}[] \\ \frac{\alpha_{12}}{2}\{\} & \frac{\alpha_{12}}{2}\{] \end{pmatrix}, \tag{9}$$

respectively, where a symbol before the arrow is a decomposed subgraph. Elements of a matrix after the arrow are emerged subgraphs with their numbers. The left column of a matrix includes subgraphs revealed from the left of the location of the split and the right column includes subgraphs revealed from the right. Elements in a row of a matrix corresponds to subgraphs with the same kinds of ends adjacent to the location of the split. In accordance with (9) and the second recurrence of (6) we have

$$\begin{aligned} \alpha_{21} &= \frac{\alpha_{11}}{2} = \frac{\alpha_{11}}{2} + 0 = \frac{\alpha_{11}}{2} + \frac{\alpha_{31}}{2}, \\ \alpha_{22} &= \frac{\alpha_{11}}{2} + \frac{\alpha_{12}}{2} = \frac{\alpha_{12}}{2} + \frac{\alpha_{32}}{2}, \\ \alpha_{23} &= \frac{\alpha_{12}}{2} = 0 + \frac{\alpha_{12}}{2} = \frac{\alpha_{13}}{2} + \frac{\alpha_{33}}{2}. \end{aligned}$$

Thus

$$\begin{aligned} T_1(m) &= \alpha_{21} T\left(\frac{m}{2}\right) + \alpha_{22} T_1\left(\frac{m}{2}\right) + \alpha_{23} T_2\left(\frac{m}{2}\right) + \alpha \\ &= \left(\frac{\alpha_{11}}{2} + \frac{\alpha_{31}}{2}\right) T\left(\frac{m}{2}\right) + \left(\frac{\alpha_{12}}{2} + \frac{\alpha_{32}}{2}\right) T_1\left(\frac{m}{2}\right) + \end{aligned}$$

$$\left(\frac{\alpha_{13}}{2}+\frac{\alpha_{33}}{2}\right)T_2\left(\frac{m}{2}\right)+\alpha$$

$$= \frac{1}{2}\left(\alpha_{11}T\left(\frac{m}{2}\right)+\alpha_{12}T_1\left(\frac{m}{2}\right)+\alpha_{13}T_2\left(\frac{m}{2}\right)+\alpha\right)+$$

$$\frac{1}{2}\left(\alpha_{31}T\left(\frac{m}{2}\right)+\alpha_{32}T_1\left(\frac{m}{2}\right)+\alpha_{33}T_2\left(\frac{m}{2}\right)+\alpha\right)$$

$$= \frac{1}{2}T(m)+\frac{1}{2}T_2(m).$$

The proof of the theorem is complete. ■

Remark 1 *Suppose the initial graph in Theorem 1 is an one-one sided subgraph of G, i.e., not skew-symmetric. It is clear that this graph's expression is also defined by the system like (6) with analogous restrictions.*

One can see that system (6) with restrictions (7) and (8) is a special case of (2) and generalization of (3). As shown above (Fig. 6, system (5)), an overlapping decomposition can also ultimately give system (2).

Therefore, system (2) appears in solving a problem of deriving the explicit form of graph expression complexity for various graphs of regular structure. Solving this system being a rather special problem for a discrete mathematics as a whole, is a common problem from the perspective of the algorithmic theory.

It is possible to divide system (2) into separate recurrences [4] and further to use general methods of linear recurrences solving. However, these methods can be very cumbersome and lead to appearance of high-degree equations. We propose a simpler way that accommodates the restrictions imposed on the coefficients and directly gives closed forms for solutions of (2). As an intermediate step we solve a special simultaneous system of two linear recurrences.

4 A system of two recurrences

Lemma 2 *Given a system*

$$\begin{cases} a_n = \alpha_{11}a_{n-1}+\alpha_{12}b_{n-1}+\alpha_1 \\ b_n = \alpha_{21}a_{n-1}+\alpha_{22}b_{n-1}+\alpha_2, \end{cases}$$

when

$$\alpha_{11}+\alpha_{12}=\alpha_{21}+\alpha_{22} \tag{10}$$

and where a_0 and b_0 are initial values of a and b, respectively, it holds that

Case 1. $\alpha_{12} \neq -\alpha_{21}$, $\alpha_{11}+\alpha_{12} \neq 1$, $\alpha_{11}-\alpha_{21} \neq 1$:

$$\begin{aligned}
a_n &= (\alpha_{11}+\alpha_{12})^n a_0 + \alpha_{12}(b_0-a_0)\frac{(\alpha_{11}+\alpha_{12})^n-(\alpha_{11}-\alpha_{21})^n}{\alpha_{12}+\alpha_{21}} + \\
&\quad \alpha_1\frac{(\alpha_{11}+\alpha_{12})^n-1}{\alpha_{11}+\alpha_{12}-1} + \frac{\alpha_{12}(\alpha_2-\alpha_1)}{\alpha_{12}+\alpha_{21}} \times \\
&\quad \left(\frac{(\alpha_{11}+\alpha_{12})^n-\alpha_{11}-\alpha_{12}}{\alpha_{11}+\alpha_{12}-1} - \frac{(\alpha_{11}-\alpha_{21})^n-\alpha_{11}+\alpha_{21}}{\alpha_{11}-\alpha_{21}-1}\right) \\
b_n &= (\alpha_{11}+\alpha_{12})^n b_0 + \alpha_{21}(a_0-b_0)\frac{(\alpha_{11}+\alpha_{12})^n-(\alpha_{11}-\alpha_{21})^n}{\alpha_{12}+\alpha_{21}} + \\
&\quad \alpha_2\frac{(\alpha_{11}+\alpha_{12})^n-1}{\alpha_{11}+\alpha_{12}-1} + \frac{\alpha_{21}(\alpha_1-\alpha_2)}{\alpha_{12}+\alpha_{21}} \times \\
&\quad \left(\frac{(\alpha_{11}+\alpha_{12})^n-\alpha_{11}-\alpha_{12}}{\alpha_{11}+\alpha_{12}-1} - \frac{(\alpha_{11}-\alpha_{21})^n-\alpha_{11}+\alpha_{21}}{\alpha_{11}-\alpha_{21}-1}\right).
\end{aligned}$$

Case 2. $\alpha_{12} \neq -\alpha_{21}$, $\alpha_{11}+\alpha_{12} = 1$ $(\alpha_{11}-\alpha_{21} \neq 1)$:

$$\begin{aligned}
a_n &= a_0 + \alpha_{12}(b_0-a_0)\frac{1-(\alpha_{11}-\alpha_{21})^n}{\alpha_{12}+\alpha_{21}} + \\
&\quad \alpha_1 n + \frac{\alpha_{12}(\alpha_2-\alpha_1)}{\alpha_{11}-\alpha_{21}-1}\left(\frac{(\alpha_{11}-\alpha_{21})^n-\alpha_{11}+\alpha_{21}}{\alpha_{11}-\alpha_{21}-1} - n + 1\right) \\
b_n &= b_0 + \alpha_{21}(a_0-b_0)\frac{1-(\alpha_{11}-\alpha_{21})^n}{\alpha_{12}+\alpha_{21}} + \\
&\quad \alpha_2 n + \frac{\alpha_{21}(\alpha_1-\alpha_2)}{\alpha_{11}-\alpha_{21}-1}\left(\frac{(\alpha_{11}-\alpha_{21})^n-\alpha_{11}+\alpha_{21}}{\alpha_{11}-\alpha_{21}-1} - n + 1\right).
\end{aligned}$$

Case 3. $\alpha_{12} \neq -\alpha_{21}$, $\alpha_{11}-\alpha_{21} = 1$ $(\alpha_{11}+\alpha_{12} \neq 1)$:

$$\begin{aligned}
a_n &= (\alpha_{11}+\alpha_{12})^n a_0 + (\alpha_{12}(b_0-a_0)+\alpha_1)\frac{(\alpha_{11}+\alpha_{12})^n-1}{\alpha_{12}+\alpha_{21}} + \\
&\quad \frac{\alpha_{12}(\alpha_2-\alpha_1)}{\alpha_{12}+\alpha_{21}}\left(\frac{(\alpha_{11}+\alpha_{12})^n-\alpha_{11}-\alpha_{12}}{\alpha_{12}+\alpha_{21}} - n + 1\right). \\
b_n &= (\alpha_{11}+\alpha_{12})^n b_0 + (\alpha_{21}(a_0-b_0)+\alpha_2)\frac{(\alpha_{11}+\alpha_{12})^n-1}{\alpha_{12}+\alpha_{21}} + \\
&\quad \frac{\alpha_{21}(\alpha_1-\alpha_2)}{\alpha_{12}+\alpha_{21}}\left(\frac{(\alpha_{11}+\alpha_{12})^n-\alpha_{11}-\alpha_{12}}{\alpha_{12}+\alpha_{21}} - n + 1\right).
\end{aligned}$$

Case 4. $\alpha_{12} = -\alpha_{21}$, $\alpha_{11}+\alpha_{12} \neq 1$ $(\alpha_{11}-\alpha_{21} \neq 1)$:

$$a_n = (\alpha_{11}+\alpha_{12})^{n-1}((\alpha_{11}+\alpha_{12})a_0 + \alpha_{12}(b_0-a_0)n) +$$

$$
\begin{aligned}
& \alpha_1 \frac{(\alpha_{11}+\alpha_{12})^n - 1}{\alpha_{11}+\alpha_{12}-1} + \frac{\alpha_{12}(\alpha_2-\alpha_1)}{(\alpha_{11}+\alpha_{12}-1)^2} \times \\
& \left((n-1)(\alpha_{11}+\alpha_{12})^n - n(\alpha_{11}+\alpha_{12})^{n-1}+1\right) \\
b_n \quad = \quad & (\alpha_{11}+\alpha_{12})^{n-1}((\alpha_{11}+\alpha_{12})b_0 + \alpha_{21}(a_0-b_0)n) + \\
& \alpha_2 \frac{(\alpha_{11}+\alpha_{12})^n - 1}{\alpha_{11}+\alpha_{12}-1} + \frac{\alpha_{21}(\alpha_1-\alpha_2)}{(\alpha_{11}+\alpha_{12}-1)^2} \times \\
& \left((n-1)(\alpha_{11}+\alpha_{12})^n - n(\alpha_{11}+\alpha_{12})^{n-1}+1\right).
\end{aligned}
$$

Case 5. $\alpha_{12} = -\alpha_{21}$, $\alpha_{11}+\alpha_{12} = 1$ $(\alpha_{11}-\alpha_{21} = 1)$:

$$
\begin{aligned}
a_n &= a_0 + (\alpha_{12}(b_0-a_0)+\alpha_1)n + \alpha_{12}(\alpha_2-\alpha_1)\frac{n(n-1)}{2} \\
b_n &= b_0 + (\alpha_{21}(a_0-b_0)+\alpha_2)n + \alpha_{21}(\alpha_1-\alpha_2)\frac{n(n-1)}{2}.
\end{aligned}
$$

Proof. Denote $\Delta_n = b_n - a_n$, $S = \alpha_{11}+\alpha_{12}$, $D = \alpha_{11}-\alpha_{21}$, $\delta = \alpha_2 - \alpha_1$.

Case 1. $\alpha_{12} \neq -\alpha_{21}$, $\alpha_{11}+\alpha_{12} \neq 1$, $\alpha_{11}-\alpha_{21} \neq 1$.

$$
\begin{aligned}
a_n &= \alpha_{11}a_{n-1} + \alpha_{12}(a_{n-1}+\Delta_{n-1}) + \alpha_1 \\
&= (\alpha_{11}+\alpha_{12})a_{n-1} + \alpha_{12}\Delta_{n-1} + \alpha_1 \\
&= Sa_{n-1} + \alpha_{12}\Delta_{n-1} + \alpha_1.
\end{aligned}
$$

As follows from (10)

$$\alpha_{11} - \alpha_{21} = \alpha_{22} - \alpha_{12}.$$

Therefore,

$$b_n - a_n = (\alpha_{11}-\alpha_{21})(b_{n-1}-a_{n-1}) + \alpha_2 - \alpha_1$$

or

$$\Delta_n = D\Delta_{n-1} + \delta.$$

Hence, we have two simultaneous recurrences:

$$
\begin{cases}
a_n = Sa_{n-1} + \alpha_{12}\Delta_{n-1} + \alpha_1 \\
\Delta_n = D\Delta_{n-1} + \delta.
\end{cases}
\tag{11}
$$

$$S - D = \alpha_{11} + \alpha_{12} - \alpha_{11} + \alpha_{21} = \alpha_{12} + \alpha_{21}. \tag{12}$$

Based on (11) and (12) we get:

$$a_n \quad = \quad Sa_{n-1} + \alpha_{12}\Delta_{n-1} + \alpha_1$$

$$
\begin{aligned}
&= S\left(Sa_{n-2}+\alpha_{12}\Delta_{n-2}+\alpha_1\right)+\alpha_{12}\left(D\Delta_{n-2}+\delta\right)+\alpha_1 \\
&= S^2a_{n-2}+\alpha_{12}\left(S+D\right)\Delta_{n-2}+\left(S+1\right)\alpha_1+\alpha_{12}\delta \\
&= S^2\left(Sa_{n-3}+\alpha_{12}\Delta_{n-3}+\alpha_1\right)+\alpha_{12}\left(S+D\right)\left(D\Delta_{n-3}+\delta\right)+ \\
&\quad \left(S+1\right)\alpha_1+\alpha_{12}\delta \\
&= S^3a_{n-3}+\alpha_{12}\left(S^2+SD+D^2\right)\Delta_{n-3}+\left(S^2+S+1\right)\alpha_1+ \\
&\quad \alpha_{12}\left(S+D+1\right)\delta \\
&= S^4a_{n-4}+\alpha_{12}\left(S^3+S^2D+SD^2+D^3\right)\Delta_{n-4}+ \\
&\quad \left(S^3+S^2+S+1\right)\alpha_1+\alpha_{12}\left(S^2+SD+D^2+S+D+1\right)\delta \\
&= \ldots = S^na_0+\alpha_{12}\Delta_0\sum_{i=0}^{n-1}S^iD^{n-1-i}+\alpha_1\sum_{i=0}^{n-1}S^i+ \qquad (13) \\
&\quad \alpha_{12}\delta\sum_{j=0}^{n-2}\sum_{i=0}^{j}S^iD^{j-i} \qquad (14) \\
&= S^na_0+\alpha_{12}\Delta_0D^{n-1}\sum_{i=0}^{n-1}\left(\frac{S}{D}\right)^i+\alpha_1\frac{S^n-1}{S-1}+\alpha_{12}\delta\sum_{j=0}^{n-2}D^j\sum_{i=0}^{j}\left(\frac{S}{D}\right)^i \\
&= S^na_0+\alpha_{12}\Delta_0D^{n-1}\frac{\left(\frac{S}{D}\right)^n-1}{\frac{S}{D}-1}+\alpha_1\frac{S^n-1}{S-1}+\alpha_{12}\delta\sum_{j=0}^{n-2}D^j\frac{\left(\frac{S}{D}\right)^{j+1}-1}{\frac{S}{D}-1} \\
&= S^na_0+\alpha_{12}\Delta_0\frac{S^n-D^n}{S-D}+\alpha_1\frac{S^n-1}{S-1}+\alpha_{12}\delta\sum_{j=0}^{n-2}\frac{S^{j+1}-D^{j+1}}{S-D} \\
&= S^na_0+\alpha_{12}\Delta_0\frac{S^n-D^n}{S-D}+\alpha_1\frac{S^n-1}{S-1}+ \\
&\quad \frac{\alpha_{12}\delta}{S-D}\left(\sum_{j=0}^{n-2}S^{j+1}-\sum_{j=0}^{n-2}D^{j+1}\right) \\
&= S^na_0+\alpha_{12}\Delta_0\frac{S^n-D^n}{S-D}+\alpha_1\frac{S^n-1}{S-1}+ \qquad (15) \\
&\quad \frac{\alpha_{12}\delta}{S-D}\left(\frac{S^n-S}{S-1}-\frac{D^n-D}{D-1}\right) \qquad (16) \\
&= \left(\alpha_{11}+\alpha_{12}\right)^na_0+\alpha_{12}\left(b_0-a_0\right)\frac{\left(\alpha_{11}+\alpha_{12}\right)^n-\left(\alpha_{11}-\alpha_{21}\right)^n}{\alpha_{12}+\alpha_{21}}+ \\
&\quad \alpha_1\frac{\left(\alpha_{11}+\alpha_{12}\right)^n-1}{\alpha_{11}+\alpha_{12}-1}+\frac{\alpha_{12}\left(\alpha_2-\alpha_1\right)}{\alpha_{12}+\alpha_{21}}\times \\
&\quad \left(\frac{\left(\alpha_{11}+\alpha_{12}\right)^n-\alpha_{11}-\alpha_{12}}{\alpha_{11}+\alpha_{12}-1}-\frac{\left(\alpha_{11}-\alpha_{21}\right)^n-\alpha_{11}+\alpha_{21}}{\alpha_{11}-\alpha_{21}-1}\right).
\end{aligned}
$$

The result for b_n can be derived on the basis of symmetry.

Case 2. $\alpha_{12} \neq -\alpha_{21}$, $\alpha_{11} + \alpha_{12} = 1$ $(\alpha_{11} - \alpha_{21} \neq 1)$.

The proof is similar to Case 1. In lines (13-14) of deriving the explici form for a_n we substitute $S = \alpha_{11} + \alpha_{12} = 1$ and obtain

$$\begin{aligned}
a_n &= S^n a_0 + \alpha_{12}\Delta_0 \sum_{i=0}^{n-1} S^i D^{n-1-i} + \alpha_1 \sum_{i=0}^{n-1} S^i + \alpha_{12}\delta \sum_{j=0}^{n-2}\sum_{i=0}^{j} S^i D^{j-i} \\
&= a_0 + \alpha_{12}\Delta_0 \sum_{i=0}^{n-1} D^{n-1-i} + \alpha_1 n + \alpha_{12}\delta \sum_{j=0}^{n-2}\sum_{i=0}^{j} D^{j-i} \\
&= a_0 + \alpha_{12}\Delta_0 \frac{D^n - 1}{D - 1} + \alpha_1 n + \frac{\alpha_{12}\delta}{D-1}\left(\frac{D^n - D}{D-1} - n + 1\right) \\
&= a_0 + \alpha_{12}(b_0 - a_0)\frac{1 - (\alpha_{11} - \alpha_{21})^n}{\alpha_{12} + \alpha_{21}} + \\
&\quad \alpha_1 n + \frac{\alpha_{12}(\alpha_2 - \alpha_1)}{\alpha_{11} - \alpha_{21} - 1}\left(\frac{(\alpha_{11} - \alpha_{21})^n - \alpha_{11} + \alpha_{21}}{\alpha_{11} - \alpha_{21} - 1} - n + 1\right).
\end{aligned}$$

The result for b_n can be derived on the basis of symmetry.

Case 3. $\alpha_{12} \neq -\alpha_{21}$, $\alpha_{11} - \alpha_{21} = 1$ $(\alpha_{11} + \alpha_{12} \neq 1)$.

The proof is similar to Case 1. In lines (13-14) of deriving the explicit form for a_n we substitute $D = \alpha_{11} - \alpha_{21} = 1$ and obtain

$$\begin{aligned}
a_n &= S^n a_0 + \alpha_{12}\Delta_0 \sum_{i=0}^{n-1} S^i D^{n-1-i} + \alpha_1 \sum_{i=0}^{n-1} S^i + \alpha_{12}\delta \sum_{j=0}^{n-2}\sum_{i=0}^{j} S^i D^{j-i} \\
&= S^n a_0 + \alpha_{12}\Delta_0 \sum_{i=0}^{n-1} S^i + \alpha_1 \sum_{i=0}^{n-1} S^i + \alpha_{12}\delta \sum_{j=0}^{n-2}\sum_{i=0}^{j} S^i \\
&= S^n a_0 + \frac{S^n - 1}{S - 1}(\alpha_{12}\Delta_0 + \alpha_1) + \frac{\alpha_{12}\delta}{S-1}\left(\frac{S^n - S}{S-1} - n + 1\right) \\
&= (\alpha_{11} + \alpha_{12})^n a_0 + (\alpha_{12}(b_0 - a_0) + \alpha_1)\frac{(\alpha_{11} + \alpha_{12})^n - 1}{\alpha_{12} + \alpha_{21}} + \\
&\quad \frac{\alpha_{12}(\alpha_2 - \alpha_1)}{\alpha_{12} + \alpha_{21}}\left(\frac{(\alpha_{11} + \alpha_{12})^n - \alpha_{11} - \alpha_{12}}{\alpha_{12} + \alpha_{21}} - n + 1\right).
\end{aligned}$$

The result for b_n can be derived on the basis of symmetry.

Case 4. $\alpha_{12} = -\alpha_{21}$, $\alpha_{11} + \alpha_{12} \neq 1$ $(\alpha_{11} - \alpha_{21} \neq 1)$

The proof is similar to Case 1.

$$S = \alpha_{11} + \alpha_{12} = \alpha_{11} - \alpha_{21} = D \tag{17}$$

Based on (17),

$$\sum_{i=0}^{n-1} S^i D^{n-1-i} = \sum_{i=0}^{n-1} S^{n-1} = nS^{n-1} \tag{18}$$

and

$$\begin{aligned}
\sum_{j=0}^{n-2}\sum_{i=0}^{j} S^i D^{j-i} &= \sum_{j=0}^{n-2} (j+1) S^j \\
&= \sum_{j=0}^{n-2} S^j + \sum_{j=1}^{n-2} S^j + \sum_{j=2}^{n-2} S^j + \ldots + S^{n-2} \\
&= \frac{S^{n-1}-1}{S-1} + \frac{S\left(S^{n-2}-1\right)}{S-1} + \frac{S^2\left(S^{n-3}-1\right)}{S-1} + \ldots + S^{n-2} \\
& \frac{1}{S-1}\left(S^{n-1} - 1 + S^{n-1} - S + S^{n-1} - S^2 + \ldots + S^{n-1}\right. \\
& \left. -S^{n-2}\right) \\
&= \frac{1}{S-1}\left((n-1) S^{n-1} - \frac{S^{n-1}-1}{S-1}\right) \\
&= \frac{1}{S-1} \frac{(n-1) S^{n-1}(S-1) - S^{n-1} + 1}{S-1} \\
&= \frac{(n-1)\left(S^n - S^{n-1}\right) - S^{n-1} + 1}{(S-1)^2} \\
&= \frac{(n-1) S^n - nS^{n-1} + 1}{(S-1)^2} \qquad (19)
\end{aligned}$$

In lines (13-14) of deriving the explicit form for a_n we substitute (18) and (19) and obtain

$$\begin{aligned}
a_n &= S^n a_0 + \alpha_{12}\Delta_0 \sum_{i=0}^{n-1} S^i D^{n-1-i} + \alpha_1 \sum_{i=0}^{n-1} S^i + \alpha_{12}\delta \sum_{j=0}^{n-2}\sum_{i=0}^{j} S^i D^{j-i} \\
&= S^n a_0 + \alpha_{12}\Delta_0 n S^{n-1} + \alpha_1 \frac{S^n - 1}{S-1} + \alpha_{12}\delta \frac{(n-1) S^n - nS^{n-1} + 1}{(S-1)^2} \\
&= S^{n-1}\left(a_0 S + \alpha_{12}\Delta_0 n\right) + \alpha_1 \frac{S^n - 1}{S-1} + \frac{\alpha_{12}\delta}{(S-1)^2}\left((n-1) S^n - nS^{n-1}\right. \\
& \left. +1\right) \\
&= \left(\alpha_{11} + \alpha_{12}\right)^{n-1}\left(\left(\alpha_{11} + \alpha_{12}\right) a_0 + \alpha_{12}\left(b_0 - a_0\right) n\right) + \\
& \alpha_1 \frac{\left(\alpha_{11} + \alpha_{12}\right)^n - 1}{\alpha_{11} + \alpha_{12} - 1} + \frac{\alpha_{12}\left(\alpha_2 - \alpha_1\right)}{\left(\alpha_{11} + \alpha_{12} - 1\right)^2} \times
\end{aligned}$$

$$\left((n-1)(\alpha_{11}+\alpha_{12})^{n}-n(\alpha_{11}+\alpha_{12})^{n-1}+1\right).$$

The result for b_n can be derived on the basis of symmetry.

Case 5. $\alpha_{12}=-\alpha_{21}$, $\alpha_{11}+\alpha_{12}=1$ $(\alpha_{11}-\alpha_{21}=1)$.

The proof is similar to Case 1. In lines (13-14) of deriving the explicit form for a_n we substitute $S=D=1$ and obtain

$$\begin{aligned}
a_n &= S^n a_0+\alpha_{12}\Delta_0\sum_{i=0}^{n-1}S^iD^{n-1-i}+\alpha_1\sum_{i=0}^{n-1}S^i+\alpha_{12}\delta\sum_{j=0}^{n-2}\sum_{i=0}^{j}S^iD^{j-i}\\
&= a_0+\alpha_{12}\Delta_0 n+\alpha_1 n+\alpha_{12}\delta\frac{n(n-1)}{2}\\
&= a_0+\left(\alpha_{12}(b_0-a_0)+\alpha_1\right)n+\alpha_{12}(\alpha_2-\alpha_1)\frac{n(n-1)}{2}.
\end{aligned}$$

The result for b_n can be derived on the basis of symmetry. ■

Case 1 is general. Since expressions in denominators may be equal to zero, there are additional Cases 2–5. The conditions in parentheses in Cases 2–5 follow from previous ones. Thus Cases 1–5 cover all possible combinations.

5 A solution of systems of three recurrences by their transformation to systems of two recurrences

Lemma 3 *Given system (2) and the following conditions:*

1. $\alpha_{11}+\alpha_{12}+\alpha_{13}=\alpha_{21}+\alpha_{22}+\alpha_{23}=\alpha_{31}+\alpha_{32}+\alpha_{33}$,
2. $\exists$ *real constants* w_1, w_2, $w_1+w_2=1$, *that* $\forall$ n, $b_n=w_1a_n+w_2c_n$,

three simultaneous recurrences (2) can be presented by means of representations of c through a and b, b through a and c, and a through b and c as the following three pairs of simultaneous recurrences, respectively:

$$\begin{cases} a_n=\alpha_{11}'a_{n-1}+\alpha_{12}'b_{n-1}+\alpha_1\\ b_n=\alpha_{21}'a_{n-1}+\alpha_{22}'b_{n-1}+\alpha_2\end{cases}\tag{20}$$

where $\alpha_{11}'=\alpha_{11}-\frac{w_1}{w_2}\alpha_{13}$, $\alpha_{12}'=\alpha_{12}+\frac{1}{w_2}\alpha_{13}$, $\alpha_{21}'=\alpha_{21}-\frac{w_1}{w_2}\alpha_{23}$, $\alpha_{22}'=\alpha_{22}+\frac{1}{w_2}\alpha_{23}$;

$$\begin{cases} a_n=\alpha_{11}'a_{n-1}+\alpha_{12}'c_{n-1}+\alpha_1\\ c_n=\alpha_{21}'a_{n-1}+\alpha_{22}'c_{n-1}+\alpha_3\end{cases}\tag{21}$$

where $\alpha'_{11} = \alpha_{11} + w_1\alpha_{12}$, $\alpha'_{12} = w_2\alpha_{12} + \alpha_{13}$, $\alpha'_{21} = \alpha_{31} + w_1\alpha_{32}$, $\alpha'_{22} = w_2\alpha_{32} + \alpha_{33}$;

$$\begin{cases} b_n = \alpha'_{11} b_{n-1} + \alpha'_{12} c_{n-1} + \alpha_2 \\ c_n = \alpha'_{21} b_{n-1} + \alpha'_{22} c_{n-1} + \alpha_3 \end{cases} \tag{22}$$

where $\alpha'_{11} = \frac{1}{w_1}\alpha_{21} + \alpha_{22}$, $\alpha'_{12} = -\frac{w_2}{w_1}\alpha_{21} + \alpha_{23}$, $\alpha'_{21} = \frac{1}{w_1}\alpha_{31} + \alpha_{32}$, $\alpha'_{22} = -\frac{w_2}{w1}\alpha_{31} + \alpha_{33}$, *and for all these pairs of simultaneous recurrences*

$$\alpha'_{11} + \alpha'_{12} = \alpha'_{21} + \alpha'_{22}. \tag{23}$$

Proof. Since $b_n = w_1 a_n + w_2 c_n$ then $c_n = -\frac{w_1}{w_2} a_n + \frac{1}{w_2} b_n$. We substitute c in (2) and get:

$$\begin{aligned} a_n &= \alpha_{11} a_{n-1} + \alpha_{12} b_{n-1} + \alpha_{13} c_{n-1} + \alpha_1 \\ &= \alpha_{11} a_{n-1} + \alpha_{12} b_{n-1} + \alpha_{13} \left(\frac{1}{w_2} b_{n-1} - \frac{w_1}{w_2} a_{n-1} \right) + \alpha_1 \\ &= \left(\alpha_{11} - \frac{w_1}{w_2}\alpha_{13} \right) a_{n-1} + \left(\alpha_{12} + \frac{1}{w_2}\alpha_{13} \right) b_{n-1} + \alpha_1 \\ b_n &= \alpha_{21} a_{n-1} + \alpha_{22} b_{n-1} + \alpha_{23} c_{n-1} + \alpha_2 \\ &= \alpha_{21} a_{n-1} + \alpha_{22} b_{n-1} + \alpha_{23} \left(\frac{1}{w_2} b_{n-1} - \frac{w_1}{w_2} a_{n-1} \right) + \alpha_2 \\ &= \left(\alpha_{21} - \frac{w_1}{w_2}\alpha_{23} \right) a_{n-1} + \left(\alpha_{22} + \frac{1}{w_2}\alpha_{23} \right) b_{n-1} + \alpha_2. \end{aligned}$$

Here

$$\begin{aligned} \alpha'_{11} + \alpha'_{12} &= \alpha_{11} - \frac{w_1}{w_2}\alpha_{13} + \alpha_{12} + \frac{1}{w_2}\alpha_{13} \\ &= \alpha_{11} + \alpha_{12} + \alpha_{13} \\ \alpha'_{21} + \alpha'_{22} &= \alpha_{21} - \frac{w_1}{w_2}\alpha_{23} + \alpha_{22} + \frac{1}{w_2}\alpha_{23} \\ &= \alpha_{21} + \alpha_{22} + \alpha_{23}. \end{aligned}$$

Since $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{21} + \alpha_{22} + \alpha_{23}$ then $\alpha'_{11} + \alpha'_{12} = \alpha'_{21} + \alpha'_{22}$.

We substitute b in (2) and get:

$$\begin{aligned} a_n &= \alpha_{11} a_{n-1} + \alpha_{12} b_{n-1} + \alpha_{13} c_{n-1} + \alpha_1 \\ &= \alpha_{11} a_{n-1} + \alpha_{12} (w_1 a_{n-1} + w_2 c_{n-1}) + \alpha_{13} c_{n-1} + \alpha_1 \\ &= (\alpha_{11} + w_1\alpha_{12}) a_{n-1} + (w_2\alpha_{12} + \alpha_{13}) c_{n-1} + \alpha_1 \\ c_n &= \alpha_{31} a_{n-1} + \alpha_{32} b_{n-1} + \alpha_{33} c_{n-1} + \alpha_3 \\ &= \alpha_{31} a_{n-1} + \alpha_{32} (w_1 a_{n-1} + w_2 c_{n-1}) + \alpha_{33} c_{n-1} + \alpha_3 \end{aligned}$$

$$= \left(\alpha_{31} + w_1\alpha_{32}\right) a_{n-1} + \left(w_2\alpha_{32} + \alpha_{33}\right) c_{n-1} + \alpha_3.$$

Here

$$\begin{aligned}
\alpha'_{11} + \alpha'_{12} &= \alpha_{11} + w_1\alpha_{12} + w_2\alpha_{12} + \alpha_{13} \\
&= \alpha_{11} + \alpha_{12} + \alpha_{13} \\
\alpha'_{21} + \alpha'_{22} &= \alpha_{31} + w_1\alpha_{32} + w_2\alpha_{32} + \alpha_{33} \\
&= \alpha_{31} + \alpha_{32} + \alpha_{33}.
\end{aligned}$$

Since $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{31} + \alpha_{32} + \alpha_{33}$ then $\alpha'_{11} + \alpha'_{12} = \alpha'_{21} + \alpha'_{22}$.

The pair of simultaneous recurrences for b and c and the equality $\alpha'_{11} + \alpha'_{12} = \alpha'_{21} + \alpha'_{22}$ for this case are derived in the same way. ■

The following theorem results from Lemmas 2 and 3.

Theorem 4 *Given system (2) and the following conditions:*

1. $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{21} + \alpha_{22} + \alpha_{23} = \alpha_{31} + \alpha_{32} + \alpha_{33}$,
2. $\exists$ *real constants* w_1, w_2, $w_1 + w_2 = 1$, *that* $\forall\ n$, $b_n = w_1 a_n + w_2 c_n$,

then with $C_1 = \alpha_{11} + \alpha_{12} + \alpha_{13}$, $C_2 = \alpha_{12} + \frac{1}{w_2}\alpha_{13}$, $C_3 = \alpha_{11} - \frac{w_1}{w_2}\alpha_{13} - \alpha_{21} + \frac{w_1}{w_2}\alpha_{23}$, $C_4 = \alpha_{12} + \frac{1}{w_2}\alpha_{13} + \alpha_{21} - \frac{w_1}{w_2}\alpha_{23}$, $C_5 = \alpha_{21} - \frac{w_1}{w_2}\alpha_{23}$, *and* a_0 *and* b_0 *which are initial values of* a *and* b, *respectively, it holds that*

Case 1. $C_4 \neq 0,\ C_1 \neq 1,\ C_3 \neq 1:$

$$\begin{aligned}
a_n &= \left(C_1\right)^n a_0 + C_2\frac{\left(C_1\right)^n - \left(C_3\right)^n}{C_4}\left(b_0 - a_0\right) + \alpha_1\frac{\left(C_1\right)^n - 1}{C_1 - 1} + \\
&\quad \frac{\left(\alpha_2 - \alpha_1\right) C_2}{C_4}\left(\frac{\left(C_1\right)^n - C_1}{C_1 - 1} - \frac{\left(C_3\right)^n - C_3}{C_3 - 1}\right) \\
b_n &= \left(C_1\right)^n b_0 + C_5\frac{\left(C_1\right)^n - \left(C_3\right)^n}{C_4}\left(a_0 - b_0\right) + \alpha_2\frac{\left(C_1\right)^n - 1}{C_1 - 1} + \\
&\quad \frac{\left(\alpha_1 - \alpha_2\right) C_5}{C_4}\left(\frac{\left(C_1\right)^n - C_1}{C_1 - 1} - \frac{\left(C_3\right)^n - C_3}{C_3 - 1}\right) \\
c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n.
\end{aligned}$$

Case 2. $C_4 \neq 0,\ C_1 = 1\ \left(C_3 \neq 1\right):$

$$\begin{aligned}
a_n &= a_0 + C_2\frac{1 - \left(C_3\right)^n}{C_4}\left(b_0 - a_0\right) + \alpha_1 n + \\
&\quad \frac{\left(\alpha_2 - \alpha_1\right) C_2}{C_3 - 1}\left(\frac{\left(C_3\right)^n - C_3}{C_3 - 1} - n + 1\right)
\end{aligned}$$

$$
\begin{aligned}
b_n &= b_0 + C_5 \frac{1-(C_3)^n}{C_4}(a_0 - b_0) + \alpha_2 n + \\
&\quad \frac{(\alpha_1-\alpha_2)C_5}{C_3-1}\left(\frac{(C_3)^n - C_3}{C_3-1} - n + 1\right) \\
c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n.
\end{aligned}
$$

Case 3. $C_4 \neq 0,\ C_3 = 1\ (C_1 \neq 1)$:

$$
\begin{aligned}
a_n &= (C_1)^n a_0 + \frac{(C_1)^n - 1}{C_4}(C_2(b_0 - a_0) + \alpha_1) + \\
&\quad \frac{(\alpha_2-\alpha_1)C_2}{C_4}\left(\frac{(C_1)^n - C_1}{C_4} - n + 1\right) \\
b_n &= (C_1)^n b_0 + \frac{(C_1)^n - 1}{C_4}(C_5(a_0 - b_0) + \alpha_2) + \\
&\quad \frac{(\alpha_1-\alpha_2)C_5}{C_4}\left(\frac{(C_1)^n - C_1}{C_4} - n + 1\right) \\
c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n.
\end{aligned}
$$

Case 4. $C_4 = 0,\ C_1 \neq 1\ (C_3 \neq 1)$:

$$
\begin{aligned}
a_n &= (C_1)^{n-1}(C_1 a_0 + C_2(b_0 - a_0)n) + \alpha_1 \frac{(C_1)^n - 1}{C_1 - 1} + \\
&\quad \frac{(\alpha_2-\alpha_1)C_2}{(C_1-1)^2}\left((n-1)(C_1)^n - n(C_1)^{n-1} + 1\right) \\
b_n &= (C_1)^{n-1}(C_1 b_0 + C_5(a_0 - b_0)n) + \alpha_2 \frac{(C_1)^n - 1}{C_1 - 1} + \\
&\quad \frac{(\alpha_1-\alpha_2)C_5}{(C_1-1)^2}\left((n-1)(C_1)^n - n(C_1)^{n-1} + 1\right) \\
c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n.
\end{aligned}
$$

Case 5. $C_4 = 0,\ C_1 = 1\ (C_3 = 1)$:

$$
\begin{aligned}
a_n &= a_0 + (C_2(b_0 - a_0) + \alpha_1)n + C_2(\alpha_2 - \alpha_1)\frac{n(n-1)}{2} \\
b_n &= b_0 + (C_5(a_0 - b_0) + \alpha_2)n + C_5(\alpha_1 - \alpha_2)\frac{n(n-1)}{2} \\
c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n.
\end{aligned}
$$

Proof. As follows from Lemma 3, three simultaneous recurrences (2) accompanied by Conditions 1 and 2 specified in the lemma, can be reduced, specifically, to the pair of simultaneous recurrences (20) accompanied by restriction (23). Based on Lemma 2 and after corresponding substitutions we obtain the expressions for a_n and b_n. The result for c_n follows directly from Condition 2 of the theorem. ■

The conditions in parentheses in Cases 2–5 follow from previous ones. Thus Cases 1–5 cover all possible combinations.

Remark 2 *It is not the only way to determine a_n, b_n, and c_n that is presented in Theorem 4. Each of recurrent variables a_n, b_n, and c_n can be determined through two of three pairs of simultaneous recurrences (20 - 22).*

The equality $b_n = w_1 a_n + w_2 c_n$ does not automatically mean the existence of the same proportion for coefficients ($\alpha_{21} = w_1\alpha_{11} + w_2\alpha_{31}$, $\alpha_{22} = w_1\alpha_{12} + w_2\alpha_{32}$, $\alpha_{23} = w_1\alpha_{13} + w_2\alpha_{33}$, $\alpha_2 = w_1\alpha_1 + w_3\alpha_3$). In the special case, the affine combination $b_n = w_1 a_n + w_2 c_n$ can be provided by the corresponding proportion for initial values of a, b, and c without observing the same proportion for coefficients (e.g., system (5)).

Suppose we have stronger conditions when the proportion takes place between the coefficients as well.

Lemma 5 *If*

1. $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{31} + \alpha_{32} + \alpha_{33}$

   and

2. $\exists$ *real constants* w_1, w_2, $w_1 + w_2 = 1$, *that*

$$\alpha_{21} = w_1\alpha_{11} + w_2\alpha_{31},\ \alpha_{22} = w_1\alpha_{12} + w_2\alpha_{32},\ \alpha_{23} = w_1\alpha_{13} + w_2\alpha_{33},$$

then

$$\alpha_{21} + \alpha_{22} + \alpha_{23} = \alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{31} + \alpha_{32} + \alpha_{33}. \tag{24}$$

Proof.

$$\begin{aligned}
\alpha_{21} + \alpha_{22} + \alpha_{23} &= w_1\alpha_{11} + w_2\alpha_{31} + w_1\alpha_{12} + w_2\alpha_{32} + w_1\alpha_{13} + w_2\alpha_{33} \\
&= w_1(\alpha_{11} + \alpha_{12} + \alpha_{13}) + w_2(\alpha_{31} + \alpha_{32} + \alpha_{33}) \\
&= (w_1 + w_2)(\alpha_{11} + \alpha_{12} + \alpha_{13}) \\
&= \alpha_{11} + \alpha_{12} + \alpha_{13}.
\end{aligned}$$

■

Hence, the equality such as (24) is a redundant condition if the condition $b_n = w_1 a_n + w_2 c_n$ is provided by the corresponding proportion between coefficients. The equality of any two sums of coefficients from three ones in (24) is sufficient in this case.

Consider a situation with weaker conditions when it is only known that

$$\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{31} + \alpha_{32} + \alpha_{33}$$

and $b_n = w_1 a_n + w_2 c_n$, where $w_1 + w_2 = 1$. In this case, only one pair of simultaneous recurrences adduced in Lemma 3 can be generated. However, the solution of this pair of recurrences is sufficient for finding a_n, b_n, and c_n. On the other hand, two other pairs of simultaneous recurrences (not as in Lemma 3) can be generated in this situation as well. Indeed,

$$\begin{aligned} b_n &= w_1 a_n + w_2 c_n \\ &= w_1(\alpha_{11} a_{n-1} + \alpha_{12} b_{n-1} + \alpha_{13} c_{n-1} + \alpha_1) + \\ &\quad w_2(\alpha_{31} a_{n-1} + \alpha_{32} b_{n-1} + \alpha_{33} c_{n-1} + \alpha_3) \\ &= (w_1\alpha_{11} + w_2\alpha_{31})\, a_{n-1} + (w_1\alpha_{12} + w_2\alpha_{32})\, b_{n-1} + \\ &\quad (w_1\alpha_{13} + w_2\alpha_{33})\, c_{n-1} + w_1\alpha_1 + w_2\alpha_3 \\ &= \alpha^*_{21} a_{n-1} + \alpha^*_{22} b_{n-1} + \alpha^*_{23} c_{n-1} + \alpha^*_2, \end{aligned}$$

where $\alpha^*_{21} = w_1\alpha_{11} + w_2\alpha_{31}$, $\alpha^*_{22} = w_1\alpha_{12} + w_2\alpha_{32}$, $\alpha^*_{23} = w_1\alpha_{13} + w_2\alpha_{33}$, $\alpha^*_2 = w_1\alpha_1 + w_3\alpha_3$. Hence, the recurrence for b_n in (2) can be replaced so that we have the following three simultaneous recurrences:

$$\begin{cases} a_n = \alpha_{11} a_{n-1} + \alpha_{12} b_{n-1} + \alpha_{13} c_{n-1} + \alpha_1 \\ b_n = \alpha^*_{21} a_{n-1} + \alpha^*_{22} b_{n-1} + \alpha^*_{23} c_{n-1} + \alpha^*_2 \\ c_n = \alpha_{31} a_{n-1} + \alpha_{32} b_{n-1} + \alpha_{33} c_{n-1} + \alpha_3 \ . \end{cases} \tag{25}$$

By Lemma 5,

$$\alpha^*_{21} + \alpha^*_{22} + \alpha^*_{23} = \alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{31} + \alpha_{32} + \alpha_{33}.$$

For this reason, by Lemma 3, (25) can be presented as three pairs of simultaneous recurrences. Thus Theorem 4 can be generalized as follows.

Theorem 6 *Given system (2) and the following conditions:*

1. $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{31} + \alpha_{32} + \alpha_{33}$
2. *∃ real constants* w_1, w_2, $w_1 + w_2 = 1$, *that* $\forall\ n$, $b_n = w_1 a_n + w_2 c_n$,

then with $\alpha^*_{21} = w_1\alpha_{11} + w_2\alpha_{31}$, $\alpha^*_{23} = w_1\alpha_{13} + w_2\alpha_{33}$, $\alpha^*_2 = w_1\alpha_1 + w_3\alpha_3$, $C_1 = \alpha_{11} + \alpha_{12} + \alpha_{13}$, $C_2 = \alpha_{12} + \frac{1}{w_2}\alpha_{13}$, $C_3 = \alpha_{11} - \frac{w_1}{w_2}\alpha_{13} - \alpha^*_{21} + \frac{w_1}{w_2}\alpha^*_{23}$,

$C_4 = \alpha_{12} + \frac{1}{w_2}\alpha_{13} + \alpha_{21}^* - \frac{w_1}{w_2}\alpha_{23}^*$, $C_5 = \alpha_{21}^* - \frac{w_1}{w_2}\alpha_{23}^*$, *and* a_0 *and* b_0 *which are initial values of a and b, respectively, it holds that*

Case 1. $C_4 \neq 0,\ C_1 \neq 1,\ C_3 \neq 1:$

$$\begin{aligned} a_n &= (C_1)^n a_0 + C_2 \frac{(C_1)^n - (C_3)^n}{C_4}(b_0 - a_0) + \alpha_1 \frac{(C_1)^n - 1}{C_1 - 1} + \\ &\quad \frac{(\alpha_2^* - \alpha_1)C_2}{C_4}\left(\frac{(C_1)^n - C_1}{C_1 - 1} - \frac{(C_3)^n - C_3}{C_3 - 1}\right) \\ b_n &= (C_1)^n b_0 + C_5 \frac{(C_1)^n - (C_3)^n}{C_4}(a_0 - b_0) + \alpha_2^* \frac{(C_1)^n - 1}{C_1 - 1} + \\ &\quad \frac{(\alpha_1 - \alpha_2^*)C_5}{C_4}\left(\frac{(C_1)^n - C_1}{C_1 - 1} - \frac{(C_3)^n - C_3}{C_3 - 1}\right) \\ c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n. \end{aligned}$$

Case 2. $C_4 \neq 0,\ C_1 = 1\ (C_3 \neq 1):$

$$\begin{aligned} a_n &= a_0 + C_2 \frac{1 - (C_3)^n}{C_4}(b_0 - a_0) + \alpha_1 n + \\ &\quad \frac{(\alpha_2 - \alpha_1)C_2}{C_3 - 1}\left(\frac{(C_3)^n - C_3}{C_3 - 1} - n + 1\right) \\ b_n &= b_0 + C_5 \frac{1 - (C_3)^n}{C_4}(a_0 - b_0) + \alpha_2^* n + \\ &\quad \frac{(\alpha_1 - \alpha_2^*)C_5}{C_3 - 1}\left(\frac{(C_3)^n - C_3}{C_3 - 1} - n + 1\right) \\ c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n. \end{aligned}$$

Case 3. $C_4 \neq 0,\ C_3 = 1\ (C_1 \neq 1):$

$$\begin{aligned} a_n &= (C_1)^n a_0 + \frac{(C_1)^n - 1}{C_4}\left(C_2(b_0 - a_0) + \alpha_1\right) + \\ &\quad \frac{(\alpha_2^* - \alpha_1)C_2}{C_4}\left(\frac{(C_1)^n - C_1}{C_4} - n + 1\right) \\ b_n &= (C_1)^n b_0 + \frac{(C_1)^n - 1}{C_4}\left(C_5(a_0 - b_0) + \alpha_2^*\right) + \\ &\quad \frac{(\alpha_1 - \alpha_2^*)C_5}{C_4}\left(\frac{(C_1)^n - C_1}{C_4} - n + 1\right) \\ c_n &= -\frac{w_1}{w_2}a_n + \frac{1}{w_2}b_n. \end{aligned}$$

Case 4. $C_4 = 0$, $C_1 \neq 1$ $(C_3 \neq 1)$:

$$\begin{aligned} a_n &= (C_1)^{n-1}(C_1 a_0 + C_2(b_0 - a_0)n) + \alpha_1 \frac{(C_1)^n - 1}{C_1 - 1} + \\ &\quad \frac{(\alpha_2^* - \alpha_1) C_2}{(C_1 - 1)^2}\left((n-1)(C_1)^n - n(C_1)^{n-1} + 1\right) \\ b_n &= (C_1)^{n-1}(C_1 b_0 + C_5(a_0 - b_0)n) + \alpha_2^* \frac{(C_1)^n - 1}{C_1 - 1} + \\ &\quad \frac{(\alpha_1 - \alpha_2^*) C_5}{(C_1 - 1)^2}\left((n-1)(C_1)^n - n(C_1)^{n-1} + 1\right) \\ c_n &= -\frac{w_1}{w_2} a_n + \frac{1}{w_2} b_n. \end{aligned}$$

Case 5. $C_4 = 0$, $C_1 = 1$ $(C_3 = 1)$:

$$\begin{aligned} a_n &= a_0 + (C_2(b_0 - a_0) + \alpha_1)n + C_2(\alpha_2^* - \alpha_1)\frac{n(n-1)}{2} \\ b_n &= b_0 + (C_5(a_0 - b_0) + \alpha_2^*)n + C_5(\alpha_1 - \alpha_2^*)\frac{n(n-1)}{2} \\ c_n &= -\frac{w_1}{w_2} a_n + \frac{1}{w_2} b_n. \end{aligned}$$

Remark 3 *As in Theorem 4, the way presented in Theorem 6 is not the only one to determine a_n, b_n, and c_n. Each of recurrent variables a_n, b_n, and c_n can be determined through two of three pairs of simultaneous recurrences derived from system (25).*

In the special case, when (24) is true, Theorem 6 is reduced to Theorem 4.

6 Additional findings

Presentation of systems of equations in *matrix form* allows to consider sums of coefficients in recurrences as sums of coefficients in rows. It might also be of interest to investigate simultaneous recurrences with equal sums of coefficients in columns.

Lemma 7 *Suppose*

$$\begin{cases} a_n = \alpha_{11} a_{n-1} + \alpha_{12} b_{n-1} + \alpha_1 \\ b_n = \alpha_{21} a_{n-1} + \alpha_{22} b_{n-1} + \alpha_2 \end{cases}$$

and

$$\alpha_{11}+\alpha_{21}=\alpha_{12}+\alpha_{22}.$$

If $\alpha_{12}\neq-\alpha_{21}$, $\alpha_{11}-\alpha_{12}\neq 1$, $\alpha_{11}+\alpha_{21}\neq 1$, *then*

$$\begin{aligned}
a_n &= (\alpha_{11}-\alpha_{12})^n a_0+\alpha_{12}(a_0+b_0)\frac{(\alpha_{11}+\alpha_{21})^n-(\alpha_{11}-\alpha_{12})^n}{\alpha_{12}+\alpha_{21}}+\\
&\alpha_1\frac{(\alpha_{11}-\alpha_{12})^n-1}{\alpha_{11}-\alpha_{12}-1}+\frac{\alpha_{12}(\alpha_1+\alpha_2)}{\alpha_{12}+\alpha_{21}}\times\\
&\left(\frac{(\alpha_{11}+\alpha_{21})^n-\alpha_{11}-\alpha_{21}}{\alpha_{11}+\alpha_{21}-1}-\frac{(\alpha_{11}-\alpha_{12})^n-\alpha_{11}+\alpha_{12}}{\alpha_{11}-\alpha_{12}-1}\right)\\
b_n &= (\alpha_{11}-\alpha_{12})^n b_0+\alpha_{21}(a_0+b_0)\frac{(\alpha_{11}+\alpha_{21})^n-(\alpha_{11}-\alpha_{12})^n}{\alpha_{12}+\alpha_{21}}+\\
&\alpha_2\frac{(\alpha_{11}-\alpha_{12})^n-1}{\alpha_{11}-\alpha_{12}-1}+\frac{\alpha_{21}(\alpha_1+\alpha_2)}{\alpha_{12}+\alpha_{21}}\times\\
&\left(\frac{(\alpha_{11}+\alpha_{21})^n-\alpha_{11}-\alpha_{21}}{\alpha_{11}+\alpha_{21}-1}-\frac{(\alpha_{11}-\alpha_{12})^n-\alpha_{11}+\alpha_{12}}{\alpha_{11}-\alpha_{12}-1}\right).
\end{aligned}$$

Proof. Denote $\sigma_n=b_n-a_n$, $S=\alpha_{11}+\alpha_{21}$, $D=\alpha_{11}-\alpha_{12}$, $\varsigma=\alpha_1+\alpha_2$.

$$\begin{aligned}
a_n &= \alpha_{11}a_{n-1}+\alpha_{12}(\sigma_{n-1}-a_{n-1})+\alpha_1\\
&= (\alpha_{11}-\alpha_{12})a_{n-1}+\alpha_{12}\sigma_{n-1}+\alpha_1\\
&= Da_{n-1}+\alpha_{12}\sigma_{n-1}+\alpha_1.
\end{aligned}$$

$$a_n+b_n=(\alpha_{11}+\alpha_{21})(a_{n-1}+b_{n-1})+\alpha_1+\alpha_2$$

or

$$\sigma_n=S\sigma_{n-1}+\varsigma.$$

Hence, we have two simultaneous recurrences:

$$\begin{cases} a_n=Da_{n-1}+\alpha_{12}\sigma_{n-1}+\alpha_1\\ \sigma_n=S\sigma_{n-1}+\varsigma.\end{cases}\tag{26}$$

One can see that system (26) is the same as system (11) in the proof of Lemma 2. Hence, in lines (15-16) of deriving the explicit form for a_n we swap S and D, replace Δ_0 with σ_0 and δ with ς, and obtain

$$\begin{aligned}
a_n &= D^n a_0+\alpha_{12}\sigma_0\frac{D^n-S^n}{D-S}+\alpha_1\frac{D^n-1}{D-1}+\frac{\alpha_{12}\varsigma}{D-S}\left(\frac{D^n-D}{D-1}-\frac{S^n-S}{S-1}\right)\\
&= (\alpha_{11}-\alpha_{12})^n a_0+\alpha_{12}(a_0+b_0)\frac{(\alpha_{11}+\alpha_{21})^n-(\alpha_{11}-\alpha_{12})^n}{\alpha_{12}+\alpha_{21}}+\\
&\alpha_1\frac{(\alpha_{11}-\alpha_{12})^n-1}{\alpha_{11}-\alpha_{12}-1}+\frac{\alpha_{12}(\alpha_1+\alpha_2)}{\alpha_{12}+\alpha_{21}}\times
\end{aligned}$$

$$\left(\frac{(\alpha_{11}+\alpha_{21})^n-\alpha_{11}-\alpha_{21}}{\alpha_{11}+\alpha_{21}-1}-\frac{(\alpha_{11}-\alpha_{12})^n-\alpha_{11}+\alpha_{12}}{\alpha_{11}-\alpha_{12}-1}\right).$$

ie result for b_n can be derived on the basis of symmetry. ■

Analogous results can be obtained for special cases as in Lemma 2.

Conclusions and future work

e have proposed an approach that gives closed forms for solutions of spe-
il simultaneous systems of three linear recurrences. Sums of coefficients these recurrences are equal and each recurrent variable is an affine com-
iation of two other recurrent variables. The systems are solved by their ssection into pairs of recurrences with equal sums of coefficients.

The solutions are applied for deriving explicit forms of complexities of aph expression generated by decomposition methods. Specifically, apply-
g this way to systems (3) and (5), we have obtained the following explicit rmulae for the number of labels in expressions of full square rhomboids 7) and square rhomboids (28), respectively, of size m $(m = 2^r)$:

$$T(m) = \frac{154}{135}m^{\log_2 6}+\frac{1}{27}m^{\log_2 3}-\frac{2}{5} \quad (27)$$

$$T(m) = \frac{89}{45}m^{\log_2 6}-\frac{20}{9}m^{\log_2 3}-\frac{1}{5}. \quad (28)$$

Our intent is to determine the class of graphs for which the complexities their expressions can be expressed with these recurrences.

We are going to generalize the presented technique to a system of an bitrary (or, at least, a larger) number of recurrences.

Specifically, as shown in Section 3, each of simultaneous recurrences rresponds to decomposition of a subgraph of a given kind. A subgraph's id is determined by kinds of its ends (the left and the right). Suppose, bgraphs with z kinds of ends are revealed in the course of decomposition. r a given kind of the left (right) end there are z possible kinds of the right ft) end, i.e., z kinds of subgraphs. A subgraph with the left end of kind i $\leq i \leq z$) and the right end of kind j $(1 \leq j \leq z)$ and a subgraph with the t end of kind j and the right end of kind i are considered as subgraphs the same kind. Thus subgraphs of Z kinds are emerged throughout composition, where

$$Z = z + z - 1 + z - 2 + \ldots + 1 = \frac{z(z+1)}{2}. \quad (29)$$

Accordingly, corresponding system will consist of Z recurrences.

For example, in the case of a Fibonacci graph $z = 1$ and we have a single recurrence. For square and full square rhomboids $z = 2$ and, therefore, three simultaneous recurrences appear. Decomposition of graphs of more complicated structure may give systems of 6 recurrences, 10 recurrences, 15 recurrences, and so on, in accordance with (29).

Each of these systems will be characterized by equal sums of coefficients in recurrences. The question is what will be the additional invariant specific for the systems which have more than three recurrences.

We have presented a similar technique for systems of two recurrences with equal sums of coefficients in columns. It is of interest to develop a method capable to handle three or more simultaneous recurrences with equal sums of coefficients in columns. The problem is to find additional restrictions which are to be imposed on the coefficients.

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