

The Spectrum Problem for Two Multigraphs with Four Vertices and Seven Edges

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Abstract

Let G be one of the two multigraphs obtained from $K_4 - e$ by replacing two edges with a double-edge while maintaining a minimum degree of 2. We find necessary and sufficient conditions on n and λ for the existence of a G -decomposition of ${}^\lambda K_n$.

1 Introduction

Throughout this paper, we may refer to a multigraph as a graph; however, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For a graph G , we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set (or multiset) of G , respectively. For a simple graph G and a positive integer λ , we use ${}^\lambda G$ to denote the graph obtained from G by replacing each edge in $E(G)$ with λ parallel edges. Alternatively, we let λG denote the graph consisting of λ vertex-disjoint copies of G . For edge-disjoint graphs G and H , we use $G \cup H$ to represent the graph with edge set (or multiset) $E(G) \cup E(H)$ and vertex set $V(G) \cup V(H)$. We define the *join* of two vertex-disjoint graphs G and H , denoted $G \vee H$, as the graph with vertex

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set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{a, b\} : a \in V(G), b \in V(H)\}$. We use $K_{s \times t}$ to denote the complete multipartite simple graph with s parts of size t , and we use $K_{r, s \times t}$ to denote the complete multipartite simple graph with one part of size r and s parts of size t . If G is a subgraph of H , we use $H \setminus G$ to denote the graph obtained from H by deleting the edges in $E(G)$.

1.1 The Spectrum Problem

Let K and G be graphs with G a subgraph of K . A G -decomposition of K is a set (or multiset) $\Delta = \{G_1, G_2, \dots, G_t\}$ of subgraphs of K such that each $G_i \in \Delta$ is isomorphic to G and such that each edge of K appears in exactly one such G_i . Similarly, if G and H are each subgraphs of K , then a $\{G, H\}$ -decomposition of K is defined to be a set $\{H_1, H_2, \dots, H_t\}$ of subgraphs of K such that each $H_i \in \Delta$ is isomorphic to either G or H and such that each edge of K appears in exactly one such H_i . A G -decomposition of K is also known as a (K, G) -design or, if K is the complete graph on n vertices, a G -design of order n .

A classic problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a G -decomposition of ${}^\lambda K_n$. This is known as the *spectrum problem* for G because the set of all such n is called the *spectrum for G -designs of index λ* . The spectrum for G -designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and simple graphs with at most 5 vertices (see [3]).

In recent years, there have been some investigations of G -designs of index λ where G is a multigraph that is not simple. For example, in [8] Carter determined the spectra for G -designs of index λ for all connected cubic multigraphs G of order at most 6. The spectra for G -designs of index λ have been investigated for various multigraphs G of small order, e.g., certain subgraphs of 2K_4 with 5 edges (see [12, 6, 13]), 6 edges (see [2, 7]), 7 edges (see [4]), or 8 edges (see [5]). In this paper we consider two multigraphs, G_1 and G_2 , each with 7 edges and minimum degree 2 obtained by replacing two edges of $K_4 - e$ with a double edge (see Figure 1). We settle the spectrum problem for these multigraphs.

1.2 Some Basic Results

The necessary conditions for the existence of a G -decomposition of ${}^\lambda K_n$ include the following:

- $|V(G)| \leq n$,
- $|E(G)|$ divides $|E({}^\lambda K_n)| = \lambda n(n-1)$, and
- $\gcd\{\deg(v) : v \in V(G)\}$ divides $\lambda(n-1)$.

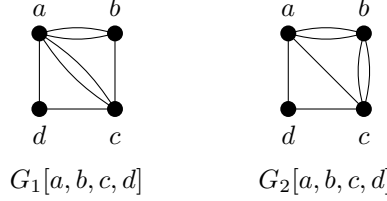


Figure 1: The two multigraphs consisting of $K_4 - e$ with two double edges and minimum degree 2.

Applying these necessary conditions to the two multigraphs under consideration, we obtain the following necessary conditions on their spectra.

Lemma 1.1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_1 -decomposition of ${}^\lambda K_n$ only if the following hold:*

- if $\gcd(\lambda, 7) = 1$, then $n \equiv 0$ or $1 \pmod{7}$;
- if $\gcd(\lambda, 7) = 7$, then $n \geq 4$.

Proof. Let $\lambda \geq 2$ and $n \geq 4$ and suppose there exists a G_1 -decomposition of ${}^\lambda K_n$. Since $|E(G_1)|$ must divide $|E({}^\lambda K_n)|$ for such a G_1 -decomposition to exist, we must have that $7 \mid \lambda n(n-1)/2$, and thus $14 \mid \lambda n(n-1)$. First, if $\gcd(\lambda, 7) = 1$, then $14 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod{7}$. Finally, if $\gcd(\lambda, 7) = 7$, then $2 \mid n(n-1)$, which is true for any $n \geq 4$. ■

Lemma 1.2. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_2 -decomposition of ${}^\lambda K_n$ only if the following hold:*

- if $\gcd(\lambda, 14) = 1$, then $n \equiv 1$ or $7 \pmod{14}$;
- if $\gcd(\lambda, 14) = 2$, then $n \equiv 0$ or $1 \pmod{7}$;
- if $\gcd(\lambda, 14) = 7$, then $n \equiv 1 \pmod{2}$;
- if $\gcd(\lambda, 14) = 14$, then $n \geq 4$.

Proof. Let $\lambda \geq 2$ and $n \geq 4$ and suppose there exists a G_2 -decomposition of ${}^\lambda K_n$. Since $|E(G_2)|$ must divide $|E({}^\lambda K_n)|$ for such a G_2 -decomposition to exist, we must have that $7 \mid \lambda n(n-1)/2$, and thus $14 \mid \lambda n(n-1)$. Also, since all the vertices of G_2 have even degree, each vertex of ${}^\lambda K_n$ must similarly have even degree; thus, $2 \mid \lambda(n-1)$. First, if $\gcd(\lambda, 14) = 1$, then $14 \mid n(n-1)$ and $2 \mid (n-1)$, and thus $n \equiv 1$ or $7 \pmod{14}$. Second, if $\gcd(\lambda, 14) = 2$, then λ is even and $7 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod{7}$. Third, if $\gcd(\lambda, 14) = 7$, then $2 \mid n(n-1)$ but also $2 \mid (n-1)$, and thus $n \equiv 1 \pmod{2}$. Finally, if $\gcd(\lambda, 14) = 14$, then $14 \mid \lambda$, and thus there are no further restrictions on n . ■

The following theorems on decompositions of complete graphs and complete multipartite graphs are used extensively in proving our main results. All of these results can be found in the *Handbook of Combinatorial Designs* [9] (see [1] and [10]).

Theorem 1.3 ([1]). *If n is an odd positive integer, then there exists a $\{K_3, K_5\}$ -decomposition of K_n .*

Theorem 1.4 ([10]). *The necessary and sufficient conditions for the existence of a K_3 -decomposition of $K_{t \times m}$ are (i) $t \geq 3$, (ii) $(t-1)m \equiv 0 \pmod{2}$, and (iii) $t(t-1)m^2 \equiv 0 \pmod{6}$.*

Theorem 1.5 ([10]). *If $t \geq 3$ and $t \equiv 0 \pmod{3}$, then there exists a K_3 -decomposition of $K_{4, t \times 2}$.*

Combining the previous two results, we have the following corollary that is more directly applicable in our general constructions.

Corollary 1.6. *Let $t \geq 3$. There exists a K_3 -decomposition of $K_{t \times 2}$ if $t \equiv 0$ or $1 \pmod{3}$ and of $K_{4, (t-2) \times 2}$ if $t \equiv 2 \pmod{3}$.*

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [11]).

Theorem 1.7. *Let m, n, r, s , and t be positive integers. If there exists a $(K_{t \times m}, K_n)$ -design, then there exists a $(K_{t \times ms}, K_{n \times s})$ -design. Similarly, if there exists a $(K_{r, t \times m}, K_n)$ -design, then there exists a $(K_{rs, t \times ms}, K_{n \times s})$ -design.*

2 Some Small Examples

In this section we present G_1 - and G_2 -decompositions of various graphs that are needed for the constructions presented in Section 3. Let $G \in \{G_1, G_2\}$ and let $G[a, b, c, d]$ denote the graph with vertex set $\{a, b, c, d\}$ and edge set as represented in Figure 1. For example, $G_2[0, 1, 2, 3]$ denotes the graph with vertex set $\{0, 1, 2, 3\}$ and edge multiset $\{\{0, 1\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 2\}, \{2, 3\}\}$. Given the graphs represented by the notation $G[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $G[a, b, c, d] + i = G[a+i, b+i, c+i, d+i]$ where all addition is performed in \mathbb{Z}_n . By convention, define $\infty + 1 = \infty$.

2.1 Small Designs of Index 2

Example 2.1. Let $V({}^2K_7) = \mathbb{Z}_6 \cup \{\infty\}$ and let $\Delta_1 = \{G_1[1, 0, 3, \infty] + i : i \in \mathbb{Z}_6\}$ and $\Delta_2 = \{G_2[0, 1, 3, \infty] + i : i \in \mathbb{Z}_6\}$. Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of 2K_7 .

Example 2.2. Let $V({}^2K_8) = \mathbb{Z}_8$ and let $\Delta_1 = \{G_1[0, 2, 3, 7] + i : i \in \mathbb{Z}_8\}$ and $\Delta_2 = \{G_2[1, 0, 3, 5] + i : i \in \mathbb{Z}_8\}$. Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of 2K_8 .

Example 2.3. Let $V({}^2K_{14}) = \mathbb{Z}_{13} \cup \{\infty\}$ and let $\Delta_1 = \{G_1[5, 3, 0, \infty] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0, 4, 1, 7] + i : i \in \mathbb{Z}_{13}\}$ and $\Delta_2 = \{G_2[0, 3, 1, \infty] + i : i \in \mathbb{Z}_{13}\} \cup \{G_2[0, 5, 1, 7] + i : i \in \mathbb{Z}_{13}\}$. Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^2K_{14}$.

Example 2.4. Let $V({}^2K_{15}) = \mathbb{Z}_{15}$ and let $\Delta_1 = \{G_1[5, 0, 1, 3] + i : i \in \mathbb{Z}_{15}\} \cup \{G_1[7, 0, 1, 4] + i : i \in \mathbb{Z}_{15}\}$ and $\Delta_2 = \{G_2[0, 3, 1, 7] + i : i \in \mathbb{Z}_{15}\} \cup \{G_2[0, 5, 1, 7] + i : i \in \mathbb{Z}_{15}\}$. Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^2K_{15}$.

Example 2.5. Let $V({}^2K_{28}) = \mathbb{Z}_{27} \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[17, 0, 11, \infty] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[1, 0, 15, 23] + i : i \in \mathbb{Z}_{27}\} \\ &\quad \cup \{G_1[9, 0, 12, 17] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[4, 11, 0, 2] + i : i \in \mathbb{Z}_{27}\}, \\ \Delta_2 &= \{G_2[0, 3, 1, \infty] + i : i \in \mathbb{Z}_{27}\} \cup \{G_2[0, 8, 1, 14] + i : i \in \mathbb{Z}_{27}\} \\ &\quad \cup \{G_2[0, 11, 5, 9] + i : i \in \mathbb{Z}_{27}\} \cup \{G_2[10, 0, 15, 6] + i : i \in \mathbb{Z}_{27}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^2K_{28}$.

Example 2.6. Let $V({}^2K_{29}) = \mathbb{Z}_{29}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 2, 8, 5] + i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0, 14, 11, 4] + i : i \in \mathbb{Z}_{29}\} \\ &\quad \cup \{G_1[0, 12, 13, 6] + i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0, 10, 9, 5] + i : i \in \mathbb{Z}_{29}\}, \\ \Delta_2 &= \{G_2[14, 2, 0, 3] + i : i \in \mathbb{Z}_{29}\} \cup \{G_2[13, 4, 0, 5] + i : i \in \mathbb{Z}_{29}\} \\ &\quad \cup \{G_2[5, 6, 0, 13] + i : i \in \mathbb{Z}_{29}\} \cup \{G_2[3, 10, 0, 14] + i : i \in \mathbb{Z}_{29}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^2K_{29}$.

Example 2.7. Let $V({}^2K_{3 \times 7}) = \mathbb{Z}_{21}$ with partition $\{\{i \in \mathbb{Z}_{21} : i \equiv j \pmod{3}\} : j \in \mathbb{Z}_3\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[1, 5, 0, 8] + i : i \in \mathbb{Z}_{21}\} \cup \{G_1[0, 10, 2, 7] + i : i \in \mathbb{Z}_{21}\}, \\ \Delta_2 &= \{G_2[0, 8, 7, 2] + i : i \in \mathbb{Z}_{21}\} \cup \{G_2[0, 11, 7, 2] + i : i \in \mathbb{Z}_{21}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^2K_{3 \times 7}$.

Example 2.8. Let $V({}^2K_{5 \times 7}) = \mathbb{Z}_{35}$ with partition $\{\{i \in \mathbb{Z}_{35} : i \equiv j \pmod{5}\} : j \in \mathbb{Z}_5\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[9, 12, 0, 1] + i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0, 21, 13, 1] + i : i \in \mathbb{Z}_{35}\} \\ &\quad \cup \{G_1[6, 13, 2, 0] + i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0, 19, 17, 11] + i : i \in \mathbb{Z}_{35}\}, \\ \Delta_2 &= \{G_2[0, 11, 4, 16] + i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0, 6, 4, 13] + i : i \in \mathbb{Z}_{35}\} \\ &\quad \cup \{G_2[0, 17, 9, 12] + i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0, 14, 13, 16] + i : i \in \mathbb{Z}_{35}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^2K_{5 \times 7}$.

2.2 Small Designs of Index 3

Example 2.9. Let $V({}^3K_7) = \mathbb{Z}_2 \times \mathbb{Z}_3 \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[(0, 0 + i), (1, 0 + i), (1, 2 + i), (0, 1 + i)] : i \in \mathbb{Z}_3\} \\ &\quad \cup \{G_1[(1, 2 + i), \infty, (1, 1 + i), (0, 1 + i)] : i \in \mathbb{Z}_3\} \\ &\quad \cup \{G_1[(0, 2 + i), \infty, (0, 1 + i), (1, 0 + i)] : i \in \mathbb{Z}_3\}, \\ \Delta_2 &= \{G_2[\infty, (0, 1 + i), (0, 0 + i), (1, 1 + i)] : i \in \mathbb{Z}_3\} \\ &\quad \cup \{G_2[(1, 1 + i), (0, 1 + i), (1, 0 + i), \infty] : i \in \mathbb{Z}_3\} \\ &\quad \cup \{G_2[(0, 0 + i), (1, 1 + i), (1, 2 + i), (0, 2 + i)] : i \in \mathbb{Z}_3\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of 3K_7 .

Example 2.10. Let $V({}^3K_8) = \mathbb{Z}_8$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[4, 1, 3, 6], G_1[2, 6, 0, 3], G_1[2, 3, 7, 0], G_1[0, 1, 3, 4], \\ &\quad G_1[0, 4, 5, 1], G_1[0, 6, 7, 5], G_1[1, 6, 5, 2], G_1[2, 4, 1, 7], \\ &\quad G_1[3, 5, 7, 1], G_1[4, 7, 5, 2], G_1[6, 3, 5, 2], G_1[6, 4, 7, 1]\}.\end{aligned}$$

Then Δ_1 is a G_1 -decomposition of 3K_8 .

Example 2.11. Let $V({}^3K_{14}) = \mathbb{Z}_{13} \cup \{\infty\}$ and let $\Delta_1 = \{G_1[0, \infty, 10, 4] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0, 2, 5, 1] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0, 1, 6, 2] + i : i \in \mathbb{Z}_{13}\}$. Then Δ_1 is a G_1 -decomposition of ${}^3K_{14}$.

Example 2.12. Let $V({}^3K_{15}) = \mathbb{Z}_{15}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 2, 5, 6] + i : i \in \mathbb{Z}_{15}\} \cup \{G_1[0, 4, 7, 5] + i : i \in \mathbb{Z}_{15}\} \\ &\quad \cup \{G_1[1, 7, 0, 4] + i : i \in \mathbb{Z}_{15}\}, \\ \Delta_2 &= \{G_2[0, 1, 7, 2] + i : i \in \mathbb{Z}_{15}\} \cup \{G_2[0, 2, 7, 3] + i : i \in \mathbb{Z}_{15}\} \\ &\quad \cup \{G_2[0, 3, 7, 1] + i : i \in \mathbb{Z}_{15}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^3K_{15}$.

Example 2.13. Let $V({}^3K_{28}) = \mathbb{Z}_{27} \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, \infty, 13, 26] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[0, 12, 16, 5] + i : i \in \mathbb{Z}_{27}\} \\ &\quad \cup \{G_1[0, 10, 6, 12] + i : i \in \mathbb{Z}_{27}\} \\ &\quad \cup \{G_1[0, 9, 5, 8] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[0, 7, 8, 10] + i : i \in \mathbb{Z}_{27}\} \\ &\quad \cup \{G_1[0, 3, 2, 9] + i : i \in \mathbb{Z}_{27}\}.\end{aligned}$$

Then Δ_1 is a G_1 -decomposition of ${}^3K_{28}$.

Example 2.14. Let $V({}^3K_{29}) = \mathbb{Z}_{29}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 15, 13, 10] + i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0, 12, 10, 14] + i : i \in \mathbb{Z}_{29}\} \\ &\quad \cup \{G_1[0, 11, 9, 18] + i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0, 8, 7, 13] + i : i \in \mathbb{Z}_{29}\} \\ &\quad \cup \{G_1[0, 6, 5, 12] + i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0, 4, 3, 8] + i : i \in \mathbb{Z}_{29}\}, \\ \Delta_2 &= \{G_2[0, 13, 27, 14] + i : i \in \mathbb{Z}_{29}\} \cup \{G_2[0, 12, 2, 19] + i : i \in \mathbb{Z}_{29}\} \\ &\quad \cup \{G_2[0, 11, 2, 20] + i : i \in \mathbb{Z}_{29}\} \cup \{G_2[0, 8, 1, 22] + i : i \in \mathbb{Z}_{29}\} \\ &\quad \cup \{G_2[0, 6, 1, 24] + i : i \in \mathbb{Z}_{29}\} \cup \{G_2[0, 4, 1, 26] + i : i \in \mathbb{Z}_{29}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^3K_{29}$.

Example 2.15. Let $V({}^3K_{3 \times 7}) = \mathbb{Z}_{21}$ with partition $\{\{i \in \mathbb{Z}_{21} : i \equiv j \pmod{3}\} : j \in \mathbb{Z}_3\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 5, 10, 14] + i : i \in \mathbb{Z}_{21}\} \cup \{G_1[0, 1, 14, 13] + i : i \in \mathbb{Z}_{21}\} \\ &\quad \cup \{G_1[0, 4, 2, 10] + i : i \in \mathbb{Z}_{21}\}, \\ \Delta_2 &= \{G_2[0, 8, 7, 11] + i : i \in \mathbb{Z}_{21}\} \cup \{G_2[0, 2, 7, 8] + i : i \in \mathbb{Z}_{21}\} \\ &\quad \cup \{G_2[0, 11, 7, 2] + i : i \in \mathbb{Z}_{21}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^3K_{3 \times 7}$.

Example 2.16. Let $V({}^3K_{5 \times 7}) = \mathbb{Z}_{35}$ with partition $\{\{i \in \mathbb{Z}_{35} : i \equiv j \pmod{5}\} : j \in \mathbb{Z}_5\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 17, 16, 14] + i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0, 13, 14, 1] + i : i \in \mathbb{Z}_{35}\} \\ &\quad \cup \{G_1[0, 12, 9, 16] + i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0, 8, 11, 17] + i : i \in \mathbb{Z}_{35}\} \\ &\quad \cup \{G_1[0, 7, 4, 12] + i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0, 6, 2, 11] + i : i \in \mathbb{Z}_{35}\}, \\ \Delta_2 &= \{G_2[0, 4, 1, 7] + i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0, 9, 1, 4] + i : i \in \mathbb{Z}_{35}\} \\ &\quad \cup \{G_2[0, 7, 1, 9] + i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0, 13, 2, 14] + i : i \in \mathbb{Z}_{35}\} \\ &\quad \cup \{G_2[0, 18, 2, 13] + i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0, 14, 2, 18] + i : i \in \mathbb{Z}_{35}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^3K_{5 \times 7}$.

2.3 Small Designs of Index 7

Example 2.17. Let $V({}^7K_4) = \mathbb{Z}_3 \cup \{\infty\}$ and let $\Delta_1 = \{G_1[\infty, 0, 1, 2] + i : i \in \mathbb{Z}_3\} \cup \{G_1[0, 1, 2, \infty] + i : i \in \mathbb{Z}_3\}$. Then Δ_1 is a G_1 -decomposition of 7K_4 .

Example 2.18. Let $V({}^7K_5) = \mathbb{Z}_5$ and let $\Delta_1 = \{G_1[0, 4, 3, 1] + i : i \in \mathbb{Z}_5\} \cup \{G_1[0, 3, 1, 2] + i : i \in \mathbb{Z}_5\}$ and $\Delta_2 = \{G_2[0, 3, 2, 1] + i : i \in \mathbb{Z}_5\} \cup \{G_2[0, 2, 3, 1] + i : i \in \mathbb{Z}_5\}$. Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of 7K_5 .

Example 2.19. Let $V({}^7K_6) = \mathbb{Z}_5 \cup \{\infty\}$ and let $\Delta_1 = \{G_1[0, 1, 2, \infty] + i : i \in \mathbb{Z}_5\} \cup \{G_1[\infty, 0, 1, 3] + i : i \in \mathbb{Z}_5\} \cup \{G_1[0, 2, 3, 4] + i : i \in \mathbb{Z}_5\}$. Then Δ_1 is a G_1 -decomposition of 7K_6 .

Example 2.20. Let $V({}^7K_9) = \mathbb{Z}_9$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 1, 3, 4] + i : i \in \mathbb{Z}_9\} \cup \{G_1[0, 2, 4, 3] + i : i \in \mathbb{Z}_9\} \\ &\quad \cup \{G_1[0, 5, 6, 8] + i : i \in \mathbb{Z}_9\} \cup \{G_1[0, 6, 5, 7] + i : i \in \mathbb{Z}_9\}, \\ \Delta_2 &= \{G_2[2, 0, 6, 5] + i : i \in \mathbb{Z}_9\} \cup \{G_2[1, 0, 4, 8] + i : i \in \mathbb{Z}_9\} \\ &\quad \cup \{G_2[1, 0, 2, 3] + i : i \in \mathbb{Z}_9\} \cup \{G_2[3, 0, 5, 8] + i : i \in \mathbb{Z}_9\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of 7K_9 .

Example 2.21. Let $V({}^7K_{10}) = \mathbb{Z}_9 \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, \infty, 3, 4] + i : i \in \mathbb{Z}_9\} \cup \{G_1[2, 0, \infty, 1] + i : i \in \mathbb{Z}_9\} \\ &\quad \cup \{G_1[0, 1, 4, 3] + i : i \in \mathbb{Z}_9\} \cup \{G_1[0, 4, 3, 1] + i : i \in \mathbb{Z}_9\} \\ &\quad \cup \{G_1[4, 6, 0, 2] + i : i \in \mathbb{Z}_9\}.\end{aligned}$$

Then Δ_1 is a G_1 -decomposition of ${}^7K_{10}$.

Example 2.22. Let $V({}^7K_{11}) = \mathbb{Z}_{11}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 1, 8, 9] + i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0, 4, 3, 9] + i : i \in \mathbb{Z}_{11}\} \\ &\quad \cup \{G_1[0, 3, 5, 1] + i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0, 5, 4, 2] + i : i \in \mathbb{Z}_{11}\} \\ &\quad \cup \{G_1[0, 2, 5, 4] + i : i \in \mathbb{Z}_{11}\}, \\ \Delta_2 &= \{G_2[1, 0, 2, 6] + i : i \in \mathbb{Z}_{11}\} \cup \{G_2[3, 0, 4, 9] + i : i \in \mathbb{Z}_{11}\} \\ &\quad \cup \{G_2[4, 0, 5, 7] + i : i \in \mathbb{Z}_{11}\} \cup {}^2\{G_2[2, 0, 3, 7] + i : i \in \mathbb{Z}_{11}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^7K_{11}$.

Example 2.23. Let $V({}^7K_{3 \times 2}) = \mathbb{Z}_6$ with partition $\{\{0, 3\}, \{1, 4\}, \{2, 5\}\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0, 5, 1, 2] + i : i \in \mathbb{Z}_6\} \cup \{G_1[0, 4, 2, 1] + i : i \in \mathbb{Z}_6\}, \\ \Delta_2 &= \{G_2[0, 1, 2, 4], G_2[0, 1, 5, 4], G_2[0, 2, 1, 5], G_2[0, 5, 4, 2], \\ &\quad G_2[3, 4, 2, 1], G_2[3, 4, 5, 1], G_2[3, 2, 4, 5], G_2[3, 5, 1, 2], \\ &\quad G_2[1, 3, 2, 0], G_2[1, 3, 5, 0], G_2[4, 0, 2, 3], G_2[4, 0, 5, 3]\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^7K_{3 \times 2}$.

Example 2.24. Let $V({}^7K_6 \setminus {}^7K_2) = \mathbb{Z}_6$ with $V({}^7K_2) = \{0, 1\}$ and let

$$\begin{aligned} \Delta_1 = \{ & G_1[5, 3, 2, 4], G_1[3, 2, 4, 5], G_1[2, 1, 3, 5], G_1[0, 2, 3, 4], G_1[0, 2, 3, 4], \\ & G_1[0, 2, 4, 3], G_1[0, 3, 5, 4], G_1[1, 3, 5, 2], G_1[1, 3, 5, 2], G_1[4, 1, 5, 0], \\ & G_1[4, 2, 1, 3], G_1[4, 2, 1, 3], G_1[5, 0, 2, 1], G_1[5, 0, 4, 1]\}. \end{aligned}$$

Then Δ_1 is a G_1 -decomposition of ${}^7K_6 \setminus {}^7K_2$.

Example 2.25. Let $V({}^7K_7 \setminus {}^7K_3) = \mathbb{Z}_7$ with $V({}^7K_3) = \{0, 1, 2\}$ and let

$$\begin{aligned} \Delta_1 = \{ & G_1[0, 6, 4, 3], G_1[6, 3, 2, 5], G_1[0, 6, 3, 4], G_1[0, 3, 4, 5], G_1[0, 3, 4, 5], \\ & G_1[1, 3, 4, 5], G_1[1, 3, 4, 5], G_1[1, 3, 5, 4], G_1[2, 3, 5, 4], G_1[2, 4, 6, 3], \\ & G_1[2, 4, 6, 3], G_1[5, 0, 6, 1], G_1[5, 0, 6, 3], G_1[5, 2, 3, 1], G_1[5, 3, 2, 4], \\ & G_1[6, 1, 4, 2], G_1[6, 1, 4, 3], G_1[6, 1, 5, 0]\}, \\ \Delta_2 = \{ & G_2[2, 5, 4, 6], G_2[2, 3, 6, 4], G_2[2, 5, 3, 4], G_2[0, 3, 4, 5], G_2[0, 3, 4, 5], \\ & G_2[0, 3, 4, 5], G_2[0, 4, 5, 6], G_2[0, 6, 5, 3], G_2[2, 3, 6, 4], G_2[3, 1, 5, 2], \\ & G_2[3, 1, 5, 2], G_2[3, 1, 5, 6], G_2[3, 6, 5, 1], G_2[4, 0, 6, 1], G_2[4, 1, 6, 2], \\ & G_2[4, 1, 6, 2], G_2[4, 1, 6, 2], G_2[5, 0, 6, 2]\}. \end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1 - and G_2 -decompositions of ${}^7K_7 \setminus {}^7K_3$.

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Example 2.26. Let $V({}^{14}K_4) = \mathbb{Z}_4$ and let $\Delta_2 = \{G_2[0, 1, 2, 3] + i : i \in \mathbb{Z}_4\} \cup {}^2\{G_2[0, 2, 1, 3] + i : i \in \mathbb{Z}_4\}$. Then Δ_2 is a G_2 -decomposition of ${}^{14}K_4$.

Example 2.27. Let $V({}^{14}K_6) = \mathbb{Z}_5 \cup \{\infty\}$ and let $\Delta_2 = {}^3\{G_2[\infty, 0, 2, 4] + i : i \in \mathbb{Z}_5\} \cup \{G_2[0, 1, 2, \infty] + i : i \in \mathbb{Z}_5\} \cup {}^2\{G_2[0, 1, 2, 3] + i : i \in \mathbb{Z}_5\}$. Then Δ_2 is a G_2 -decomposition of ${}^{14}K_6$.

Example 2.28. Let $V({}^{14}K_{10}) = \mathbb{Z}_9 \cup \{\infty\}$ and let

$$\begin{aligned} \Delta_2 = & {}^2\{G_2[0, 4, \infty, 2] + i : i \in \mathbb{Z}_9\} \cup \{G_2[0, 2, \infty, 4] + i : i \in \mathbb{Z}_9\} \\ & \cup \{G_2[0, 2, 4, \infty] + i : i \in \mathbb{Z}_9\} \cup \{G_2[0, 4, 2, 7] + i : i \in \mathbb{Z}_9\} \\ & \cup \{G_2[0, 1, 4, 2] + i : i \in \mathbb{Z}_9\} \cup {}^4\{G_2[0, 1, 4, 3] + i : i \in \mathbb{Z}_9\}. \end{aligned}$$

Then Δ_2 is a G_2 -decomposition of ${}^{14}K_{10}$.

Example 2.29. Let $V({}^{14}K_6 \setminus {}^{14}K_2) = \mathbb{Z}_4 \cup \{\infty_1, \infty_2\}$ with $V({}^{14}K_2) = \{\infty_1, \infty_2\}$ and let

$$\begin{aligned} \Delta_2 = & {}^3\{G_2[0 + i, \infty_1, 2 + i, \infty_2] : i \in \mathbb{Z}_4\} \\ & \cup \{G_2[0 + i, \infty_2, 2 + i, \infty_1] : i \in \mathbb{Z}_4\} \\ & \cup \{G_2[0 + i, \infty_2, 2 + i, 1 + i] : i \in \mathbb{Z}_4\} \\ & \cup {}^2\{G_2[0 + i, 1 + i, 2 + i, 3 + i] : i \in \mathbb{Z}_4\}. \end{aligned}$$

Then Δ_2 is a G_2 -decomposition of ${}^{14}K_6 \setminus {}^{14}K_2$.

3 Main Results

Through judicious use of the examples from the previous section, we show that the necessary conditions on G_1 - and G_2 -designs that we established in Section 1.2 are sufficient for any index $\lambda \geq 2$. In the following constructions, we make extensive use of the join of complete graphs. Of special note is our use of the null graph K_0 , which has an empty vertex set. For example, $K_7 \vee K_0$ is simply K_7 . Similarly, $K_7 \setminus K_0$ is also K_7 . On the other hand, $K_7 \vee K_1 = K_8$, but $K_7 \setminus K_1 = K_7$.

First, we settle the spectra for G_1 - and G_2 -designs of certain indices.

Lemma 3.1. *Let $G \in \{G_1, G_2\}$. There exists a G -decomposition of 2K_n if $n \equiv 0$ or $1 \pmod{7}$.*

Proof. Let $G \in \{G_1, G_2\}$ and let $n = 7r + t$ for some positive integer r and $t \in \{0, 1\}$. If (r, t) is $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(4, 0)$, or $(4, 1)$, then the result follows from Examples 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6, respectively. The remainder of the proof breaks into two cases.

CASE 1: r is odd with $r \geq 3$.

By Theorem 1.3 there exists a $\{K_3, K_5\}$ -decomposition of K_r . Thus by Theorem 1.7 there exists a $\{K_{3 \times 7}, K_{5 \times 7}\}$ -decomposition of $K_{r \times 7}$. Since there exist G -decompositions of both ${}^2K_{3 \times 7}$ and ${}^2K_{5 \times 7}$ (by Examples 2.7 and 2.8, respectively), a G -decomposition of ${}^2K_{r \times 7}$ also exists by transitivity. Finally, we note that $K_{7r+t} = (rK_7 \vee K_t) \cup K_{r \times 7} = K_{r \times 7} \cup \bigcup_{i=1}^r K_{7+t}$. Thus ${}^2K_{7r+t} = {}^2K_{r \times 7} \cup \bigcup_{i=1}^r {}^2K_{7+t}$, and the result follows from the existence of G -decompositions of ${}^2K_{r \times 7}$, 2K_7 , and 2K_8 .

CASE 2: r is even with $r \geq 6$.

Let $r = 2s$ for some integer $s \geq 3$; hence, $n = 14s + t$. By Corollary 1.6 there exists a K_3 -decomposition either of $K_{s \times 2}$ if $s \not\equiv 2 \pmod{3}$ or of $K_{4, (s-2) \times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3 \times 7}$ -decomposition of either $K_{s \times 14}$ or $K_{28, (s-2) \times 14}$. Since there exists a G -decomposition of ${}^2K_{3 \times 7}$ (by Example 2.7), a G -decomposition of either ${}^2K_{s \times 14}$ or ${}^2K_{28, (s-2) \times 14}$ also exists by transitivity. Finally, we note that K_{14s+t} can be described as either $(sK_{14} \vee K_t) \cup K_{s \times 14} = K_{s \times 14} \cup \bigcup_{i=1}^s K_{14+t}$ or $((K_{28} \cup (s-2)K_{14}) \vee K_t) \cup K_{28, (s-2) \times 14} = K_{28, (s-2) \times 14} \cup K_{28+t} \cup \bigcup_{i=1}^{s-2} K_{14+t}$. Thus, we describe ${}^2K_{14s+t}$ as ${}^2K_{s \times 14} \cup \bigcup_{i=1}^s {}^2K_{14+t}$ when $s \not\equiv 2 \pmod{3}$ and as ${}^2K_{28, (s-2) \times 14} \cup {}^2K_{28+t} \cup \bigcup_{i=1}^{s-2} {}^2K_{14+t}$ when $s \equiv 2 \pmod{3}$, and the result follows from the existence of G -decompositions of ${}^2K_{s \times 14}$ or ${}^2K_{28, (s-2) \times 14}$, ${}^2K_{14}$, ${}^2K_{15}$, ${}^2K_{28}$, and ${}^2K_{29}$. ■

Lemma 3.2. *There exists a G_1 -decomposition of 3K_n if $n \equiv 0$ or $1 \pmod{7}$.*

Proof. Let $n = 7r + t$ for some positive integer r and $t \in \{0, 1\}$. If (r, t) is $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(4, 0)$, or $(4, 1)$, then the result follows from Examples 2.9, 2.10, 2.11, 2.12, 2.13, and 2.14, respectively. The proof then follows as in the proof of Lemma 3.1, where the requisite G_1 -decompositions of the multipartite graphs ${}^3K_{3 \times 7}$ and ${}^3K_{5 \times 7}$ can be found in Examples 2.15 and 2.16, respectively. ■

Lemma 3.3. *There exists a G_2 -decomposition of 3K_n if $n \equiv 1$ or $7 \pmod{14}$.*

Proof. If n is 7, 15, or 29, then the result follows from Examples 2.9, 2.12, and 2.14, respectively. The remainder of the proof breaks into two cases.

CASE 1: $n \equiv 1 \pmod{14}$ with $n \geq 43$.

Let $n = 14r + 1$ for some integer $r \geq 3$; hence, $n = 7(2r) + 1$. By Corollary 1.6 there exists a K_3 -decomposition either of $K_{r \times 2}$ if $r \not\equiv 2 \pmod{3}$ or of $K_{4, (r-2) \times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3 \times 7}$ -decomposition of either $K_{r \times 14}$ or $K_{28, (r-2) \times 14}$. Since there exists a G_2 -decomposition of ${}^3K_{3 \times 7}$ (by Example 2.15), a G_2 -decomposition of either ${}^3K_{r \times 14}$ or ${}^3K_{28, (r-2) \times 14}$ also exists by transitivity. Finally, we note that K_{14r+1} can be described as either $(rK_{14} \vee K_1) \cup K_{r \times 14} = K_{r \times 14} \cup \bigcup_{i=1}^r K_{15}$ or $((K_{28} \cup (r-2)K_{14}) \vee K_1) \cup K_{28, (r-2) \times 14} = K_{28, (r-2) \times 14} \cup K_{29} \cup \bigcup_{i=1}^{r-2} K_{15}$. Thus, we describe ${}^3K_{14r+1}$ as ${}^3K_{r \times 14} \cup \bigcup_{i=1}^r {}^3K_{15}$ when $r \not\equiv 2 \pmod{3}$ and as ${}^3K_{28, (r-2) \times 14} \cup {}^3K_{29} \cup \bigcup_{i=1}^{r-2} {}^3K_{15}$ when $r \equiv 2 \pmod{3}$, and the result follows from the existence of G_2 -decompositions of ${}^3K_{r \times 14}$ or ${}^3K_{28, (r-2) \times 14}$, ${}^3K_{15}$, and ${}^3K_{29}$.

CASE 2: $n \equiv 7 \pmod{14}$ with $n \geq 21$.

Let $n = 14r + 7$ for some positive integer r ; hence, $n = 7(2r + 1)$. By Theorem 1.3 there exists a $\{K_3, K_5\}$ -decomposition of K_{2r+1} . Thus by Theorem 1.7 there exists a $\{K_{3 \times 7}, K_{5 \times 7}\}$ -decomposition of $K_{(2r+1) \times 7}$. Since there exist G_2 -decompositions of both ${}^3K_{3 \times 7}$ and ${}^3K_{5 \times 7}$ (by Examples 2.15 and 2.16, respectively), a G_2 -decomposition of ${}^3K_{(2r+1) \times 7}$ also exists by transitivity. Finally, we note that $K_{14r+7} = K_{(2r+1) \times 7} \cup (2r + 1)K_7 = K_{(2r+1) \times 7} \cup \bigcup_{i=1}^{2r+1} K_7$. Thus ${}^3K_{14r+7} = {}^3K_{(2r+1) \times 7} \cup \bigcup_{i=1}^{2r+1} {}^3K_7$, and the result follows from the existence of G_2 -decompositions of ${}^3K_{(2r+1) \times 7}$ and 3K_7 . ■

Lemma 3.4. *There exists a G_1 -decomposition of 7K_n if $n \geq 4$.*

Proof. Let $n = 4r + t$ for some positive integer r and $t \in \{0, 1, 2, 3\}$. If (r, t) is $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, or $(2, 3)$, then the result follows from Examples 2.17, 2.18, 2.19, 2.20, 2.21, and 2.22, respectively. If (r, t)

is $(1, 3)$ or $(2, 0)$, then $n \equiv 0$ or $1 \pmod{7}$, and the result follows from overlaying 2 copies of a G_1 -decomposition of 2K_n (see Lemma 3.1) and 1 copy of a G_1 -decomposition of 3K_n (see Lemma 3.2). For the remainder of the proof, we assume $r \geq 3$. By Corollary 1.6 there exists a K_3 -decomposition either of $K_{r \times 2}$ if $r \not\equiv 2 \pmod{3}$ or of $K_{4, (r-2) \times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3 \times 2}$ -decomposition of either $K_{r \times 4}$ or $K_{8, (r-2) \times 4}$. Since there exists a G_1 -decomposition of ${}^7K_{3 \times 2}$ (by Example 2.23), a G_1 -decomposition of either ${}^7K_{r \times 4}$ or ${}^7K_{8, (r-2) \times 4}$ also exists by transitivity. Finally, we note that K_{4r+t} can be described as either $(rK_4 \vee K_t) \cup K_{r \times 4} = K_{r \times 4} \cup K_{4+t} \cup \bigcup_{i=1}^{r-1} (K_{4+t} \setminus K_t)$ or $((K_8 \cup (r-2)K_4) \vee K_t) \cup K_{8, (r-2) \times 4} = K_{8, (r-2) \times 4} \cup K_{8+t} \cup \bigcup_{i=1}^{r-2} (K_{4+t} \setminus K_t)$. Thus, we describe ${}^7K_{4r+t}$ as ${}^7K_{r \times 4} \cup {}^7K_{4+t} \cup \bigcup_{i=1}^{r-1} ({}^7K_{4+t} \setminus {}^7K_t)$ when $r \not\equiv 2 \pmod{3}$ and as ${}^7K_{8, (r-2) \times 4} \cup {}^7K_{8+t} \cup \bigcup_{i=1}^{r-2} ({}^7K_{4+t} \setminus {}^7K_t)$ when $r \equiv 2 \pmod{3}$, and the result follows from the existence of G_1 -decompositions of ${}^7K_{r \times 4}$ or ${}^7K_{8, (r-2) \times 4}$, 7K_4 , 7K_5 , 7K_6 , 7K_7 , 7K_8 , 7K_9 , ${}^7K_{10}$, ${}^7K_{11}$, ${}^7K_6 \setminus {}^7K_2$, and ${}^7K_7 \setminus {}^7K_3$, where the latter two decompositions are shown to exist in Examples 2.24 and 2.25, respectively. \blacksquare

Lemma 3.5. *There exists a G_2 -decomposition of 7K_n if $n \geq 5$ and n is odd.*

Proof. Let $n = 4r + t$ for some positive integer r and $t \in \{1, 3\}$. If (r, t) is $(1, 1)$, $(2, 1)$, or $(2, 3)$, then the result follows from Examples 2.18, 2.20, and 2.22, respectively. If (r, t) is $(1, 3)$, then $n = 7$, and the result follows from overlaying 2 copies of a G_2 -decomposition of 2K_7 (see Lemma 3.1) and 1 copy of a G_2 -decomposition of 3K_7 (see Lemma 3.3). For the remainder of the proof, we assume $r \geq 3$, and the proof then follows as in the proof of Lemma 3.4, where the requisite G_2 -decompositions of ${}^7K_{3 \times 2}$ and ${}^7K_7 \setminus {}^7K_3$ can also be found in Examples 2.23 and 2.25, respectively. \blacksquare

Lemma 3.6. *There exists a G_2 -decomposition of ${}^{14}K_n$ if $n \geq 4$.*

Proof. If n is odd, then the result follows from overlaying 2 copies of a G_2 -decomposition of 7K_n (see Lemma 3.5). For the remainder of the proof, we assume n is even. Let $n = 4r + t$ for some positive integer r and $t \in \{0, 2\}$. If (r, t) is $(1, 0)$, $(1, 2)$, or $(2, 2)$, then the result follows from Examples 2.26, 2.27, and 2.28, respectively. If (r, t) is $(2, 0)$, then $n = 8$, and the result follows from overlaying 7 copies of a G_2 -decomposition of 2K_n (see Lemma 3.1). For the remainder of the proof, we assume $r \geq 3$. By Corollary 1.6 there exists a K_3 -decomposition either of $K_{r \times 2}$ if $r \not\equiv 2 \pmod{3}$ or of $K_{4, (r-2) \times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3 \times 2}$ -decomposition of either $K_{r \times 4}$ or $K_{8, (r-2) \times 4}$. Since there exists a G_2 -decomposition of ${}^7K_{3 \times 2}$ (by Example 2.23), a G_2 -decomposition of either ${}^{14}K_{r \times 4}$ or ${}^{14}K_{8, (r-2) \times 4}$ also exists by transitivity. Finally, we note that

K_{4r+t} can be described as either $(rK_4 \vee K_t) \cup K_{r \times 4} = K_{r \times 4} \cup K_{4+t} \cup \bigcup_{i=1}^{r-1} (K_{4+t} \setminus K_t)$ or $((K_8 \cup (r-2)K_4) \vee K_t) \cup K_{8, (r-2) \times 4} = K_{8, (r-2) \times 4} \cup K_{8+t} \cup \bigcup_{i=1}^{r-2} (K_{4+t} \setminus K_t)$. Thus, we describe ${}^{14}K_{4r+t}$ as ${}^{14}K_{r \times 4} \cup {}^{14}K_{4+t} \cup \bigcup_{i=1}^{r-1} ({}^{14}K_{4+t} \setminus {}^{14}K_t)$ when $r \not\equiv 2 \pmod{3}$ and as ${}^{14}K_{8, (r-2) \times 4} \cup {}^{14}K_{8+t} \cup \bigcup_{i=1}^{r-2} ({}^{14}K_{4+t} \setminus {}^{14}K_t)$ when $r \equiv 2 \pmod{3}$, and the result follows from the existence of G_2 -decompositions of ${}^{14}K_{r \times 4}$ or ${}^{14}K_{8, (r-2) \times 4}$, ${}^{14}K_4$, ${}^{14}K_6$, ${}^{14}K_8$, ${}^{14}K_{10}$, and ${}^{14}K_6 \setminus {}^{14}K_2$, where the latter decompositions is shown to exist in Example 2.29. ■

Finally, we settle the spectra for G_1 - and G_2 -designs of any index λ (at least 2).

Theorem 3.7. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_1 -decomposition of ${}^\lambda K_n$ if and only if the following hold:*

- if $\gcd(\lambda, 7) = 1$, then $n \equiv 0$ or $1 \pmod{7}$;
- if $\gcd(\lambda, 7) = 7$, then $n \geq 4$.

Proof. The necessity of the given conditions is established in Lemma 1.1. We now show sufficiency. Let $n \geq 4$ and let $\lambda = 7r + t$ for some integers $r \geq 0$ and $t \in \{2, 3, \dots, 8\}$. In the case where $t = 7$, the result follows from overlaying $r + 1$ copies of a G_1 -decomposition of 7K_n (see Lemma 3.4). For the remainder of the proof, we assume $n \equiv 0$ or $1 \pmod{7}$. In the case where t is even, the result follows from overlaying r copies of a G_1 -decomposition of 7K_n (see Lemma 3.4) and $t/2$ copies of a G_1 -decomposition of 2K_n (see Lemma 3.1). In the case where t is odd, the result follows from overlaying r copies of a G_1 -decomposition of 7K_n (see Lemma 3.4), 1 copy of a G_1 -decomposition of 3K_n (see Lemma 3.2), and $(t-3)/2$ copies of a G_1 -decomposition of 2K_n (see Lemma 3.1). ■

Theorem 3.8. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_2 -decomposition of ${}^\lambda K_n$ if and only if the following hold:*

- if $\gcd(\lambda, 14) = 1$, then $n \equiv 1$ or $7 \pmod{14}$;
- if $\gcd(\lambda, 14) = 2$, then $n \equiv 0$ or $1 \pmod{7}$;
- if $\gcd(\lambda, 14) = 7$, then $n \equiv 1 \pmod{2}$;
- if $\gcd(\lambda, 14) = 14$, then $n \geq 4$.

Proof. The necessity of the given conditions is established in Lemma 1.2. We now show sufficiency. Let $n \geq 4$ and let $\lambda = 14r + t$ for some integers $r \geq 0$ and $t \in \{2, 3, \dots, 15\}$. In the case where $t = 14$, the result follows from overlaying $r + 1$ copies of a G_2 -decomposition of ${}^{14}K_n$ (see Lemma 3.6). In the case where $t = 7$, we assume that n is odd, and the result follows from overlaying $2r + 1$ copies of a G_2 -decomposition of 7K_n

(see Lemma 3.5). For the remainder of the proof, we assume $n \equiv 0$ or $1 \pmod{7}$. In the case where t is even, the result follows from overlaying r copies of a G_2 -decomposition of 14K_n (see Lemma 3.6) and $t/2$ copies of a G_2 -decomposition of 2K_n (see Lemma 3.1). In the case where t is odd, we assume that n is odd, and the result follows from overlaying r copies of a G_2 -decomposition of 14K_n (see Lemma 3.6), 1 copy of a G_2 -decomposition of 3K_n (see Lemma 3.3), and $(t-3)/2$ copies of a G_2 -decomposition of 2K_n (see Lemma 3.1). ■

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