The 3-Xline Graph of a Given Graph

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Abstract

For a given graph G, a variation of its line graph is the 3-xline graph, where two 3-paths P and Q are adjacent in G if $V(P) \cap V(Q) = \{v\}$ when v is the interior vertex of both P and Q. The vertices of the 3-xline graph $XL_3(G)$ correspond to the 3-paths in G, and two distinct vertices of the 3-xline graph are adjacent if and only if they are adjacent 3-paths in G. In this paper, we study 3-xline graphs for several classes of graphs, and show that for a connected graph G, the 3-xline graph is never isomorphic to G and is connected only when G is the star $K_{1,n}$ for $n = 2$ or $n \geq 5$. We also investigate cycles in 3-xline graphs and characterize those graphs G where $XL_3(G)$ is Eulerian.

1 Introduction

For a given graph G, a derived graph of G is a graph obtained from G by some type of a graph operation. The study of the structural properties of derived graphs is a popular area of research in graph theory. While one of the most familiar derived graphs of a graph is the line graph, various generalizations of line graphs also have been introduced and studied (see [1], [3], [8] and [10]).

The line graph $L(G)$ of a nonempty graph G has the set of edges in G as its vertex set with two vertices of $L(G)$ adjacent if the corresponding edges of G are adjacent. This concept was introduced in 1932 by Whitney when he was investigating graph isomorphisms [11]. Two recent generalizations of line graphs were introduced by Chartrand.

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The ℓ-line graph was introduced in 2015 and studied extensively in [1 and [2], where the emphasis was on 3-line graphs. Let G be a connecte graph of order at least 3. Two nontrivial paths P and Q in G are said t be adjacent paths in G if $V(P) \cap V(Q) = \{v\}$ where v is an end-vertex o both P and Q. For an integer $\ell \geq 2$, the ℓ-line graph $L_\ell(G)$ of a graph G is the graph whose vertex set is the set of ℓ-paths (paths of order l) of G where two vertices of $L_\ell(G)$ are adjacent if they are adjacent ℓ-paths in G Since the 2-line graph is the line graph $L(G)$ for every graph G, this is a generalization of line graphs.

The k-path graph was first discussed in 2018 and studied in [4] and [9]. Let $k \geq 2$ be an integer and let G be a graph containing k-paths. The k-path graph $P_k(G)$ of G has the set of k-paths of G as its vertex set and where two distinct vertices of $P_k(G)$ are adjacent if the corresponding k-paths of G have a $(k-1)$-path in common. Specifically, if $k = 3$, then vertices in $P_3(G)$ correspond to 3-paths in G and distinct vertices are adjacent in $P_3(G)$ when the corresponding 3-paths have an edge in common. Again, this is a generalization of line graphs because $P_2(G)$ is the line graph of G.

In this paper we investigate a variation of 3-line graphs and 3-path graphs called 3-xline graphs, where for a given graph G, two 3-paths P and Q are adjacent in G if $V(P) \cap V(Q) = \{v\}$ where v is the interior vertex of both P and Q. The 3-xline graph of a graph G, denoted by $XL_3(G)$ has the 3-paths in G as its vertex set as before, but where two distinct vertices of the 3-xline graph are adjacent if and only if they are adjacent 3-paths in G. For instance, if u, v, w, x, y and z are (not necessarily distinct) vertices of a graph G, then the two vertices (denoted by uvw and xyz to indicate the 3-paths u, v, w and x, y, z) in $XL_3(G)$ are adjacent if and only if $v = y$ and u, w, x and z are distinct vertices in G. Note that, just as with the 3-line graphs and 3-path graphs, some vertex of G must have degree 2 or more for $XL_3(G)$ to exist.

These concepts are illustrated in Figure 1 where the 3-path graph and 3-xline graph of a graph G are shown while the 3-line graph of G is empty in this example. Note that in the figure the vertices in the derived graphs are labeled using the original edges. A vertex in the derived graphs also could be labeled as uvw where the vertex corresponds to the 3-path u, v, w in G.

In [2], the following formula for the order of $L_3(G)$ was given for a connected graph G with the convention that if $a < b$, then $\binom{a}{b} = 0$.

Proposition 1.1 *If G is a connected graph of order $n \geq 2$ with degree sequence $d_1, d_2 \ldots, d_n$, then the order of $L_3(G)$ is $\sum_{i=1}^{n} \binom{d_i}{2}$.*

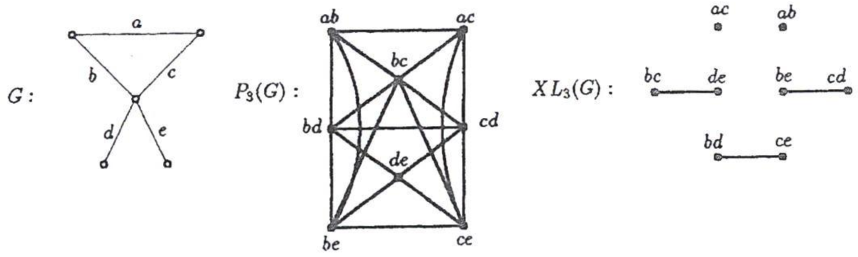

Figure 1: A graph G, its 3-path graph $P_3(G)$ and 3-xline graph $XL_3(G)$

This is true based on the following argument: Let G be a nontrivial connected graph and v a vertex of G. If $\deg v = 1$, then there is no P_3 in G whose interior vertex is v; while if $\deg_G v \geq 2$, then there are exactly $\binom{\deg_G v}{2}$ copies of P_3 whose interior vertex is v. Note that this formula also gives the orders of the 3-path graph $P_3(G)$ and the 3-xline graph $XL_3(G)$.

We refer to [5] for graph theory notation and terminology not described in this paper.

2 Preliminary Results of 3-Xline Graphs

Let G be a graph of order $n \geq 3$. If a vertex v in the graph G has degree less than 4, then there are no pairs of 3-paths that have only v in common. Thus, there are no edges in $XL_3(G)$ between vertices of the form uvw. Proposition 1.1 gives a formula for the order of $XL_3(G)$ based on the degrees of the vertices in G. The next result gives a formula for the size of $XL_3(G)$ also based on the degrees of the vertices of G.

Proposition 2.1 *If G is a connected graph of order n with degree sequence $d_1, d_2, \ldots, d_n$, where at least one of the $d_i \geq 2$, then the size of $XL_3(G)$ is* $\sum_{i=1}^{n} \left[\frac{d_i(d_i-1)(d_i-2)(d_i-3)}{8}\right]$.

Proof. As noted above, if $\deg_G v = d_v < 4$ for a vertex v of G, then no edges of $XL_3(G)$ are generated by 3-paths of the form uvw and $\binom{d_v-2}{2} = 0$. Thus, consider vertices v where $\deg_G \geq 4$. For such a vertex v, there are $\binom{d_v}{2}$ vertices in $XL_3(G)$ of the form uvw and for each of these vertices there are exactly $\binom{d_v-2}{2}$ vertices in $XL_3(G)$ that are adjacent to it creating an edge. Since the product $\binom{d_v}{2}.\binom{d_v-2}{2}$ counts an edge twice, we divide it by 2. By considering every vertex of G, we see that the number of edges in $XL_3(G)$ is $\sum_{i=1}^{n} \left[\frac{\binom{d_i}{2}\binom{d_i-2}{2}}{2}\right]$. This can also be written as

Table 1: The order and size of the 3-xline graph of some graphs.

Class	G	$\|V(XL_3(G))\|$	$\|E(XL_3(G))\|$
Paths	$P_n,\ n \geq 2$	$n-2$	0
Cycles	$C_n,\ n \geq 3$	n	0
Complete Graphs	$K_n,$ $2 \leq n \leq 4$	$n\binom{n-1}{2} = \frac{n(n-1)(n-2)}{2}$	0
Complete Graphs	$K_n,\ n \geq 5$	$n\binom{n-1}{2} = \frac{n(n-1)(n-2)}{2}$	$\frac{n(n-1)(n-2)(n-3)(n-4)}{8}$
Stars	$K_{1,n},\ n \geq 2$	$\binom{n}{2}$	$\frac{n(n-1)(n-2)(n-3)}{8}$
Wheels	W_n	$\binom{n}{2} + n\binom{3}{2} = \frac{n(n+5)}{2}$	$\frac{n(n-1)(n-2)(n-3)}{8}$
Petersen Graph	P	30	0
Complete Bipartite Graphs	$K_{m,n},\ m \geq 2,$ $n \geq 2$	$m\binom{n}{2} + n\binom{m}{2} = \frac{mn(m+n-2)}{2}$	$\frac{mn}{8}[(n-1)(n-2)(n-3)$ $+(m-1)(m-2)(m-3)]$

$\sum_{i=1}^{n} \left[\frac{d_i(d_i-1)(d_i-2)(d_i-3)}{8}\right]$ for all i where $d_i \geq 4$. ■

Using the formulas in Propositions 1.1 and 2.1, we summarize results for several classes of graphs in Table 1.

It is easy to verify that $XL_3(K_n)$ is regular of degree $\binom{n-3}{2}$ and $XL_3(K_{1,n})$ is regular of degree $\binom{n-2}{2}$. For stars, the 3-xline graph $XL_3(K_{1,2}) \cong K_1$, the graph $XL_3(K_{1,3}) \cong 3K_1$ and $XL_3(K_{1,4}) \cong 3K_2$. This leads to the following interesting result.

Theorem 2.2 *The 3-xline graph of $G = K_{1,5}$ is the Petersen graph.*

Proof. Suppose the vertices of $G = K_{1,5}$ are labeled such that the vertex u is adjacent to the vertices v_1, v_2, v_3, v_4 and v_5. Since each vertex of $XL_3(K_{1,5})$ is of the form $v_i u v_j$, we can simplify the notation to ij. By Proposition 1.1, the order of $XL_3(K_{1,5})$ is $\binom{5}{2} = 10$ and the degree of each vertex is $\binom{3}{2} = 3$. To show that $XL_3(K_{1,5}) \cong P$, consider the graph of $XL_3(K_{1,5})$ shown in Figure 2 drawn as the Petersen graph. ■

We have seen that $XL_3(K_{1,2})$, $XL_3(K_{1,3})$ and $XL_3(K_{1,4})$ are disconnected while $XL_3(K_{1,1})$ and $XL_3(K_{1,5})$ are connected. The next result determines for which stars the 3-xline graphs are connected.

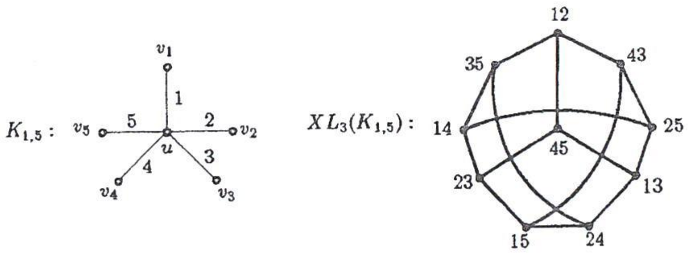

Figure 2: The 3-xline graph of $K_{1,5}$.

Lemma 2.3 *The 3-xline graph for the star $K_{1,n}$ is connected if and only if $n = 2$ or $n \geq 5$.*

Proof. For $n = 3$ or 4, we have already seen that the 3-xline graph is disconnected and for $n = 2$ or 5, we have $XL_3(K_{1,2}) \cong K_1$ and $XL_3(K_{1,5}) \cong P$, the Petersen graph, which both are connected. Now suppose that $n \geq 6$ and suppose that the vertices of $K_{1,n}$ are labeled such that the vertex u is adjacent to the vertices $v_1, v_2, v_3, \ldots, v_n$. Every pair of vertices of $XL_3(K_{1,n})$ are of the form v_iuv_j and v_kuv_l. To show that $XL_3(K_{1,n})$ always has a path from v_iuv_j to v_kuv_l when $n \geq 6$, consider two cases:

Case 1: Suppose i, j, k and ℓ are distinct. Then vertices v_iuv_j and v_kuv_ℓ are adjacent.

Case 2: Suppose that $i = k$ or ℓ or $j = k$ or ℓ. Without loss of generality, suppose that $i = k$. Then vertices v_iuv_j and v_iuv_ℓ are not adjacent; however, since $n \geq 6$, there exist positive integers α and β less than or equal to 6 such that $\{i, j, k, \ell\} \cap \{\alpha, \beta\} = \emptyset$. Thus, vertices v_iuv_j and v_iuv_ℓ are both adjacent to $v_\alpha uv_\beta$ creating a desired path.

In both cases, there is a v_iuv_j - v_kuv_ℓ path when $n \geq 6$, which completes the proof. ■

The next result characterizes all graphs with connected 3-xline graphs.

Theorem 2.4 *If G is a graph where some vertex has degree at least 2, then $XL_3(G)$ is connected if and only if G is one of the following:*

(*i*) *a star $K_{1,n}$ with $n = 2$ or $n \geq 5$,*

(*ii*) *a disconnected graph where one component is a star $K_{1,n}$ with $n = 2$ or $n \geq 5$ and the remaining components contain only vertices v where* $\deg v \leq 1$.

Proof. First, suppose that G is a star $K_{1,n}$ with $n = 2$ or $n \geq 5$. The graph $XL_3(G)$ is connected by Lemma 2.3. Next, let G be a disconnected graph where one component C is a star $K_{1,n}$ with $n = 2$ or $n \geq 5$ and the remaining components contain only vertices v where $\deg v \leq 1$. The only vertices and edges in $XL_3(G)$ will correspond to vertices and adjacencies in C. Thus, $XL_3(G)$ is connected by Lemma 2.3.

To show the converse, we use the contrapositive. Suppose that G is not a graph satisfying conditions (i) or (ii). If G is isomorphic to $K_{1,3} \cup aK_1 \cup bK_2$ or $K_{1,4} \cup aK_1 \cup bK_2$, for some nonnegative integers a and b, then $XL_3(G)$ is disconnected by Lemma 2.3. If G is not a star and does not satisfy condition (ii) , then G has at least two vertices u and v such that $\deg u \geq 2$, $\deg v \geq 2$ and the graph $XL_3(G)$ has at least two vertices, say wux and yvz. Since vertices with interior vertex u are adjacent only to vertices with the same interior vertex, there is no path from wux to yvz, so $XL_3(G)$ is disconnected. ∎

The next result presents a formula for the number of components in $XL_3(G)$ for any given graph G where some vertex of G has degree at least 2.

Corollary 2.5 *If G is a graph with n_1 vertices of degree 2 or degree at least 5, and n_2 vertices of degree 3 or 4, then $XL_3(G)$ has $(n_1 + 3n_2)$ components.*

Proof. If a vertex v of G has degree 2, then $XL_3(G)$ has a component that consists of a single vertex and if $\deg v \geq 5$, then $XL_3(G)$ has a connected component. If $\deg v = 3$ then $XL_3(G)$ has three components which are isolated vertices and if $\deg v = 4$, then $XL_3(G)$ has three components isomorphic to K_2. Finally, if $\deg v < 2$, then $XL_3(G)$ is not affected. Thus, $XL_3(G)$ has $(n_1 + 3n_2)$ components as desired. ∎

For instance, consider the graphs G and $XL_3(G)$ in Figure 1. Since G has two vertices of degree 2 and one vertex of degree 4, the graph $XL_3(G)$ has $(2 + 3(1)) = 5$ components.

Next, we show that $XL_3(G)$ is never a complete graph unless $G \cong P_3 + aK_1 + bK_2$ for some nonnegative integers a and b.

Corollary 2.6 *If G is a graph where some vertex has degree at least 2, then $XL_3(G)$ is never complete unless $G \cong P_3 + aK_1 + bK_2$ for some nonnegative integers a and b.*

Proof. If $G \cong P_3 + aK_1 + bK_2$ for some nonnegative integers a and b, then $XL_3(G) \cong K_1$. If G has at least two vertices of degree 2 and $\Delta(G) = 2$, then $XL_3(G)$ is disconnected and not complete by Corollary 2.5. On the other hand, suppose some vertex v satisfies $\deg v \geq 3$, and say that v is adjacent to vertices u, w, and x. Now uvw and uvx are vertices in $XL_3(G)$ that are not adjacent, so $XL_3(G)$ is not complete. ■

We conclude this section by investigating the following question: For which graphs G is $XL_3(G)$ isomorphic to G?

Theorem 2.7 *No graph G is isomorphic to $XL_3(G)$.*

Proof. If G is isomorphic to $XL_3(G)$ then $\Delta(G) = \binom{\Delta(G)-2}{2}$ for $\Delta(G) \geq 2$. This equation has an integer solution $\Delta(G) = 6$. Thus, consider $\Delta(G) = 6$. If G is a graph with n_1 vertices of degree 6, then $XL_3(G)$ has $15n_1$ vertices of degree 6. Clearly G is not isomorphic to $XL_3(G)$. If $\Delta(G) < 6$, there does not exist a graph G isomorphic to $XL_3(G)$. Thus, no graph G is isomorphic to $XL_3(G)$. ■

As we saw in Theorem 2.4, multiple graphs can have the same 3-xline graph. For instance, if G is a connected graph with at least one vertex of degree 2 and a is any positive integer, then $XL_3(G) = XL_3(G \cup aK_1) = XL_3(G \cup aK_2)$. This example also illustrates that multiple disconnected graphs can have the same 3-xline graph. Also, $XL_3(K_{1,3})$ is isomorphic to $XL_3(P_5)$ because both are isomorphic to the empty graph $\overline{K_3}$. However, we still have the following questions:

Question: Do there exist nonisomorphic connected graphs G and H such that $XL_3(G) \cong XL_3(H)$ and $XL_3(G)$ is nonempty?

Question: Which graphs H are isomorphic to $XL_3(G)$ for some graph G?

3 Cycles and Circuits in the 3-Xline Graph

As seen in several of the proofs from the previous section, the stars $K_{1,n}$ form an important class of graphs that is useful in understanding the structure of 3-xline graphs in general. In this section we focus on several properties of $XL_3(K_{1,n})$ for various values of n.

As observed earlier, the graphs $XL_3(K_{1,2})$, $XL_3(K_{1,3})$ and $XL_3(K_{1,4})$ contain no cycles, and $XL_3(K_{1,5}) \cong P$ contains no 3-cycles or 4-cycles. However, for $n \geq 6$, the graph $XL_3(K_{1,n})$ always contains copies of C_3.

Theorem 3.1 *If $G = K_{1,n}$ with $n \geq 6$, then S_Δ, the number of triangles (or copies of K_3) in $XL_3(G)$ is*

$$S_\Delta = \frac{\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}}{3!} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{48}.$$

Proof. Let $G = K_{1,n}$ and let v be the vertex of degree n. A triangle is formed in $XL_3(G)$ when three 3-paths in G share only the interior vertex v. That is, the three 3-paths would be of the form tvu, wvx and yvz where t, u, w, x, y and z are distinct vertices in G adjacent to v. The vertex tvu can be selected in $\binom{n}{2}$ ways, the vertex wvx then can be selected from the remaining $n-2$ vertices adjacent to v (in $\binom{n-2}{2}$ ways), and yvz then can be selected from the $n-4$ unused vertices adjacent to v (in $\binom{n-4}{2}$ ways). Since the same triangle can be created in six ways, we divide the product of the three factors by 6 to derive the formula. ■

Since each vertex of G of degree 6 or more generates its own component in $XL_3(G)$, the number of triangles of $XL_3(G)$ can be found by summing the number of triangles in each component.

Corollary 3.2 *Let G be a graph where the degree of vertex v is d_v. Let $V^*(G) = \{v : d_v \geq 6\}$. The number of triangles S_Δ in $XL_3(G)$ is given by*

$$S_\Delta = \begin{cases} 0 & \text{if } \Delta(G) \leq 5, \\ \sum_{v \in V^*(G)} \frac{\binom{d_v}{2}\binom{d_v-2}{2}\binom{d_v-4}{2}}{6} & \text{if } \Delta(G) > 5. \end{cases}$$

Proof. Since adjacent vertices in $XL_3(G)$ correspond to 3-paths in G with a common interior vertex, triangles will consist only of three distinct 3-paths in G with a common interior vertex. Therefore, if the degree of a vertex v in $V(G)$ is d_v, then the number of triangles in $XL_3(G)$ is $\sum_{v \in V^*(G)} \frac{\binom{d_v}{2}\binom{d_v-2}{2}\binom{d_v-4}{2}}{6}$ where $V^*(G) = \{v : d_v \geq 6\}$. ■

Theorem 3.3 *For any integer $n \geq 6$, if G has a vertex u such that $d_u \geq n$, then $XL_3(G)$ contains an n-cycle.*

Proof. Let $n \geq 6$ be an integer and let G be a graph with vertex u such that $d_u \geq n$. Label any n vertices adjacent to u as $v_1, v_2, \ldots, v_n$. Now, using the notation introduced in the proof of Theorem 2.2, create an n-cycle as follows:

Case 1: If $n = 6$, then one 6-cycle is $(12, 34, 16, 32, 14, 36, 12)$.

Case 2: If $n \geq 8$ and $n = 4t$ for some integer $t \geq 2$, then one n-cycle is $(12, 34, 16, 38, \ldots, 1k, 3(k+2), \ldots, 1(n-2), 3n, 52, 74, 56, 78, \ldots, 5k, 7(k+2), \ldots, 5(n-2), 7n, 12)$ where $k \equiv 2 \pmod 4$.

Case 3: If $n \geq 10$ and $n = 4t+2$ for some integer $t \geq 2$,, then one n-cycle is $(12, 34, 16, 38, \ldots, 1k, 3(k+2), \ldots, 3(n-2), 1n, 52, 74, 56, 78, \ldots, 5k, 7(k+2), \ldots, 7(n-2), 5n, 12)$ where $k \equiv 2 \pmod 4$.

Case 4: If n is odd, insert the vertex 57 between vertices 12 and 34 in ch of the cycles in the three previous cases to create cycles of length 7, $+1$ and $4t+3$ respectively. ∎

If G is a graph with a vertex of degree at least 6, then G also contains cycles and 5-cycles. For instance, using the notation from the previous oof, a 4-cycle in G is $12, 34, 15, 36, 12$ and a 5-cycle is $12, 34, 15, 36, 45, 12$. y combining this with Corollary 3.2 and Theorem 3.3, we have the follow- g result.

orollary 3.4 *If G is a graph with maximum degree $\Delta(G) \geq 6$, then $L_3(G)$ contains an n-cycle for every value of n such that $3 \leq n \leq \Delta(G)$.*

Next, we consider which 3-xline graphs have Hamiltonian cycles or Eu- rian circuits. Dirac's well-known sufficient condition for a graph to be amiltonian is the following: If G is a graph of order $m \geq 3$ such that eg $v \geq \frac{m}{2}$ for every vertex v of G, then G is Hamiltonian [6]. When is $L_3(K_{1,n})$ Hamiltonian? We know for $n = 2, 3$ or 4, that $XL_3(K_{1,n})$ con- ins no cycles and is not Hamiltonian. We also know that $XL_3(K_{1,5})$ is the etersen graph which is not Hamiltonian (see [7] for a proof). If we apply irac's condition to this situation, then we obtain the following result.

heorem 3.5 *If $n \geq 6$, then $XL_3(K_{1,n})$ is Hamiltonian.*

roof. If $n = 6$, then label the central vertex of $K_{1,6}$ as u and the end- ertices as $v_1, v_2, \ldots, v_6$. There exists a Hamiltonian cycle through the fif- een vertices of $XL_3(K_{1,6})$ as following: (12, 36, 24, 56, 14, 25, 16, 45, 13, 26, 5, 46, 23, 15, 34, 12). (Recall that consecutive vertices on this cycle are ad- acent because for each pair of vertices the four digits are distinct.)

If $n = 7$, we can label the end-vertices v_1 through v_7 and using the same otation as when $n = 6$, a Hamiltonian cycle through the twenty-one ver- ices of $XL_3(K_{1,7})$ is the following: (12, 34, 57, 23, 14, 67, 13, 45, 36, 47, 26, 5, 27, 46, 25, 37, 16, 35, 24, 17, 56, 12).

Let $n \geq 8$. The graph $XL_3(K_{1,n})$ is regular with order $\binom{n}{2}$ and common legree $\binom{n-1}{2}$. By Dirac's condition, if $\binom{n-1}{2} \geq \frac{\binom{n}{2}}{2}$, then $XL_3(K_{1,n})$ is Hamiltonian. The inequality reduces to $n(n-9)+12 \geq 0$ which is true when $n \geq 8$. ∎

Based on Theorem 2.4, a natural extension to all graphs is the following:

Corollary 3.6 *If G is a graph where some vertex has degree at least 2, then $XL_3(G)$ is Hamiltonian if and only if G is one of the following:*

(i) *a star* $K_{1,n}$ *with* $n \geq 6$*,*

(ii) *a disconnected graph where one component is a star* $K_{1,n}$ *with* $n \geq 6$ *and the remaining components contain only vertices* v *where* $\deg v \leq 1$.

A related question is if $XL_3(K_{1,n})$ is ever Eulerian. Recall that a graph is Eulerian if it is connected and the degree of every vertex is even. Since $XL_3(K_{1,n})$ is connected and regular with common degree $\binom{n-2}{2}$ for $n \geq 5$, we must determine which values of $n \geq 5$ make $\binom{n-2}{2}$ even. Since $\binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$, the value is even when $(n-2)(n-3)$ is divisible by 4. Since $n-2$ and $n-3$ are consecutive, only one is even. Therefore, either $4|(n-2)$ or $4|(n-3)$. That is, $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$

Combining the previous argument with Theorem 2.4 we can characterize all graphs G where $XL_3(G)$ is Eulerian.

Theorem 3.7 *If* G *is a graph where some vertex has degree at least* 2*, then* $XL_3(G)$ *is Eulerian if and only if* G *is one of the following:*

(i) *a star* $K_{1,n}$ *with* $n \geq 5$ *such that* $n \equiv 2 \pmod 4$ *or* $n \equiv 3 \pmod 4$*,*

(ii) *a disconnected graph where one component is a star* $K_{1,n}$ *with* $n \geq 5$ *such that* $n \equiv 2 \pmod 4$ *or* $n \equiv 3 \pmod 4$ *and the remaining components contain only vertices* v *where* $\deg v \leq 1$.

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