Bijections between compositions over finite groups

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Abstract

One may generalize integer compositions by replacing positive integers with elements from an additive group, giving the broader concept of compositions over a group. In this note we give some simple bijections between compositions over a finite group. It follows from these bijections that the number of compositions of a nonzero group element g is independent of g. As a result we derive a simple expression for the number of compositions of any given group element. This extends an earlier result for abelian groups which was obtained using generating functions and a multivariate multisection formula.

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1 Main result

Throughout this note, G denotes a finite additive group with $r \geq 2$ elements, and 0 denotes the identity element of G. An m-composition of $g \in G$ (or compositions of g with m parts) is a sequence $g_1, g_2, \ldots, g_m$ of nonzero elements from G such that $g_1 + g_2 + \cdots + g_m = g$. Since G might not be abelian, the additions are performed from left to right. In the following, it is convenient to allow some parts of the sequence to be 0, and we shall call such sequences *weak compositions* over G. Let $c_m(g)$ be the number of m-compositions of g. Our main result is the following.

Theorem 1. *Let G be a finite group. Then*

$$C_m(g) = \begin{cases} \frac{1}{r}((r-1)^m - (-1)^m) & \text{if } g \neq 0, \\ \frac{1}{r}((r-1)^m + (-1)^m(r-1)) & \text{if } g = 0. \end{cases}$$

The above formula was first obtained by Wang and Muratović-Ribić [1] for finite fields. Gao, MacFie and Wang [2] extended it to all finite abelian groups using generating functions and a multisection formula.

Theorem 1 shows that $c_m(g) = c_m(h)$ for any two nonzero elements $g, h \in G$. This fact will be proved in the next section using simple bijections between m-compositions over G. Theorem 1 will follow immediately from these bijections.

2 Bijections

The following proposition is easy to verify.

Proposition 1 (Bijection ϕ). *For a weak composition $(g_1, ..., g_m)$, define $\phi(g_1, ..., g_m) = (h_1, ..., h_m)$ such that $h_j = g_1 + \cdots + g_j$. Then ϕ is a bijection between weak m-compositions over G, and its inverse is given by $g_1 = h_1$,*

$g_j = -h_{j-1} + h_j$ for all $2 \leq j \leq m$. Furthermore, $(g_1, ..., g_m)$ is a composition of g if and only if $h_m = g$, $h_1 \neq 0$, and $h_j \neq h_{j-1}$ for all $2 \leq j \leq m$.

Our next result gives bijections between m-compositions of different nonzero group elements.

Proposition 2 (Bijections ψ and μ). *Let ϕ be the bijection defined in Proposition 1.*

(a) For two distinct nonzero elements $a, b \in G$, the mapping $\psi_{a,b}$ which swaps a and b in a composition is a bijection between m-compositions.

(b) The mapping $\mu_{a,b} = \phi^{-1}\psi_{a,b}\phi$ is a bijection between m-compositions of a and m-compositions of b.

Proof. Consider an m-composition $(g_1, ..., g_m)$ of a. Then $(h_1, \ldots, h_m) := \phi(g_1, ..., g_m)$ satisfies $h_m = a$, $h_1 \neq 0$, and $h_j \neq h_{j-1}$ for all $2 \leq j \leq m$. Hence $(x_1, \ldots, x_m) := \psi_{a,b}(\phi(g_1, ..., g_m))$ is a weak composition which satisfies $x_m = b$, $x_1 \neq 0$, and and $x_j \neq x_{j-1}$ for all $2 \leq j \leq m$. It follows from Proposition 1 that $\phi^{-1}(x_1, \ldots, x_m)$ is an m-composition of b. □

Proof of Theorem 1 We first note from Proposition 2 that $c_m(a) = c_m(b)$ for any two nonzero elements $a, b \in G$.
Next we note that the total number of m-compositions over G is given by

$$c_m(0) + (r-1)c_m(g) = (r-1)^m, \tag{1}$$

where g is any fixed nonzero element of G.
Finally we note that

$$\begin{aligned} c_m(0) &= \#\{(x_1, \ldots, x_m) : x_1 + \cdots + x_{m-1} + x_m = 0,\ x_1, \ldots, x_m \neq 0\} \\ &= \#\{(x_1, \ldots, x_m) : x_1 + \cdots + x_{m-1} = -x_m,\ x_1, \ldots, x_m \neq 0\} \\ &= (r-1)c_{m-1}(g), \end{aligned}$$

which together with (1) completes the proof of Theorem 1. □

References

[1] A. Muratović-Ribić and Q. Wang, Partitions and compositions over finite fields, *Electron. J. Combin.* **20(1)** (2013), Paper 34, 14 pp.

[2] Z. Gao, A. Macfie and Q. Wang Counting Compositions Over Finite Abelian Groups, *Electron. J. Combin.* **25(2)** (2018), Paper 19, 23 pp.