A survey on Roman domination parameters in directed graphs

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Abstract

Unlike undirected graphs where the concept of Roman domination has been studied very extensively over the past 15 years, Roman domination remains little studied in digraphs. However, the published works are quite varied and deal with different aspects of Roman domination, including for example, Roman bondage, Roman reinforcement, Roman dominating family of functions and the signed version of some Roman dominating functions. In this survey, we will explore some Roman domination related results on digraphs, some of which are extensions of well-known properties of the Roman domation parameters of undirected graphs.

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1 Introduction

Throughout this survey, a digraph $D = (V, A)$ consists of finite vertex set $V = V(D)$ and an arc set $A = A(D) \subseteq P$, where P is the set of all ordered pairs of distinct vertices of V. Thus D has neither loops nor multiple arcs, but it may contain pairs of opposite arcs. If $A = P$, then the digraph is *complete*. A digraph without directed cycles of length 2 is an *oriented graph*. The order $n = n(D)$ of a digraph D is the number of its vertices. If (u, v) is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* (or a *successor*) of u and u is an *in-neighbor* (or a *predecessor*) of v.

For a vertex v, the sets $N^-(v) = N_D^-(v) = \{u \mid (u,v) \in A\}$ and $N^+(v) = N_D^+(v) = \{u \mid (v,u) \in A\}$ are called the *inset* and *outset* of the vertex v. We also let the sets $N^-[v] = N^-(v) \cup \{v\}$ and $N^+[v] = N^+(v) \cup \{v\}$. The *indegree* of a vertex v is $d^-(v) = d_D^-(v) = |N^-(v)|$ and the *outdegree* of v is $d^+(v) = d_D^+(v) = |N^+(v)|$. The *minimum indegree, maximum indegree, minimum outdegree* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If $W = x_1, x_2, \ldots, x_n$ is a sequence of vertices such that every x_{i+1} is an out-neighbor of x_i, then W is a *directed walk* from x_1 to x_n of length $n-1$. If all of x_i's are different, then W is a *directed path*. Moreover, if $x_1 = x_n$, then W is a *directed cycle* (*circuit*). If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. The *complement* of a digraph $D = (V, A)$ is the digraph $\overline{D} = (V, P - A)$. For an integer $k > 0$, a digraph is *k-out-regular* if $d^+(x) = k$ for every vertex $x \in V$. A *tournament* of order n, denoted by T_n, is a complete oriented graph. A *cycle factor* C of a digraph D is a collection of t directed cycles $C_1, C_2, \ldots, C_t$ that are pairwise vertex-disjoint and $\bigcup_{i=1}^{k} V(C_i) = V(D)$. A *rooted tree* is a connected digraph with one vertex of indegree 0, called the root, and each of the remaining vertices has indegree 1. A digraph D is *contrafunctional* if each vertex of D has indegree one. In [19], Harary, Norman and Cartwright have shown that every connected contrafunctional digraph has a unique directed cycle and the removal of any arc of the directed cycle results in a rooted tree. The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N_{D(G)}^-[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1.1 *Let $D(G)$ be the associated digraph of a graph G. If $\mu(G)$ is a graph parameter and $\mu(D(G))$ is its corresponding digraph parameter, then mostly $\mu(D(G)) = \mu(G)$.*

A subset S of vertices of D is a *dominating set* if for all $v \notin S$, v is a successor of some vertex $s \in S$, that is $N^+[S] = V$. The *domination* *number* $\gamma(D)$ is the minimum cardinality of a dominating set of D. In terms of application, the questions of Facility Location and Assignment Problems etc. are related to the idea of domination or independent domination on digraphs. The domination number of a digraph was introduced by Fu [14] and has been studied by several authors [24, 25]. In his Ph.D. dissertation [24], Lee has surveyed some of the bounds on the domination number of undirected graphs and proved corresponding ones for digraphs.

2 Roman domination in digraphs

A *Roman dominating function* (RDF) on a digraph $D = (V, A)$ is defined by Kamaraj and Hemalatha [22] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ has an in-neighbor u for which $f(u) = 2$. If f is an RDF on a digraph D, then let V_i be the set of vertices assigned the value i under f for $i \in \{0, 1, 2\}$. We will write in this case $f = (V_0, V_1, V_2)$. The *Roman domination number* $\gamma_R(D)$ of D equals the minimum weight of an RDF on D. Roman domination in digraphs has been studied in [18, 30] and elsewhere. As for undirected graphs, it was observed in [22] that

$$\gamma(D) \leq \gamma_R(D) \leq \min\{2\gamma(D), n(D)\}.$$

It is worth noting that the bound $\gamma_R(D) \leq n(D)$ can be reached for connected digraphs of order at least three which makes a difference with undirected graphs. To see this, consider the digraph $K_{1,n}$ ($n \geq 2$) whose arcs are directed from the leaves towards the central vertex.

The first general bound on the Roman domination number of a digraph is given by Kamaraj et al. [22] which is a straightforward extension of a similar bound established for undirected graphs.

Proposition 2.1 ([22]) For any digraph D with n vertices, $\gamma_R(D) \leq n - \Delta^+(D) + 1$.

As an immediate consequence of Proposition 2.1, any digraph D satisfies $\gamma_R(D) < n$ if and only if $\Delta^+(D) \geq 2$. Note that there are many digraphs D including directed paths and directed cycles, having equal Roman domination number and order of D. Ouldrabah et al. [27] showed that the problem of deciding whether an oriented graph D satisfies $\gamma_R(D) = n - \Delta^+(D) + 1$ is *co*-$\mathcal{NP}$-complete. Moreover, they characterized the out-regular oriented graphs and tournaments attaining equality in Proposition 2.1. The *rotational tournament* RQT_7 is a 3-out-regular-tournament with vertex set $\{0, 1, 2, 3, 4, 5, 6\}$ and $i \to j$ if $j - i \equiv 1, 2$ or $4 \pmod 7$. Let $\mathcal{F}_1$ be the family of all 1-out-regular oriented graphs D such that every component of

D has only one circuit of order at least 3 and let $\mathcal{F}_2$ be the family of all 2-out-regular oriented graphs D for which the maximum outdegree of the subgraph induced by $V - N^+[x]$ is at most one for every $x \in V$.

Theorem 2.2 ([27]) *Let D be a k-out-regular oriented graph of order n, where $k \geq 1$. Then $\gamma_R(D) = n - \Delta^+(D) + 1$ if and only if $D \in \mathcal{F}_1 \cup \mathcal{F}_2$ or $D = RQT_7$.*

Theorem 2.3 ([27]) *Let T_n be a tournament of order $n \geq 2$. Then we have $\gamma_R(T_n) = n - \Delta^+(T_n) + 1$ if and only if $\Delta^+(T_n) \geq n-3$, or $\Delta^+(T_n) = n - 4$ and $N^+_{T_n}[u] \cup N^+_{T_n}[v] \neq V$ for all pairs of vertices $u, v \in V$.*

Since $d^+_D(v) + d^+_{\overline{D}}(v) = n - 1$ for any vertex $v \in V(D)$, the following Nordhaus-Gaddum type result for the Roman domination number of a digraph given in [18] immediately follows from Proposition 2.1.

Proposition 2.4 ([18]) *Let D be a digraph with n vertices. Then $\gamma_R(D) + \gamma_R(\overline{D}) \leq n + 3$.*

Ouldrabah et al. [27] gave a descriptive characterization of digraphs D of order $n \geq 3$ attaining equality in the upper bound of Proposition 2.4.

Theorem 2.5 ([27]) *If D is a digraph of order $n \geq 3$, then $\gamma_R(D) + \gamma_R(\overline{D}) = n+3$ if and only if D is a k-out-regular digraph with $1 \leq k \leq n-2$ and the maximum outdegrees of the subgraphs induced by $V - N^+_D[x]$ and $V - N^+_{\overline{D}}[x]$ are at most one for every $x \in V$.*

In addition, they showed that for every digraph D of order $n \geq 3$, $\gamma_R(D) + \gamma_R(\overline{D}) \geq 4$, with equality if and only if both D and $\overline{D}$ have a vertex of outdegree $n - 1$.

Digraphs D with $\gamma_R(D) = \gamma(D) + k$ for any integer k such that $1 \leq k \leq \gamma(D)$ have been characterized in [18, 30] as follows. Note that these characterizations are straightforward extension to similar results for undirected graphs.

Theorem 2.6 ([18, 30]) *Let D be a digraph of order n. Then:*

1. *$\gamma(D) = \gamma_R(D)$ if and only if $\Delta^+(D) = 0$.*

2. *if $n \geq 2$ and $\delta^-(D) \geq 1$, then $\gamma_R(D) = \gamma(D) + 1$ if and only if there is a vertex $v \in V(D)$ with $d^+(v) = n - \gamma(D)$.*

3. *if $n \geq 4$, $\delta^-(D) \geq 1$ and k is an integer with $2 \leq k \leq \gamma(D)$, then $\gamma_R(D) = \gamma(D) + k$ if and only if:*

(i) for any integer s with $1 \leq s \leq k-1$, D does not have a set U_t of t ($1 \leq t \leq s$) vertices satisfying $|N^+[U_t]| = n - \gamma(D) - s + 2t$,

(ii) there exists an integer l with $1 \leq l \leq k$ such that D has a set W_l of l vertices satisfying $|N^+[W_l]| = n - \gamma(D) - k + 2l$.

A characterization of digraphs with small Roman domination has been given in [30]. More precisely, digraphs D with $\gamma_R(D) = k$ for $k \in \{2, 3, 4, 5\}$. However, Hao, Xie and Chen [18] extended these results to arbitrary positive integer k by proving the following.

Theorem 2.7 ([18]) *For any positive integer k and a digraph D of order $n \geq k$, $\gamma_R(D) = k$ if and only if one of the following conditions holds:*

1. *$n = k$ and $\Delta^+(D) \leq 1$,*

2. *for any proper subset $X \subset V(D)$ with $1 \leq |X| \leq \lfloor k/2 \rfloor$, $N^+[X]| \leq n + 2|X| - k$. In addition, there exists some proper subset $Y \subset V(D)$ with $1 \leq |Y| \leq \lfloor k/2 \rfloor$ such that $N^+[Y]| = n + 2|Y| - k$ and $\Delta^+(D[V(D) - N^+[Y]]) \leq 1$.*

Other bounds similar to those established for undirected graphs were obtained for digraphs.

Theorem 2.8 ([30]) *For any digraph D on n vertices,*

$$\gamma_R(D) \leq n \left(\frac{2 + \ln \frac{1+\delta^-(D)}{2}}{1 + \delta^-(D)} \right).$$

Theorem 2.9 ([17]) *Let D be a digraph of order n and maximum outdegree $\Delta^+ \geq 1$. Then*

$$\gamma_R(D) \geq \left\lceil \frac{2n}{1 + \Delta^+} \right\rceil.$$

We call a set $S \subseteq V(D)$ a *2-packing* of the digraph D if $N^-[u] \cap N^-[v] = \emptyset$ for any two distinct vertices of $u, v \in S$. The maximum cardinality of a 2-packing is the *2-packing number* of D, denoted by $\rho_2(D)$. The parameters γ_R and ρ_2 are related as follows.

Proposition 2.10 ([17]) *For any digraph D of order n with $\delta^+ \geq 1$,*

$$\gamma_R(D) \leq n - (\delta^+ - 1)\rho_2(D).$$

For a positive integer k, a function $f : V(D) \longrightarrow \{-1,1\}$ is called in [15] a *signed k-dominating function* on a digraph D if $f(N^-[v]) = \sum_{x\in N^-[v]} f(x) \geq k$ for every $v \in V(D)$. The *signed k-domination number* $\gamma_{ks}(D)$ of D is the minimum weight of a signed k-dominating function on D. The special case $k = 1$ was introduced by Zelinka [44] and has been studied by several authors [23, 32] and elsewhere. An upper bound on the Roman domination number of a digraph D in terms of its order and signed domination number $\gamma_s(D) = \gamma_{1s}(D)$ is given in [17].

Proposition 2.11 ([17]) *For any digraph D of order n,*

$$\gamma_R(D) \leq \gamma_s(D)/2 + 5n/6.$$

2.1 Roman Bondage and Roman reinforcement in digraphs

2.1.1 Roman bondage

Roman bondage for digraphs was studied by Dehgardi, Meierling, Sheikholeslami and Volkmann [12]. The *Roman bondage number* $b_R(D)$ of a digraph D with maximum outdegree at least two is the minimum cardinality of all sets $A' \subset A$ for which $\gamma_R(D - A') > \gamma_R(D)$. Dehgardi et al. determined the Roman bondage number in several classes of digraphs.

Theorem 2.12 ([12]) *1) If D is a digraph of order $n \geq 3$ with exactly $k \geq 1$ vertices of outdegree $n-1$, then $b_R(D) = k$.*
2) If $n \geq 3$, then $b_R(K_n^*) = n$, where K_n^* is the complete digraph of order n.
3) Let $K^*_{n_1,n_2,\ldots,n_p}$ be the complete p-partite digraph such that $p \geq 2 \geq n_1$, $n_1 \leq n_2 \leq \cdots \leq n_p$ and $n = \sum_{i=1}^{p} n_i \geq 3$. Then

$$b_R(K^*_{n_1,n_2,\ldots,n_p}) = \begin{cases} i & \text{if } n_i = 1 < n_{i+1}, \\ 2i & \text{if } n_i = 2 < n_{i+1}. \end{cases}$$

4) Let $K^*_{m,n}$ be the complete bipartite digraph such that $n \geq m \geq 2$ and $m + n \geq 5$. Then

$$b_R(K^*_{m,n}) = \begin{cases} 2 & \text{if } m = 2, \\ m+2 & \text{if } m \geq 3. \end{cases}$$

(5) For $k \geq 2$, let C_n^k be the k-th power of the directed cycle of length $n \geq 2k+1$. If n is a multiple of $k+1$, then $b_R(C_n^k) = k+1$.

Dehgardi et al. also gave some sharp bounds for the Roman bondage number of a digraph. The *underlying graph* $G[D]$ of a digraph D is the graph obtained from D by replacing each arc (u, v) by an edge uv. Note that $G[D]$ has two parallel edges uv when D contains the arcs (u, v) and (v, u).

Theorem 2.13 ([12]) *1) If D is a digraph, and xyz a path of length 2 in $G[D]$ such that $(y, x), (y, z) \in A(D)$, then $b_R(D) \leq \deg_{G[D]}(x) + \deg_{G[D]}(y) + \deg_{G[D]}(z) - 3 - |N^-(x) \cap N^-(y)|$.*
Moreover, if x and z are adjacent in $G[D]$, then $b_R(D) \leq \deg_{G[D]}(x) + \deg_{G[D]}(y) + \deg_{G[D]}(z) - 4 - |N^-(x) \cap N^-(y)|$.
2) If D is a digraph, and xyz a path of length 2 in the graph $G[D]$ such that $(y, x), (y, z) \in A(D)$, then

$$b_R(D) \leq \deg_{G[D]}(x) + \deg_D^-(y) + \deg_{G[D]}(z) - |N^-(x) \cap N^-(y) \cap N^-(z)|.$$

Assume that $\delta^+(D) \geq 2$, and let $y \in V(D)$ be an arbitrary vertex. Then there exist two different vertices $x, z \in N^+(y)$. Thus $G[D]$ contains a path xyz such that $(y, x), (y, z) \in A(D)$ for each vertex $y \in D$. Applying Theorem 2.13 (2) for a vertex $y \in V(D)$ with $\deg^-(y) = \delta^-(D)$, it follows immediately that $b_R(D) \leq 2\Delta(G[D]) + \delta^-(D)$ for any digraph D with $\delta^+(D) \geq 2$. Moreover, since always $\delta^-(D) \leq \frac{1}{2}\Delta(G[D])$, we find that $b_R(D) \leq \frac{5}{2}\Delta(G[D])$ for any digraph D with $\delta^+(D) \geq 2$.

The next bounds involve the maximum degree and Roman domination number.

Theorem 2.14 ([12]) *1) Let D be a digraph of order $n \geq 4$ with $\delta^+(D) \geq 2$ and Roman domination number $\gamma_R(D) \geq 3$. Then $b_R(D) \leq (\gamma_R(D) - 2)\Delta(G[D]) + 1$.*
2) Let D be a vertex-transitive digraph of order n. Then $b_R(D) \geq \lceil \frac{2n}{\gamma_R(D)} \rceil$.

2.1.2 Roman reinforcement number of digraphs

In [21], Huang, Wang and Xu defined the *reinforcement number* $r(D)$ of a digraph D as the minimum number of arcs that must be added to D in order to decrease the domination number. Analogously, Dehgardi, Meierling, Sheikholeslami and Volkmann [11] introduced the *Roman reinforcement number* $r_R(D)$ of a digraph D as the minimum number of arcs that must be added to D in order to decrease the Roman domination number. Obviously, if $\gamma_R(D) \in \{1, 2\}$, then addition of arcs does not reduce the Roman domination number, and thus $r_R(D) = 0$ as defined in [11]. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, we find that $\gamma_R(G) = \gamma_R(D(G))$, $r(G) = r(D(G))$ and $r_R(G) = r_R(D(G))$ for any graph G.

Theorem 2.15 ([11]) *If D is a digraph with $\gamma_R(D) = 2\gamma(D) \geq 4$, then $r(D) = r_R(D) + 1$.*

Proposition 2.16 ([11]) *If D is a digraph of order n with $\gamma_R(D) \geq 3$, then $r_R(D) \leq n - \Delta^+(D) - \gamma_R(D) + 2$. This bound is sharp.*

The following result provides a characterization of digraphs D of order $n \geq 3$ and $\Delta^+(D) \geq 1$ such that $r_R(D) = 1$.

Theorem 2.17 ([11]) *Let D be a digraph of order $n \geq 3$ and $\Delta^+(D) \geq 1$. Then $r_R(D) = 1$ if and only if there is a $\gamma_R(D)$-function $f = (V_0, V_1, V_2)$ with $V_1 \neq \emptyset$.*

As a consequence of Theorem 2.17, for a digraph D of order $n \geq 4$ with $\gamma_R(D) = 4$ and $r_R(D) \geq 2$, we obtain $r_R(D) = n - \Delta^+(D) - 2$. Furthermore, if D is a digraph of order $n \geq 3$ and $\Delta^+(D) \geq 1$ such that $\gamma_R(D)$ is odd, then $r_R(D) = 1$.

Dehgardi et al. [11] transferred an idea from [21] to digraphs and presented an upper bound for the Roman reinforcement number. Any minimum Roman dominating function $f = (V_0, V_1, V_2)$ on a digraph D such that $|V_2|$ is maximum will be called a *nice $\gamma_R(D)$-function*. Let S be a subset of vertices of a digraph D with $|S| \geq 2$. Assume that $\eta(S) = \max\{|N^+[X]|\colon X \subseteq S, |X| = |S| - 1\}$ and define

$$\eta(D) = \max\{\eta(V_1 \cup V_2)\colon f = (V_0, V_1, V_2) \text{ is a nice } \gamma_R(D)\text{-function}\}.$$

It is clear that $\eta(V_1 \cup V_2) \leq n - 1$ for any nice $\gamma_R(D)$-function and hence $\eta(D) \leq n - 1$.

Theorem 2.18 ([11]) *Let D be a nonempty digraph of order $n \geq 3$ with $\gamma_R(D) \geq 3$. Then $r_R(D) \leq n - \eta(D)$.*

If D is a digraph and S a subset of $V(D)$, then let

$$\rho(S) = \min\{|PN(x, S)| : x \in S\}.$$

The *private neighborhood number* of D is defined by

$$\rho(D) = \min\{\rho(V_2)\colon f = (V_0, V_1, V_2) \text{ is a nice } \gamma_R(D)\text{-function}\}.$$

Note that if $A(D) \neq \emptyset$, then it is clear that $\rho(D) \geq 2$.

Theorem 2.19 ([11]) *If D is a digraph of order $n \geq 3$ with $\Delta^+(D) \geq 1$, then $r_R(D) \leq \rho(D) - 1$.*

Corollary 2.20 *1) If D is a digraph of order $n \geq 3$ with $\Delta^+(D) \geq 1$, then $r_R(D) \leq \frac{2n}{\gamma_R(D)} - 1$.*
2) For any digraph D of order $n \geq 3$ and $\Delta^+(D) \geq 1$, $r_R(D) \leq \Delta^+(D)$. Moreover, the bound is sharp for any digraph D with $\Delta^+(D) = 1$.
3) If D is a digraph of order $n \geq 3$ with $\Delta^+(D) \geq 1$, then

$$r_R(D) \leq \frac{n - \gamma_R(D) + 2}{2}.$$

Dehgardi et al. then studied Roman reinforcement number in compositions of digraphs. For two undirected graphs G and H, the *join* $G + H$ is defined as the undirected graph consisting of G and H with each vertex of G adjacent to every vertex of H. In the directed case, there are two possibilities to define the join of two digraphs. Let G and H be digraphs. The digraph $G \to H$ is obtained from G and H by adding all possible arcs from vertices of G to vertices of H, and $G \leftrightarrow H$ is be obtained from $G \to H$ by adding all possible arcs from vertices of H to vertices of G.

Proposition 2.21 ([11]) *Let G and H be two digraphs with $\Delta^+(G) \geq 1$ and $\Delta^+(H) \geq 1$. Then*

1. $r_R(G \to H) = r_R(G)$,
2. $r_R(G \leftrightarrow H) = \min\{n(G) - \Delta^+(G) - 2, n(H) - \Delta^+(H) - 2\}$.

The *corona* GoH of two undirected graphs G and H is formed from one copy of G and $n(G)$ copies of H by joining v_i to every vertex in H_i, where v_i is the ith vertex of G and H_i is the ith copy of H. For digraphs G and H, if all the additional edges are from G to H_i, then we denote the resulting digraph by $G\vec{o}H$.

Proposition 2.22 ([11]) *Let G and H be two digraphs with $n(H) \geq 2$. Then*

$$r_R(G\vec{o}H) = \begin{cases} 0 & \text{if } n(G) = 1, \\ n(H) & \text{if } G \text{ is the empty graph and } n(G) \geq 2, \\ n(H) - 1 & \text{otherwise.} \end{cases}$$

2.2 Roman domatic number in digraphs

A set $\{f_1, f_2, \ldots, f_d\}$ of Roman dominating functions on a digraph D with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(D)$, is called a *Roman dominating family* (of functions) on D. The maximum number of functions in a Roman dominating family on G is the *Roman domatic number* of D, denoted by $d_R(D)$. Clearly $d_R(D) \geq 1$. Hao et al. [17] studied the properties of Roman domatic number in digraphs.

Theorem 2.23 ([17]) *For any digraph D, $d_R(D) = 1$ if and only if D has no directed even cycle.*

Theorem 2.24 ([17]) *For any digraph D of order n, $\gamma_R(D) \cdot d_R(D) \leq 2n$.*

As a consequence of Theorem 2.24, we have the following result.

Theorem 2.25 ([17]) *For any digraph D of order $n \geq 2$, $d_R(D) \leq n$ with equality if and only if D is a complete digraph.*

Using Theorems 2.23, 2.24 and 2.25, the following upper bound on the sum $\gamma_R(D) + d_R(D)$ can be given.

Theorem 2.26 ([17]) *Let D be a digraph D of order $n \geq 2$. Then*

$$\gamma_R(D) + d_R(D) \leq n + 2$$

with equality if and only if D is a complete digraph, or $\Delta^+(D) \leq 1$ and D has a directed even cycle.

Since $\gamma_R(\overrightarrow{C}_n) = n$, it follows from Theorem 2.24 that $d_R(\overrightarrow{C}_n) = 2$ for even n.

Theorem 2.27 ([17]) *For any digraph D, we have $d_R(D) \leq \delta^-(D) + 2$.*

The reader can find an example in [17] showing the sharpness of Theorem 2.27.

Theorem 2.28 ([17]) *If D is a k-out-regular digraph of order n, where $n = p(k+1) + r$ with integers $p \geq 1$ and $0 \leq r \leq k$, then*

$$d_R(D) \leq k + \epsilon,$$

where $\epsilon = 1$ when $k = 0$, or $r = 0$, or $2r = k + 1$, and $\epsilon = 0$ otherwise.

Using Theorems 2.27 and 2.28, the following Nordhaus-Gaddum type result can be given.

Theorem 2.29 ([17]) *If D is a digraph of order $n \geq 2$, then*

$$d_R(D) + d_R(\overline{D}) \leq n + \epsilon$$

where $\epsilon = 1$ when D is out regular, $\epsilon = 2$ when D is not in-regular and $\epsilon = 3$ otherwise.

The following extension of Theorem 2.27 can be found in a note of Volkmann and Meierling [43].

Theorem 2.30 ([43]) *For any digraph D, we have $d_R(D) \le \delta^-(D) + 2$. Moreover, if $d_R(D) = \delta^-(D) + 2$, then the set of vertices of minimum in-degree is an independent set.*

Theorem 2.30 leads to the following improvement of Theorem 2.29.

Theorem 2.31 ([43]) *If D is a digraph of order n, then*

$$d_R(D) + d_R(\overline{D}) \le n + 1.$$

Using Observation 1.1 and Theorem 2.31, we obtain the following Nordhaus-Gaddum bound for graphs.

Corollary 2.32 ([43]) *If G is a graph of order n, then*

$$d_R(G) + d_R(\overline{G}) \le n + 1.$$

In [29] Corollary 2.32 was only proved for regular graphs.

2.3 {2}-Roman domination in digraphs

An {2}-*Roman dominating function* (or *Italian dominating function*) on a digraph D with vertex set $V(D)$ is defined by Volkmann [38] as a function $f : V(D) \to \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. The minimum weight of an {2}-Roman dominating function f is the {2}-*Roman domination number* (or Italian domination number), denoted by $\gamma_I(D)$. Clearly, every Roman dominating function is an {2}-Roman dominating function of D and thus $\gamma_I(D) \le \gamma_R(D)$. Hence, any upper bound on $\gamma_R(D)$ yields an upper bound on $\gamma_I(D)$, and thus Proposition 2.4 and Theorem 2.8 are true for the {2}-Roman domination number. Furthermore, since the set $V_1 \cup V_2$ in an {2}-Roman dominating function $f = (V_0, V_1, V_2)$ is a dominating set of D, we observe that $\gamma(D) \le \gamma_I(D)$. Altogether, we obtain the following inequality chain.

$$\gamma(D) \le \gamma_I(D) \le \gamma_R(D) \le \min\{n(D), 2\gamma(D)\}. \qquad (1)$$

It is observed in [38] that $\gamma_I(D) = n(D)$ if and only if

$$\max\{\Delta^+(D), \Delta^-(D)\} \le 1.$$

In [2], the *2-rainbow dominating function* of a digraph D is defined as a function f from $V(D)$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V(D)$ with $f(v) = \emptyset$ the condition $\bigcup_{x \in N^-(v)} f(x) = \{1, 2\}$ is fulfilled. The *weight* of a 2-rainbow dominating function f is the value $\sum_{v \in V(D)} |f(v)|$. The *2-rainbow domination number* $\gamma_{r2}(D)$ is the minimum weight of a 2-rainbow dominating function of D. The {2}-Roman domination and 2-rainbow domination numbers are related as follows.

Theorem 2.33 ([38]) *If D is a digraph of order $n \geq 2$, then $\gamma_I(D) \leq \gamma_{r2}(D) \leq 2\gamma_I(D) - 2$.*

The digraphs D with $\gamma_R(D) = k$ for $k \in \{2, 3\}$ are characterized in [38] as follows.

Proposition 2.34 ([38]) Let D be a digraph of order $n \geq 2$. Then $\gamma_I(D) = 2$ if and only if $\Delta^+(D) = n - 1$ or there exist two different vertices u and v such that $N^+(u) \cap N^+(v) = V(D) \setminus \{u, v\}$.

Proposition 2.35 ([38]) Let D be a digraph of order $n \geq 3$ such that $\Delta^+(D) \leq n - 2$ and there doesn't exist two different vertices a and b such that $N^+(a) \cap N^+(b) = V(D) \setminus \{a, b\}$. Then $\gamma_I(D) = 3$ if and only if $\Delta^+(D) = n - 2$ or there exist three pairwise different vertices u, v and w such that each vertex $x \in V(D) \setminus \{u, v, w\}$ has at least two in-neighbors in the set $\{u, v, w\}$.

The next lower bound is an extension of Theorem 11 in [10].

Theorem 2.36 ([38]) If D is a digraph of order n, then

$$\gamma_I(D) \geq \left\lceil \frac{2n}{\Delta^+(D) + 2} \right\rceil .$$

2.4 Double Roman domination and double $\{2\}$-Roman domination in digraphs

In [16], Hao, Chen and Volkmann continue the study of double Roman domination for directed graphs. A *double Roman dominating function* (abbreviated DRDF) on a digraph D is defined in [16] as a function $f : V(D) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3, while if $f(v) = 1$, then the vertex v must have at least one in-neighbor assigned 2 or 3. The *double Roman domination number* $\gamma_{dR}(D)$ of a digraph D is the minimum weight of a DRDF on D. Double Roman domination numbers of paths and cycles are determined by Ouldrabah, Blidia, Bouchou and Volkmann in [28].

Proposition 2.37 ([28]) *For any positive integer $n \geq 3$,*

$$\gamma_{dR}(\overrightarrow{P}_n) = \gamma_{dR}(\overrightarrow{C}_n) = \left\lceil \frac{3n}{2} \right\rceil .$$

The double Roman domination and (Roman) domination of a digraph are related as follows.

Proposition 2.38 ([16]) *Let D be a digraph. Then*

1. $2\gamma(D) \leq \gamma_{dR}(D) \leq 3\gamma(D)$. *Moreover, the right equality holds if and only if there exists a $\gamma_{dR}(D)$-function f such that $V_1 = V_2 = \emptyset$ and the left equality holds if and only if $\gamma(D) = \gamma_2(D)$, where $\gamma_2(D)$ is 2-domination number of D.*

2. $\gamma_R(D)+1 \leq \gamma_{dR}(D) \leq 2\gamma_R(D)$, *with equality in right side if and only if D is empty.*

Since $\gamma_R(D) \leq n$ for any digraph D of order n, the following corollary follows from Proposition 2.38 (Item 2).

Corollary 2.39 ([16]) *For any digraph D of order n, $\gamma_{dR}(D) \leq 2n$ with equality if and only if D is empty.*

As a consequence of Propositions 2.1 and 2.38, the next result is proved in [16].

Proposition 2.40 ([16]) For any non-empty digraph D with n vertices,

$$\gamma_{dR}(D) \leq 2(n - \Delta^+(D)) + 1. \tag{2}$$

In [28], the authors give a descriptive characterization for out-regular digraphs and tournaments satisfying (2) with equality as follows.

Theorem 2.41 ([28]) *Let D be a k-out-regular digraph of order n with $k \geq 1$. Then $\gamma_{dR}(D) = 2(n-k)+1$ if and only if $D = K^*_{k+1}$ or $D = K^*_{k+2} - A(C)$, where C is a cycle factor without a cycle of order 2.*

Theorem 2.42 ([28]) *Let T_n be a tournament of order n with maximum out-degree $\Delta^+ \geq 1$. Then $\gamma_{dR}(T_n) = 2(n-\Delta^+)+1$ if and only if $\Delta^+ \geq n-2$.*

Using the bound (2), Hao et al. proved the following Nordhaus-Gaddum type result, and recently Ouldrabah et al. [28] give a descriptive characterization of extremal digraphs D of order $n \geq 4$. From a directed cycle $\overrightarrow{C_3}$, we define two 1-out-regular digraphs denoted by $\overrightarrow{H_1}$ and $\overrightarrow{H_2}$ as follows: The digraph $\overrightarrow{H_1}$ is obtained from $\overrightarrow{C_3}$ by adding a vertex, say v, and join it by an arc from v to a vertex of $\overrightarrow{C_3}$. The digraph $\overrightarrow{H_2}$ is obtained from $\overrightarrow{H_1}$ by adding a further vertex, say u, and join it by an arc from u to v.

Theorem 2.43 ([28]) For any digraph D with $n \geq 4$ vertices,

$$6 \leq \gamma_{dR}(D) + \gamma_{dR}(\overline{D}) \leq 2n+3.$$

Furthermore, for the lower bound equality holds if and only if both D and $\overline{D}$ have a vertex of out-degree $n-1$ and for the upper bound equality holds if and only if D or $\overline{D}$ is an element of $\left\{K^*_n, \overrightarrow{C_4}, \overrightarrow{C_5}, 2\overrightarrow{C_3}, \overrightarrow{H_1}, \overrightarrow{H_2}\right\}$.

An upper bound on the double Roman domination number of a digraph D in terms of its order and signed domination number was given in [16].

Proposition 2.44 ([16]) *If D is a digraph of order n, then $\gamma_{dR}(D) \leq \gamma_s(D) + 4n/3$.*

Other bounds for the double Roman domination number of a digraph D can be stated in terms of its order, minimum indegree and maximum outdegree.

Proposition 2.45 ([16]) *For every connected digraph D of order $n \geq 4$ with $\delta^-(D) \geq 1$, $\gamma_{dR}(D) \leq (5n-1)/3$.*

Proposition 2.46 ([16]) *For any digraph D of order n with $\delta^-(D) \geq 1$,*

$$\gamma_{dR}(D) \leq n\left\{3 - 3\left(\frac{3}{2(1+\delta^-(D))}\right)^{\frac{1}{\delta^-(D)}} + 2\left(\frac{3}{2(1+\delta^-(D))}\right)^{\frac{1+\delta^-(D)}{\delta^-(D)}}\right\}.$$

Theorem 2.47 ([16]) *For any connected digraph D of order $n \geq 4$,*

$$\gamma_{dR}(D) \geq \left\lceil \frac{6n+3}{2\Delta^+ + 3} \right\rceil.$$

The next result, derived from Theorem 2.47, shows that the upper bound in (2) and the lower bound in Theorem 2.47 are sharp.

Corollary 2.48 ([16]) *Let D be a connected digraph of order $n \geq 4$. Then $\gamma_{dR}(D) = 3$ if and only if $\Delta^+(D) = n-1$.*

An improvement of the bound in Theorem 2.47 has been given by Volkmann in [37] when $\Delta^+(D) \geq 2$.

Theorem 2.49 ([37]) *If D is a digraph of order n with $\Delta^+(D) \geq 2$, then*

$$\gamma_{dR}(D) \geq \left\lceil \frac{3n}{\Delta^+(D)+1} \right\rceil.$$

A *double $\{2\}$-Roman dominating function* (or double Italian dominating function) on a digraph D is defined by Volkmann [37] as a function $f : V(D) \to \{0,1,2,3\}$ such that each vertex $u \in V(D)$ with $f(u) \in \{0,1\}$ has the property that $\sum_{x \in N^-[u]} f(x) \geq 3$. The minimum weight of a double $\{2\}$-Roman dominating function f is the *double $\{2\}$-Roman domination number* or *double Italian domination number*, denoted by $\gamma_{dI}(D)$. Clearly, any double Roman dominating function on a digraph D is a double $\{2\}$-Roman dominating function on D and so $\gamma_{dI}(D) \leq \gamma_{dR}(D)$. Thus any upper bound on $\gamma_{dR}(D)$ yields an upper bound on $\gamma_{dI}(D)$. It was observed in [37] that $\gamma_I(D) + 1 \leq \gamma_{dI}(D) \leq 2\gamma_I(D)$. An inequality chain similar to that given in Proposition 2.38 (item 1) holds for the double {2}-Roman domination number, namely, $\gamma(D) + 2 \leq \gamma_{dI}(D) \leq 3\gamma(D)$ with equality in the upper bound if and only if there exists a $\gamma_{dI}(D)$-function $f = (V_0, V_1, V_2, V_3)$ with $|V_1| = |V_2| = 0$.

The following bounds are established on the double {2}-Roman domination number in [37].

Proposition 2.50 **([37])** *If D is a digraph of order n, then $\gamma_{dI}(D) \leq 2n$ with equality if and only if D is empty.*

Proposition 2.51 **([37])** *If D is a bipartite digraph of order n with minimum indegree $\delta^-(D) \geq 1$, then $\gamma_{dI}(D) \leq \gamma_{dR}(D) \leq 3n/2$.*

If C_n^* is an oriented cycle of even length, then $\gamma_{dI}(C_n^*) = \gamma_{dR}(C_n^*) = 3n/2$, and therefore Proposition 2.51 is sharp. If C_n^* is an oriented cycle of odd length, then $\gamma_{dI}C_n^*) = \gamma_{dR}(C_n^*) = (3(n-1))/2+2$, and thus Proposition 2.51 is not valid in general.

Theorem 2.52 **([37])** *If D is a digraph with $\delta^-(D) \geq 2$, then*

$$\gamma_{dI}(D) \leq |V(D)| + 2 - \delta^-(D).$$

The complete digraph K_n^* $(n \geq 3)$, complete bipartite digraphs $K_{3,3}^*$ and $K_{4,4}^*$ demonstrate that Theorem 2.52 is sharp.

Theorem 2.53 **([37])** *If D is a digraph of order n with maximum outdegree $\Delta^+(D) = \Delta^+$, then*

$$\gamma_{dI}(D) \geq \min\left\{\frac{2n + 2\Delta^+ + 6}{\Delta^+ + 2}, \frac{2n + \Delta^+}{\Delta^+ + 1}\right\}.$$

Corollary 2.54 **([37])** *Let D be a digraph of order $n \geq 2$. Then $\gamma_{dI}(D) = 3$ if and only if $\Delta^+(D) = n - 1$.*

2.5 Twin Roman domination in digraphs

In [8], Chartrand, Dankelmann, Schultz and Swart defined a *twin dominating set* of a digraph D as a subset $S \subseteq V(D)$ such that for all $v \notin S$, v is a predecessor of some vertex $s \in S$ and v is a successor of some vertex $t \in S$, that is $N^+[S] = N^-[S] = V$. The *twin domination number* $\gamma^*(D)$ is the minimum cardinality of a twin dominating set of D. The twin domination number of a digraph has been also studied in [4, 5, 26].

Following the idea of Chartrand et al., Abdollahzadeh Ahangar et al. [1] introduced the concept of twin Roman domination in digraphs. A *twin Roman dominating function* (TRDF) on D is a Roman dominating function of D such that every vertex with label 0 has an out-neighbor with label 2. The *twin Roman domination number* of a digraph D, denoted by $\gamma_R^*(D)$, equals the minimum weight of a TRDF on D. Since the set of all vertices assigned weights 1 or 2 is a twin dominating set when f is a TRDF, and since placing weight 1 at the vertices of the digraph or weight 2 at the vertices of a twin dominating set yields a TRDF, we have

$$\gamma^*(D) \le \gamma_R^*(D) \le \min\{n(D), 2\gamma^*(D)\}.$$

Obviously $\gamma_R^*(D) = n(D)$ if and only if there exists a $\gamma_R^*(D)$-function assigning weight 1 to all vertices of D. The next result provides a condition for a digraph D to have a twin Roman domination number less than $n(D)$.

Proposition 2.55 **([1])** *Let D be a digraph of order n, maximum out-degree Δ^+ and maximum in-degree Δ^-. If $\Delta^+ + \Delta^- \ge n+3$, then $\gamma_R^*(D) < n$.*

Abdollahzadeh Ahangar et al. [1] have showed that the twin domination and the twin Roman domination numbers of a digraph D are equal if and only if every vertex of D has indegree 0 or outdegree 0. Moreover, they proved the existence of a lower bound in terms of the order or twin domination number of a digraph.

Proposition 2.56 **([1])** If D is a digraph on n vertices, then $\gamma_R^*(D) \ge \min\{n, \gamma^*(D) + 1\}$.

Further, they characterized the digraphs such that their twin Roman domination number equals $\gamma^*(D) + 1$ or $\gamma^*(D) + 2$.

Proposition 2.57 **([1])** *Let D be a digraph of order n. Then the following statements hold.*

1. *If $n \ne 3$ with $\delta(D) \ge 1$, then $\gamma_R^*(D) = \gamma^*(D) + 1$ if and only if there is a vertex $v \in V(D)$ with $|N^+(v) \cap N^-(v)| = n - \gamma^*(D)$.*
2. *If $n \ge 7$ and $\delta(D) \ge 1$, then $\gamma_R^*(D) = \gamma^*(D) + 2$ if and only if:*
   - *(i) D does not have a vertex v with with $|N^+(v) \cap N^-(v)| = n - \gamma^*(D)$.*
   - *(ii) either D has a vertex v with with $|N^+(v) \cap N^-(v)| = n - \gamma^*(D) - 1$ or D contains two vertices v, w such that $|(N^+[v] \cup N^+[w]) \cap (N^-[v] \cup N^-[w])| = n - \gamma^*(D) + 2$.*

Digraphs with twin Roman domination number k where $k \in \{2, 3, 4, 5\}$ were also characterized in [1] as follows.

Proposition 2.58 ([1]) Let D be a digraph of order n.

(i) For $n \geq 2$, $\gamma_R^*(D) = 2$ if and only if $n = 2$ or there is a vertex v with $d^+(v) = d^-(v) = n - 1$.

(ii) For $n \geq 3$, $\gamma_R^*(D) = 3$ if and only if D has no vertex v with $d^+(v) = d^-(v) = n - 1$. In addition (a) $n = 3$ or (b) D has a vertex v with $|N^+(v) \cap N^-(v)| = n - 2$.

(iii) For $n \geq 4$, $\gamma_R^*(D) = 4$ if and only if $|N^+(v) \cap N^-(v)| \leq n - 3$ for any vertex $v \in V(D)$. In addition, (a) $n = 4$ or (b) there is a vertex v with $|N^+(v) \cap N^-(v)| = n - 3$ or (c) there are two vertices $u, v \in V(D)$ such that $(N_D^+(u) \cup N_D^+(v)) \cap (N_D^-(u) \cap N_D^-(v)) = V(D) - \{u, v\}$.

(iv) For $n \geq 5$, $\gamma_R^*(D) = 5$ if and only if $|N^+(v) \cap N^-(v)| \leq n - 4$ for any vertex $v \in V(D)$ and $|(N_D^+(x) \cup N_D^+(y)) \cap (N_D^-(x) \cup N_D^-(y))| \leq n-3$ for all pairs of vertices $x, y \in V(D)$. In addition, (a) there are two vertices $u, v \in V(D)$ such that $|(N_D^+(u) \cup N_D^+(v)) \cap (N_D^-(u) \cup N_D^-(v))| = n-3$ or (b) $n = 5$ or (c) D contains a vertex w with $|N^+(w) \cap N^-(w)| = n-4$ and the induced subdigraph $H = D[V(D) - (N^+[w] \cap N^-[w])]$ does not contain a vertex x with $|N_H^+(x) \cap N_H^-(x)| = 2$.

We close this subsection by the following result which is a consequence of the definition and Theorem 2.9.

Theorem 2.59 ([1]) *Let D be a digraph of order n, maximum outdegree $\Delta^+ \geq 1$ and maximum indegree Δ^-. Then*

$$\gamma_R^*(D) \geq \max\left\{\left\lceil \frac{2n}{\Delta^+ + 1} \right\rceil, \left\lceil \frac{2n}{\Delta^- + 1} \right\rceil\right\}.$$

The complete digraph K_n^* $(n \geq 2)$ and complete bipartite digraph $K_{n,n}^*$ $(n \geq 4)$ are examples that show that the bound of Theorem 2.59 is sharp.

2.6 Roman game domination

The study of the Roman game domination number was initiated by Bahremandpour, Sheikholeslami and Volkmann in 2016 [6]. Roman game domination is a game on a simple graph G consisting of two players $\mathcal{D}$ and $\mathcal{A}$ called *Dominator* and *Avoider*, respectively, who take turns choosing an edge from G. In this game, each chosen edge must be oriented and the game stops when all the edges of the graph G have been oriented, thus giving a directed graph D. Knowing that the Dominator starts the game first, his goal is to decrease the Roman domination number of the digraph D, while the Avoider tries to increase it. The *Roman game domination number* of the graph G, denoted by $\gamma_{Rg}(G)$, is the Roman domination number of the directed graph resulting from this game, that is $\gamma_{Rg}(G) = \gamma_R(D)$. This is well defined if we suppose that both players follow their optimal strategies.

Here is one of the main results obtained in [6] which gives an upper bound on the Roman game domination number for any graph with no isolated vertex.

Proposition 2.60 ([6]) *Let G be a graph without isolated vertices and let (D, A, B) be a partition of $V(G)$ such that D is a dominating set of G and each vertex in A is adjacent to at least two vertices of D. Then* $\gamma_{Rg}(G) \leq 2|D| + \left\lfloor \frac{|B|}{2} \right\rfloor$.

An immediate consequence of Proposition 2.60 is that if D is a minimum 2-dominating set of G, then by putting $B = \emptyset$, we have $\gamma_{Rg}(G) \leq 2\gamma_2(G)$. Additional bounds on the Roman game domination number were obtained in [6]. Indeed, it was shown that if G is a graph of order n with maximum degree Δ, then $\gamma_{Rg}(G) \leq n - \left\lceil \frac{\Delta}{2} \right\rceil + 1$, while if G is a tree of order n, then $\left\lceil \frac{n}{2} \right\rceil + 1 \leq \gamma_{Rg}(G) \leq n - 1$. Moreover, it was proved that for a connected graph G of order $n \geq 4$, $\gamma_{Rg}(G) \geq 3$ while if $n \geq 6$, then $\gamma_{Rg}(G) \geq 4$. For paths P_n of order $n \geq 4$, it has been shown that $\gamma_{Rg}(P_n) \leq n - 1 - \left\lfloor \frac{n-4}{5} \right\rfloor$, and this bound is sharp for $n \in \{4, \ldots, 13\}$. Bahremandpour et al. conjectured that for $n \geq 4$, $\gamma_{Rg}(P_n) = \gamma_{Rg}(C_n) = n - 1 - \left\lfloor \frac{n-4}{5} \right\rfloor$. Another interesting question posed in [6] which remains open: Is it true that if $\gamma_{Rg}(G) \leq 4k$, then for any two non-adjacent vertices a and b, $\gamma_{Rg}(G + ab) \leq 5k$?

3 Signed Roman domination in digraphs

The purpose of this section is to present the signed version of two variations of Roman dominating functions in digraphs.

3.1 Signed Roman k-domination in digraphs

Let $k \geq 1$ be an integer. A *signed Roman k-dominating function* (abbreviated SRkDF) on a digraph D is defined by Volkmann in [33] as a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq k$ for every $v \in V(D)$ and every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The *signed Roman k-domination number* $\gamma^k_{sR}(D)$ of D is the minimum weight of an SRkDF on D. The special case $k = 1$ was introduced and investigated by Sheikholeslami and Volkmann in [31], where $\gamma_{sR}^1(D)$ is denoted by $\gamma_{sR}(D)$. The signed Roman k-domination number exists when $\delta^-(D) \geq \frac{k}{2} - 1$. The function assigning $+1$ to every vertex of D is an SRkD function of weight n and thus $\gamma_{sR}^k(D) \leq n$ for every digraph of order n with $\delta^-(D) \geq k - 1$. Moreover, it was observed in [31] that $\gamma_{sR}(G) = n$ if and only if D is the disjoint union of isolated vertices and oriented triangles C_3.

In the following, we list some results that are obtained from Observation 1.1 with other results on the signed Roman k-domination established in [20] for special classes of undirected graphs.

Proposition 3.1 *1. If $n \neq 3$, then $\gamma_{sR}(K_n^*) = 1$ and $\gamma_{sR}(K_3^*) = 2$.*

2. If $n \geq k \geq 2$, then $\gamma_{sR}^k(K_n^*) = k$.

3. If $k \geq 2$, then $\gamma_{sR}^k(K_{k-1,k-1}^*) = 2k - 2$ and $\gamma_{sR}^k(K_{k,k}^*) = 2k$ and if $k \geq 1$, then $\gamma_{sR}^k(K_{k+1,k+1}^*) = 2k + 1$.

We gather below the few exact values of the signed Roman k-domination number of digraphs that have been determined in [15, 31, 33].

Proposition 3.2 *1. If $AT(n)$ for $n \geq 3$ is an acyclic tournament, then $\gamma_{sR}(AT(n)) = 1$ for $n \neq 3$, and $\gamma_{sR}(AT(3)) = 2$.*

2. If $n = 2r + 1$, where r is a positive integer, then $\gamma_{sR}(\mathrm{CT}(n)) = 3$ for $r \neq 2$, and $\gamma_{sR}(\mathrm{CT}(5)) = 4$.

3. If C_n is an oriented cycle of order $n \geq 2$, then $\gamma_{sR}(C_n) = n/2$ when n is even and $\gamma_{sR}(C_n) = (n+3)/2$ when n is odd.

4. If P_n is an oriented path of order n, then $\gamma_{sR}(P_n) = n/2$ when n is even and $\gamma_{sR}(P_n) = (n+1)/2$ when n is odd.

5. For any positive integers p, q and k with $q \geq p \geq k+2$, $\gamma_{sR}^k(K_{p,q}^*) = 2k + 2$.

6. $\gamma_{sR}(K_{1,2}^*) = 2$, $\gamma_{sR}(K_{1,q}^*) = 1$ for $q \neq 2$, $\gamma_{sR}(K_{2,q}^*) = 3$ for $q \geq 2$ and $\gamma_{sR}(K_{p,q}^*) = 4$ for $q \geq p \geq 3$.

7. If H is a $(k-1)$-regular digraph of order n, then $\gamma_{sR}^k(H) = n$.

The next two bounds on the signed Roman k-domination number are extensions of those obtained in [20] for undirected graphs.

Proposition 3.3 ([33]) If D is a digraph of order n with $\delta^-(D) \geq k - 1$, then

$$\gamma_{sR}^k(D) \geq k + 1 + \Delta^-(D) - n.$$

Item 2 in Proposition 3.1 shows that the bound of Proposition 3.3 is sharp.

Theorem 3.4 ([33]) Let D be a digraph of order n with $\delta^-(D) \geq k-1$, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^+ < \Delta^+$, then

$$\gamma_{sR}^k(D) \geq \left(\frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+ + 3}\right) n.$$

Examples 9 and 10 in [20] together with Observation 1.1 shows that the bound presented in Theorem 3.4 is sharp.

Proposition 3.5 ([33]) If D is an r-out-regular digraph of order n with $r \geq k-1$, then $\gamma_{sR}^k(D) \geq kn/(r+1)$.

Item 7 in Proposition 3.2 shows that the bound of Proposition 3.5 is sharp. A Nordhaus-Gaddum type inequality for the signed Roman k-domination number of regular digraphs can be derived from Proposition 3.5 as follows.

Theorem 3.6 ([33]) *If D is an r-regular digraph of order n such that $r \geq k-1$ and $n-r-1 \geq k-1$, then*

$$\gamma_{sR}^k(D) + \gamma_{sR}^k(\overline{D}) \geq \frac{4kn}{n+1}.$$

If n is even, then $\gamma_{sR}^k(D) + \gamma_{sR}^k(\overline{D}) \geq 4k(n+1)/(n+2)$.

That the bound in Theorem 3.6 is sharp, may be seen by considering a $(k-1)$-regular digraph H of order $n = 2k-1$. Then $\overline{H}$ is a $(k-1)$-regular digraph too and in view of Proposition 3.2 (Item 7), we have $\gamma_{sR}^k(H) + \gamma_{sR}^k(\overline{H}) = 2n$ which leads to $\gamma_{sR}^k(H) + \gamma_{sR}^k(\overline{H}) = 2n = \frac{4kn}{n+1}$.
The parameters γ_{sR} and γ, the parameters γ_{sR}^k and γ_{ks} and the parameters γ_{sR}^2 and γ_2 are related as follows.

Proposition 3.7 ([31]) *Let D be a digraph of order n. Then $\gamma_{sR}(D) \geq 2\gamma(D) - n$, with equality if and only if D is the disjoint union of isolated vertices.*

Proposition 3.8 ([15]) *If D is a digraph of order n and k a positive integer k with $\delta^-(D) \geq k-1$, then*

$$\gamma_{sR}^k(D) \leq \gamma_{ks}(D) + \frac{n}{3}.$$

The special case $k=1$ of Proposition 3.8 is proved in [31]. To illustrate the sharpness of Proposition 3.8 in the case $k=1$, consider the digraph D with vertex set $V(D) = \{u_i, v_i \mid 1 \leq i \leq 3\}$ and arc set $A(D) = \{(v_i, v_j) \mid 1 \leq i \neq j \leq 3\} \cup \{(v_1, u_1), (v_1, u_2), (v_2, u_1), (v_2, u_3), (v_3, u_2), (v_3, u_3)\}$. It is easy to see that $\gamma_s(D) = 0$ and $\gamma_{sR}(D) = 2$.

Proposition 3.9 **([15])** *For any digraph D of order n with $\Delta^- \geq 2$,*

$$\gamma_{sR}^2(D) \geq 2\gamma_2(D) + 1 - n.$$

For rooted trees and connected contrafunctional digraphs the following results were established in [15, 33].

Theorem 3.10 **([33])** *If T is a rooted tree of order $n \geq 1$, then*

$$\gamma_{sR}(T) \leq \frac{n+1}{2}.$$

Directed paths demonstrate that Theorem 3.10 is sharp.

Theorem 3.11 **([15])** *For any rooted tree T of order n, $\gamma_{sR}^2(T) = n + 1$.*

Theorem 3.12 **([33])** *If D is a connected contrafunctional digraph order $n \geq 2$, then*

$$\gamma_{sR}(D) \leq \frac{n+3}{2}.$$

The directed cycles C_n of odd length show that Theorem 3.12 is sharp. The next result improves the bound in Theorem 3.12 for special families of contrafunctional digraphs.

Theorem 3.13 **([33])** *Let D be a connected contrafunctional digraph of order $n \geq 2$ with the unique directed cycle C. If C has even length or the maximum distance from C to $V(D) - V(C)$ is exactly one, then $\gamma_{sR}(D) \leq n/2$.*

Theorem 3.14 **([15])** *Let D be a connected contrafunctional digraph of order n. Then*

(*a*) $\gamma_{sR}^2(D) = n$.

(*b*) $n + k/2 \leq \gamma_{sR}^3(D) \leq (3n+1)/2$, *where k is the length of the unique directed cycle of D. In particular, if k is even, then $\gamma_{sR}^3(D) \leq 3n/2$.*

(*c*) $\gamma_{sR}^4(D) = 2n$.

In [15], Hao, Chen and Volkmann showed that for any oriented tree T of order n, $\gamma_{sR}^2(T) \geq (n+3)/2$, and characterized all oriented trees attaining this lower bound.

For an integer $k \geq 1$, Volkmann [39, 40] recently defined the *weak signed Roman k-dominating function* (WSRkDF) on a digraph D as a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the condition $f(N^-[v]) \geq k$ for each $v \in$

$V(D)$. The weight of a WSRkDF f is the value $\sum_{u\in V(D)} f(u)$. The *weak signed Roman k-domination number* $\gamma^k_{wsR}(D)$ of D is the minimum weight of a WSRkDF on D. The weak signed Roman k-domination number exists when $\delta^-(D) \geq \frac{k}{2}-1$. The definitions lead to $\gamma^k_{wsR}(D) \leq \gamma^k_{sR}(D)$. Therefore each lower bound of $\gamma^k_{wsR}(D)$ is a lower bound of $\gamma^k_{sR}(D)$. In [39, 40] it is shown that many lower bounds of $\gamma^k_{sR}(D)$ are also valid for $\gamma^k_{wsR}(D)$. In particular, Volkmann [39, 40] proved that Propositions 3.3, 3.5, 3.7 and Theorems 3.4, 3.6 hold for the weak signed Roman k-domination number too. In addition, Volkmann [40] presented the following general bounds.

Theorem 3.15 ([40]) *Let D be a digraph of order n with $\delta^-(D) \geq \lceil\frac{k}{2}\rceil-1$. Then $\gamma^k_{wsR}(D) \leq 2n$, with equality if and only if k is even, $\delta^-(D) = \frac{k}{2}-1$, and each vertex of D is of minimum in-degree or has an out-neighbor of minimum in-dgree.*

Theorem 3.16 ([40]) *Let $k \geq 3$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq \lceil\frac{k}{2}\rceil - 1$. Then $\gamma^k_{wsR}(D) \geq k + \lceil\frac{k}{2}\rceil - n$, with equality if and only if $D = K^*_{\lceil\frac{k}{2}\rceil}$.*

3.2 Signed total Roman k-domination in digraphs

A *signed total Roman k-dominating function* (STRkDF) on a digraph D is defined by Dehgardi and Volkmann [13] as a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x\in N^-(v)} f(x) \geq k$ for each $v \in V(D)$, and (ii) every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The *signed total Roman k-domination number* $\gamma^k_{stR}(D)$ of D is the minimum weight of an STRkDF on D. The special case $k = 1$ was introduced and investigated by Volkmann [35], where $\gamma^1_{stR}(D)$ is denoted by $\gamma_{stR}(D)$. The signed total Roman k-domination number exists when $\delta^-(D) \geq \frac{k}{2}$. As for the signed Roman k-domination, one can see that $\gamma^k_{stR}(D) \leq n$ for every digraph of order n with $\delta^-(D) \geq k$.

The following results are obtained by using Observation 1.1 with some results established for the signed total Roman k-domination for special classes of undirected graphs (see for example [34, 36]).

Proposition 3.17 ([13]) *If $n \geq k+2$, then $\gamma^k_{stR}(K^*_n) = k+2$.*

Proposition 3.18 ([13]) *If $p \geq k \geq 1$, then $\gamma^k_{stR}(K^*_{p,p}) = 2k$, with exception of the case that $k = 1$ and $p = 3$, in which case $\gamma^1_{stR}(K^*_{3,3}) = 4$.*

We recall that $D(C_n)$ and $D(P_n)$ are the associated digraphs of the cycle C_n and the path P_n, respectively.

Proposition 3.19 ([35]) *Let $n \geq 3$ be an integer. Then $\gamma_{stR}(D(C_n)) = \frac{n}{2}$ when $n \equiv 0 \pmod 4$, $\gamma_{stR}(D(C_n)) = \frac{n+3}{2}$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stR}(D(C_n)) = \frac{n+6}{2}$ when $n \equiv 2 \pmod 4$.*

Proposition 3.20 ([35]) *Let $n \geq 3$ be an integer. Then $\gamma_{stR}(D(P_n)) = \frac{n}{2}$ when $n \equiv 0 \pmod 4$ and $\gamma_{stR}(D(P_n)) = \lceil \frac{n+3}{2} \rceil$ otherwise.*

The signed total Roman k-domination number was determined for circulant tournaments in [13, 35] and for regular digraphs in [13].

Proposition 3.21

1. *If $n = 2r + 1$, where r is a positive integer, then $\gamma_{stR}(CT(3)) = 3$, $\gamma_{stR}(CT(7)) = 5$ and $\gamma_{stR}(CT(n)) = 4$ for $n \geq 5$ with $n \neq 7$.*
2. *If $n = 2r + 1$ with an integer $r \geq k \geq 2$, then $\gamma^k_{stR}(CT(n)) = n$ for $r = k$ and $\gamma^k_{stR}(CT(n)) = 2k + 2$ when $r > k$.*
3. *If D is a k-regular digraph of order n, then $\gamma^k_{stR}(D) = n$.*

Dehgardi and Volkmann [13] gave the following upper bound which is an extension of a bound that can be found in [34, 36].

Theorem 3.22 ([13]) *If D is a digraph of order n with minimum in-degree $\delta^- \geq k + 2$, then*

$$\gamma^k_{stR}(D) \leq n + 1 - 2\left\lfloor \frac{\delta^- - k}{2} \right\rfloor.$$

If $n \geq k+3$ and $n-k-1$ is even, then it follows from Proposition 3.17 that

$$\gamma^k_{stR}(K^*_n) = k + 2 = n + 1 - 2\left\lfloor \frac{\delta^-(K^*_n) - k}{2} \right\rfloor,$$

and therefore equality in the inequality of Theorem 3.22.

Proposition 3.23 ([13]) *If D is a digraph of order n with minimum in-degree $\delta^-(D) \geq k$, then*

$$\gamma^k_{stR}(D) \geq k + \Delta^-(D) - n.$$

Proposition 3.24 ([13]) If D is a digraph of order $n \geq k + 2$ with minimum in-degree $\delta^-(D) \geq k$, then

$$\gamma^k_{stR}(D) \geq k + 3 + \delta^-(D) - n.$$

Proposition 3.17 shows that Proposition 3.24 is sharp. Moreover, γ^k_{stR} and the 2-packing number ρ_2 are related as follows.

Theorem 3.25 (**[13]**) *If D is a digraph of order n such that $\delta^-(D) \geq k$, then*

$$\gamma_{stR}^k(D) \geq \rho_2(D)(\delta^-(D) + k) - n.$$

Theorem 3.26 (**[13]**) 1. *Let D be a digraph of order n with minimum indegree $\delta^- \geq k$, minimum outdegree δ^+ and maximum outdegree Δ^+. If $\Delta^+ > \delta^+$, then*

$$\gamma_{stR}^k(D) \geq \left(\frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+}\right) n.$$

2. *If D is an r-out-regular digraph of order n with $r \geq k$, then $\gamma_{stR}^k(D) \geq \frac{kn}{r}$.*

Using Theorem 3.26 (Item 2), we obtain the following Nordhaus-Gaddum type inequality for the signed total Roman k-domination number in digraphs.

Theorem 3.27 (**[13]**) *If D is an r-regular digraph of order n such that $r \geq k$ and $n - r - 1 \geq k$, then*

$$\gamma_{stR}^k(D) + \gamma_{stR}^k(\overline{D}) \geq \frac{4kn}{n-1}.$$

If n is even, then $\gamma_{stR}^k(D) + \gamma_{stR}^k(\overline{D}) \geq \frac{4k(n-1)}{n-2}$.

For an integer $k \geq 1$, Volkmann [41, 42] recently defined the *signed total Italian k-dominating function* (STIkDF) on a digraph D as a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions (i) $f(N^-(v)) \geq k$ for each $v \in V(D)$ and (ii) each vertex u for which $f(u) = -1$ has an in-neighbor v with $f(v) = 2$ or two in-neighbors w and z with $f(w) = f(z) = 1$. Note that in the case $k \geq 2$ or $\delta^-(D) \geq 2$, the second condition is superfluous. The *signed total Italian k-domination number* $\gamma_{stI}^k(D)$ of D is the minimum weight of an STkIDF on D. The signed total Italian k-domination number exists when $\delta^-(D) \geq \frac{k}{2}$. The definitions lead to $\gamma_{stI}^k(D) \leq \gamma_{stR}^k(D)$. Therefore each lower bound of $\gamma_{stI}^k(D)$ is a lower bound of $\gamma_{stR}^k(D)$. In [41, 42] Volkmann proved that Proposition 3.23, Proposition 3.24 when $\delta^-(D) - k$ is odd, Theorems 3.25, 3.26 and 3.27 also hold for the signed toal Italian k-domination number. In addition, Volkmann presented the following general bounds.

Theorem 3.28 (**[42]**) *Let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma_{stI}^k(D) \leq 2n$, with equality if and only if k is even, $\delta^-(D) = \frac{k}{2}$, and each vertex of D has an out-neighbor of minimum in-dgree.*

Theorem 3.29 ([42]) *Let D be a digraph of order n such that $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$, and let t be the number of vertices $x \in V(D)$ with $d^+(x) = 0$. Then Then $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq 2n - 3t$, and if $k \geq 3$ is odd, then Then $\gamma^k_{stI}(D) \leq \gamma^k_{stR}(D) \leq 2n - 3t - 1$.*

Theorem 3.30 ([42]) *Let $k \geq 2$ be an integer. If D is a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil$, then*

$$\gamma^k_{stI}(D) \geq k + 3 + \left\lceil \frac{k}{2} \right\rceil - n.$$

Examples in [42] demonstrate that Theorems 3.28 and 3.30 are both sharp.

4 Roman domination parameters in oriented graphs

An *orientation* of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for every edge $xy \in E(G)$. If $\mu(G)$ is a graph parameter and $\mu(D)$ is its corresponding digraph parameter, then clearly, two distinct orientations of a graph can have distinct μ values. Motivated by this observation, Chartrand, VanderJagt and Quan Yue defined in [9] the lower and upper orientable domination numbers of a graph as follows:

$$\text{dom}(G) = \min\{\gamma(D) \mid \text{ D is an orientation of } G\}, \text{ and}$$
$$\text{DOM}(G) = \max\{\gamma(D) \mid \text{ D is an orientation of } G\}.$$

The lower orientable twin domination number $\text{dom}^*(G)$ and upper orientable twin domination number $\text{DOM}^*(G)$ of a graph G have been defined in [8].

Similar concepts for Roman domination have been studied, and the main results will be presented in this subsection.

4.1 Orientable twin Roman domination

The study of lower and upper orientable twin Roman domination numbers of a graph G was initiated by Abdollazadeh Ahangar, Amjadi, Sheikholeslami, Samodivkin and L. Volkmann in [1], where the these two parameters were defined as follows:

$$\text{dom}^*_R(G) = \min\{\gamma^*_R(D) \mid \text{ D is an orientation of } G\}, \text{ and}$$

$\mathrm{Dom}^*_R(G) = \max\{\gamma^*_R(D) \mid \text{D is an orientation of } G\}.$

Trivially, $\mathrm{dom}^*_R(G) \leq \mathrm{Dom}^*_R(G) \leq n$ for every graph G of order n. The first result we cite provides a condition for a graph G to have $\mathrm{dom}^*_R(G) = n(G)$.

Proposition 4.1 ([1]) *If G is a graph of order n with at most one cycle, then $\mathrm{dom}^*_R(G) = n$. In particular, for $n \geq 1$, $\mathrm{dom}^*_R(K_{1,n}) = n$ and for $n \geq 2$, $\mathrm{dom}^*_R(C_n) = \mathrm{dom}^*_R(P_n) = \mathrm{Dom}^*_R(C_n) = \mathrm{Dom}^*_R(P_n) = n$.*

Exact values of the lower orientable twin Roman domination number of complete graphs, complete bipartite graphs and complete multipartite graphs have been also established in [1].

Proposition 4.2 ([1]) *For $n \geq 4$, we have $\mathrm{dom}^*_R(K_n) = 4$ and for $n \geq 2$, $\mathrm{dom}^*_R(K_{2,n}) = 4$.*

Proposition 4.3 ([1]) *For every two integers $r \geq s \geq 3$,*

$$\mathrm{dom}^*_R(K_{r,s}) = \begin{cases} 5 & \text{if} \quad s = 3 \\ 6 & \text{if} \quad s = 4 \\ 7 & \text{if} \quad s = 5 \\ 8 & \text{if} \quad s \geq 6. \end{cases}$$

Proposition 4.4 ([1]) *Let $G = K_{m_1,m_2,\ldots,m_r}$ $(r \geq 3)$ be the complete r-partite graph with $1 \leq m_1 \leq m_2 \leq \ldots \leq m_r$. Then*

$$\mathrm{dom}^*_R(K_{m_1,m_2,\ldots,m_r}) = \begin{cases} 4 & \text{if} \quad m_1 = \cdots = m_r = 1, \\ 4 & \text{if} \quad m_1 = m_2 = 1 \text{ or } m_i = 2 \text{ for some } i, \\ 5 & \text{if} \quad m_1 = 3 \text{ or } m_1 = 1 \text{ and } m_2 = 3, \\ 6 & \text{if} \quad m_1 \geq 4. \end{cases}$$

The next two upper bounds on the lower orientable twin Roman domination of a graph G are expressed in terms of the order, clique number and independence number of G.

Proposition 4.5 ([1]) *For any graph G of order $n \geq 4$ with clique number $c \geq 4$, $\mathrm{dom}^*_R(G) \leq n - c + 4$.*

Proposition 4.6 ([1]) *For any graph G of order $n \geq 4$ with $\delta(G) \geq 2$, $\mathrm{dom}^*_R(G) \leq 2(n - \alpha(G))$, where $\alpha(G)$ is the independence number of G.*

Proposition 4.2 shows that Proposition 4.5 is sharp for complete graphs K_n $(n \geq 4)$ and Proposition 4.6 is sharp for complete bipartite graphs $K_{2,n}$ $(n \geq 2)$.

4.2 Orientable signed Roman domination

The lower and upper orientable twin signed Roman domination numbers of a graph G were defined in [7] by Bodaghli, Sheikholeslami and Volkmann as follows:

$$\mathrm{dom}^*_{sR}(G) = \min\{\gamma^*_{sR}(D) \mid \text{D is an orientation of } G\}, \text{ and}$$
$$\mathrm{Dom}^*_{sR}(G) = \max\{\gamma^*_{sR}(D) \mid \text{D is an orientation of } G\}.$$

Clearly, $\mathrm{dom}^*_{sR}(G) \leq \mathrm{Dom}^*_{sR}(G) \leq n(G)$ for all graphs G. The next result shows that the right bound is sharp.

Proposition 4.7 **([7])** If G is a bipartite graph with n vertices, then $\mathrm{Dom}^*_{sR}(G) = n$.

Exact values of the lower orientable twin signed Roman domination number of complete graphs K_n and complete bipartite graphs $K_{m,n}$ ($n \geq m$) were determined in [7]. Indeed, for $n \geq 3$, $\mathrm{dom}^*_{sR}(K_n) = 3$ if $n \notin \{4, 6\}$, and $\mathrm{dom}^*_{sR}(K_n) = 4$ if $n \in \{4, 6\}$. Also, for $n \geq 2$, $\mathrm{dom}^*_{sR}(K_{2,n}) = 2$ if $n \neq 3$ and $\mathrm{dom}^*_{sR}(K_{2,3}) = 3$. For $n \geq 4$, $\mathrm{dom}^*_{sR}(K_{3,n}) = 4$, and $\mathrm{dom}^*_{sR}(K_{3,3}) = 5$. If $m = 4, 5$, then $\mathrm{dom}^*_{sR}(K_{m,n}) = m + 2$, and for $n \geq m \geq 6$, $\mathrm{dom}^*_{sR}(K_{m,n}) = 8$.

4.3 Orientable signed total Roman domination

Amjadi and Soroudi [3] introduced the lower and upper orientable twin signed total Roman domination numbers of a graph G defined as follows:

$$\mathrm{dom}^*_{stR}(G) = \min\{\gamma^*_{stR}(D) \mid$$
$$\text{D is an orientation of } G \text{ with } \min\{\delta^+(D), \delta^-(D)\} \geq 1\},$$
$$\mathrm{Dom}^*_{stR}(G) = \max\{\gamma^*_{stR}(D) \mid$$
$$\text{D is an orientation of } G \text{ with } \min\{\delta^+(D), \delta^-(D)\} \geq 1\}.$$

The authors determined the exact values of the lower orientable twin signed total Roman domination number for complete graphs and complete bipartite graphs as well as the upper orientable twin signed total Roman domination number of the complete bipartite graph $K_{n,n}$.

Proposition 4.8 **([3])** *For* $n \geq 4$ $\mathrm{dom}^*_{stR}(K_n) = 4$.

Proposition 4.9 **([3])** *Let* $n \geq m \geq 2$.

1. $\mathrm{dom}^*_{stR}(K_{2,2}) = 4$, $\mathrm{dom}^*_{stR}(K_{2,3}) = 5$ *and* $\mathrm{dom}^*_{stR}(K_{2,n}) = 6$ *for* $n \geq 4$.

2. $\mathrm{dom}^*_{stR}(K_{3,3}) = \mathrm{dom}^*_{stR}(K_{3,5}) = 6$ *and* $\mathrm{dom}^*_{stR}(K_{3,n}) = 5$ *for* $n \geq 4$ *and* $n \neq 5$.

3. $\mathrm{dom}^*_{stR}(K_{4,5}) = 5$ *and* $\mathrm{dom}^*_{stR}(K_{4,n}) = 4$ *for* $n \geq 4$ *and* $n \neq 5$.

4. $\mathrm{dom}^*_{stR}(K_{5,5}) = 6$ *and* $\mathrm{dom}^*_{sR}(K_{5,n}) = 5$ *for* $n \geq 6$.

5. *For* $n \geq 6$, $\mathrm{dom}^*_{stR}(K_{6,n}) = 4$.

6. *For* $n \geq m \geq 7$, $\mathrm{dom}^*_{stR}(K_{m,n}) = 4$.

Proposition 4.10 ([3]) $\mathrm{Dom}^*_{stR}(K_{n,n}) = 2n$.

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