HELLY-TYPE THEOREMS IN $\mathbb{R}^d$ FOR INFINITE INTERSECTIONS OF SETS STARSHAPED VIA STAIRCASE PATHS

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ABSTRACT. Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$. For each countable subfamily $\{K_m : m \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_m : m \geq 1\}$ is consistent relative to staircase paths and starshaped via staircase paths, with a staircase kernel that contains a convex set of dimension d. Then $\cap\{K : K \text{ in } \mathcal{K}\}$ has these properties as well.
For n fixed, $n \geq 1$, an analogous result holds for sets starshaped via staircase n-paths.

1. INTRODUCTION

We begin with a short introduction to the problem. Precise definitions for corresponding concepts appear in Section 2.

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that instead use the idea of visibility via staircase paths. For example, the familiar Kranosel'skii theorem [11] says that, for S a nonempty compact set in the plane, S is starshaped via segments if and only if every three points of S see via segments in S a common point. In the staircase analogue [5], for S a nonempty simply connected orthogonal polygon in the plane, S is staircase starshaped if and only if every two points of S see via staircase paths in S a common point. Moreover, in an interesting study involving median graphs as well as median polyhedra in the rectilinear space $\mathbb{R}^d$, Chepoi [7] has generalized the planar result to any finite union of boxes in $\mathbb{R}^d$ whose corresponding intersection graph is a tree. As he observes, every simply connected orthogonal polygon may be expressed as such a union. Appropriately, the staircase kernel of such a set will be staircase convex [4, Theorem 1].

Similarly, N. A. Bobylev [1] has established the following Helly-type theorem for starshaped sets: For $\mathcal{K}$ a family of compact sets in $\mathbb{R}^d$, if every

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$d + 1$ (not necessarity distinct) members of $\mathcal{K}$ have an intersection that is nonempty and starshaped (via segments), then the intersection of all the sets is nonempty and starshaped. In [3], a staircase analogue of Bobylev's result is obtained for a family of orthogonal polytopes in $\mathbb{R}^d$, imposing the requirement that our polytopes be finite unions of boxes whose intersection graphs are trees. For such a family $\mathcal{K}$, if every $d + 1$ (not necessarily distinct) members of $\mathcal{K}$ meet in a (nonempty) staircase starshaped set, then $S \equiv \cap\{K : K \text{ in } \mathcal{K}\}$ is nonempty and staircase starshaped.

Another familiar theorem, one by Victor Klee [10], establishes the following Helly-type theorem for countable intersections of convex sets : Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^d$. If every countable subfamily of $\mathcal{C}$ has nonempty intersection, then $\cap\{C : C \text{ in } \mathcal{C}\}$ is nonempty. Furthermore, results in [2] provide the following analogue of Klee's theorem for sets that are starshaped via segments: Let k and d be fixed integers, $0 \leq k \leq d$, and let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$. If every countable subfamily of $\mathcal{K}$ has as its intersection a starshaped set whose kernel is at least k-dimensional, then all members of $\mathcal{K}$ have such an intersection.

In case d is 2 (see [6, Theorem 1]), we have the following staircase analogue: Let $\mathcal{K}$ be a family of simply connected sets in the plane, and let k be 0, 1, or 2. For every countable subfamily $\{K_m : m \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_m : m \geq 1\}$ is starshaped via staircase paths and that its staircase kernel contains a convex set of dimension at least k. Then $\cap\{K : K \text{ in } \mathcal{K}\}$ has these properties. In this paper, we replace the notion of a simply connected set with a weaker requirement that we call *consistent relative to staircase paths*, allowing us to obtain a d-dimensional analogue of an earlier planar result. Since planar sets satisfying the new condition need not be simply connected, the new result also provides a small improvement for the planar case.

2. Definitions and Notation

This section includes definitions and comments, some of which appear in [3]. A set B in $\mathbb{R}^d$ is called a *box* if and only if B is a convex polytope (possibly degenerate) whose edges are parallel to the coordinate axes. A nonempty set S in $\mathbb{R}^d$ is an *orthogonal polytope* if and only if S is a connected union of finitely many boxes. An orthogonal polytope in $\mathbb{R}^2$ is an *orthogonal polygon*. Let λ be a simple polygonal path in $\mathbb{R}^d$ whose edges are parallel to the coordinate axes. That is, let λ be a simple rectilinear path in $\mathbb{R}^d$. For points x and y in S, the path λ is called an *$x - y$ path* if and only if λ lies in S and has endpoints x and y. The $x - y$ path λ is a *staircase path* (or simply a *staircase*) if and only if, as we travel along λ from x to y, no two edges of λ have opposite directions. That is , for each standard basis vector $e_i, 1 \leq i \leq d$, either each edge of λ parallel to e_i is a positive multiple of e_i

or each edge of λ parallel to e_i is a negative multiple of e_i. For $n \geq 1$, the staircase path λ is called a *staircase n-path* if and only if λ is a union of at most n edges. Staircase paths λ and μ are *compatible* if and only if no edge of λ is a negative multiple of an edge of μ.

For points x and y in a set S, we say x *sees* y (x *is visible* from y) *via staircase paths* if and only if S contains an $x - y$ staircase path. A set S is *staircase convex (orthogonally convex)* if and only if, for every pair of points x, y in S, x sees y via staircase paths. Similarly, a set S is *staircase starshaped (orthogonally starshaped)* if and only if, for some point p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S, $Ker\, S$. Analogous definitions hold for staircase n-paths.

Throughout the paper, we will use the following terminology and notation. We say that a planar set S is *simply connected* if and only if, for every simple closed curve $\delta \subseteq S$, the bounded region determined by δ lies in S. If δ is a simple path containing points x and y, then $\lambda(x, y)$ will denote the subpath of λ from x to y (ordered from x to y). For convenience, any vector parallel to a standard basis vector in R^d will be a *coordinate vector*. Readers may refer to Valentine [13], to Lay [12], to Danzer, Grünbaum, Klee [8], and to Eckhoff [9] for discussions concerning Helly-type theorems, visibility via straight line segments, and starshaped sets.

3. The Results

The following definitions will be helpful.

Definition 1. Let K be a set in $\mathbb{R}^d$. We say that K is *consistent relative to staircase paths* if and only if K satisfies this property: For every pair of points v, w for which $\overrightarrow{vw}$ is a coordinate vector in K, whenever v and w see a common point p of K via compatible staircase n-paths in K for some $n \geq 1$, then each point of $[v, w]$ sees p via a staircase n-path in K.

Comments:

(1) Observe that for points v, w and p described in Definition 1 above, any two staircase paths from points of $[v, w]$ to p will be compatible.

(2) In Figure 1 below, points u and w see point p via staircase 2-paths in set K. However, such paths cannot be compatible.

(3) It is easy to see that every simply connected set in the plane will be consistent relative to staircase paths. However, sets that are consistent relative to staircase paths need not be simply connected, as Example 1 below demonstrates.

Example 1. Let K be an orthogonal polygon bounded by the boundaries of two squares in Figure 1. The set K is consistent relative to staircase paths and is starshaped via staircase paths. (The staircase kernel comprises four rectangular regions, darkly shaded in the figure.) However, K certainly is not simply connected.

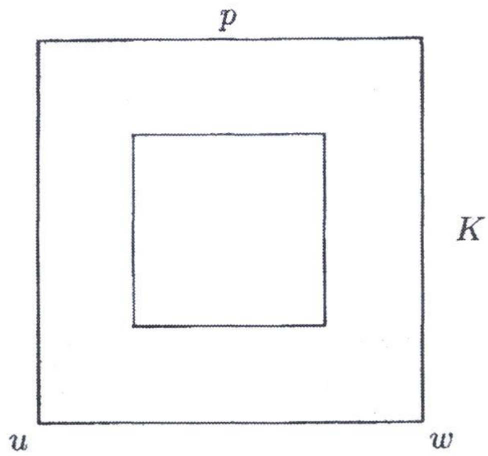

FIGURE 1

Definition 1 above allows us to obtain d-dimensional analogues of some planar results in [6]. We begin with an important lemma.

Lemma 1. Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$, with points p and s in $\cap\{K : K \text{ in } \mathcal{K}\}$. Let n be a fixed integer, $n \geq 1$. For each countable subfamily of $\mathcal{K}$, assume that the corresponding intersection is consistent relative to staircase paths and that the intersection contains a staircase n-path from p to s. Then $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path as well.

Proof. We modify a planar argument from [6, Lemma 1], using induction on n. If $n = 1$, the result is immediate. Inductively, assume that the lemma is true for natural numbers j, $1 \leq j < k$, to prove for k. Let $\mathcal{J}$ represent the family of all countable intersections of members of $\mathcal{K}$. Certainly $\mathcal{J}$ satisfies our hypotheses. Of course, every staircase path from p to s employs vectors using exactly the same directions. Since the collection of directions is finite, for one of these directions, say in the direction of the basis vector e_1, every countable intersection of members of $\mathcal{J}$ contains a staircase k-path from p to s whose first vector is in the direction of e_1. Then for each J in $\mathcal{J}$, there is a family of staircase k-paths λ from p to s whose associated first segment $[p, t(\lambda)]$ is in the direction of e_1, $p \neq t(\lambda)$. Let $T(J)$ represent the associated collection of points $t(\lambda)$.

We assert that each set $T(J)$ is convex: For t and t' in $T(J)$, t and t' see s via compatible staircase $(k-1)$-paths in J. By hypothesis, J is consistent relative to staircase paths. Hence each point u of $[t, t']$ sees s via such a path, and $[p, u]$ serves as the first vector in a staircase k-path from p to s in J. That is, $T(J)$ is convex. Moreover, using our earlier comments, every countable collection of sets $T(J)$, J in $\mathcal{J}$, will have a nonempty intersection. Thus we may apply Klee's theorem [10] to conclude that $\cap\{T(J) : J \text{ in } \mathcal{J}\}$ is nonempty.

Select t_0 in $\cap\{T(J) : J \text{ in } \mathcal{J}\}$. For every J in $\mathcal{J}$, J contains a staircase k-path from p to s having first segment $[p, t_0]$. Then for every J in $\mathcal{J}$, J contains a staircase $(k-1)$-path from t_0 to s. Using our induction hypothesis, $\cap\{K : K \text{ in } \mathcal{K}\}$ contains a staircase $(k-1)$-path from t_0 to s. Let $\delta(t_0, s)$ represent such a path. Then $[p, t_0] \cup \delta(t_0, s)$ is a staircase k-path from p to s in $\cap\{K : K \text{ in } \mathcal{K}\}$. The lemma holds for k and, by induction, holds for every integer $n \geq 1$, finishing the proof. □

Corollary 1. Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$, with points p and s in $\cap\{K : K \text{ in } \mathcal{K}\}$. For each countable subfamily of $\mathcal{K}$, assume that the corresponding intersection is consistent relative to staircase paths and that the intersection contains a staircase path from p to s. Then $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a path as well.

Proof. The proof follows the argument in [6, Corollary] and is included for completeness. We use a contrapositive argument. Suppose that $\cap\{K : K \text{ in } \mathcal{K}\}$ contains no staircase path from p to s. Then for every $n \geq 1$, $\cap\{K : K \text{ in } \mathcal{K}\}$ contains no staircase n-path from p to s. Furthermore, using Lemma 1, there is a countable subfamily $\mathcal{K}_n$ of $\mathcal{K}$ such that $\cap\{K : K \text{ in } \mathcal{K}_n\}$ contains no staircase n-path from p to s. Thus $\cap\{K : K \text{ in } \mathcal{K}_n \text{ for some } n \geq 1\}$ is a countable intersection of members of $\mathcal{K}$ containing no staircase $p - s$ path. The contrapositive of the statement establishes the result. □

Both Lemma 1 and its corollary fail if we delete the requirement that countable intersections of members of $\mathcal{K}$ be consistent relative to staircase paths. Consider the following example, adapted from [6, Example 1].

Example 2. For every real number r, let $(r, 0)$ be the associated point on the x-axis, and let $K_r = \mathbb{R}^2 \setminus \{(r, 0)\}$. Certainly countable intersections of members of $\mathcal{K}$ are not consistent relative to staircase paths. For example, letting $v = (-\sqrt{5}, 1)$, $p = (-2, 1)$, $w = (-\sqrt{2}, 1)$, $s = (1, -1)$, the set $\cap\{K_r : r \text{ rational}\}$ contains a staircase 2-path from v to s and a staircase 2-path from w to s but no staircase 2-path from p to s. It is easy to see that every countable intersection of members of $\mathcal{K}$ contains a staircase 3-path from p to s. However, p and s lie in distinct components of $\cap\{K_r : r \text{ real}\}$, so $\cap\{K_r : r \text{ real}\}$, contains no $p - s$ path at all.

The following easy proposition from [6, Proposition 1] will be helpful.

Proposition 1. Let $\mathcal{K}$ be any family of sets in $\mathbb{R}^d$. If every countable intersection of members of $\mathcal{K}$ has a nonempty interior, then $\cap\{K : K \text{ in } \mathcal{K}\}$ has a nonempty interior as well.

We are ready for our first theorem.

Theorem 1. Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$. For each countable subfamily $\{K_m : m \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_m : m \geq 1\}$ is consistent relative to staircase paths, that $\cap\{K_m : m \geq 1\}$ is starshaped via staircase paths, and that the corresponding staircase kernel contains a convex set of dimension d. Then $\cap\{K : K \text{ in } \mathcal{K}\}$ has these properties as well.

Proof. Using Lemma 1, it is easy to see that $\cap\{K : K \text{ in } \mathcal{K}\}$ is consistent relative to staircase paths. The proof will show that the other properties hold. Since countable intersections of members of $\mathcal{K}$ have nonempty interiors, by Proposition 1, $\cap\{K : K \text{ in } \mathcal{K}\}$ has nonempty interior, too. Let $S = \cap\{K : K \text{ in } \mathcal{K}\} \neq \varnothing$. To establish the theorem, we modify a strategy used by Bobylev [1], employed in [6]. Let $\mathcal{J}$ represent the family of all countable intersections of members of $\mathcal{K}$. For each J_α in $\mathcal{J}$, define $M_\alpha = \{x : x \text{ in } J_\alpha, x \text{ sees via staircase paths in } J_\alpha \text{ each point of } S\}$. Let $\mathcal{M}$ represent the family of all the M_α sets.

We will show that each countable intersection of members of $\mathcal{M}$ has nonempty interior, as does $\cap\{M_\alpha : M_\alpha \text{ in } \mathcal{M}\}$. First we observe that, for any countable subfamily $\{M_m : m \geq 1\}$ of $\mathcal{M}$ and corresponding subfamily $\{J_m : m \geq 1\}$ of $\mathcal{J}$, $Ker \cap \{J_m : m \geq 1\} \subseteq \cap\{M_m : m \geq 1\}$. Let z belong to $Ker \cap \{J_m : m \geq 1\} \neq \varnothing$. Then z sees via staircase paths in $\cap\{J_m : m \geq 1\}$ each point of S. Hence for each $m \geq 1$, z sees via staircase paths in J_m each point of S, so $z \in \cap\{M_m : m \geq 1\}$. That is, $Ker \cap \{J_m : m \geq 1\} \subseteq \cap\{M_m : m \geq 1\}$. Since $Ker \cap \{J_m : m \geq 1\}$ has nonempty interior, so does $\cap\{M_m : m \geq 1\}$, and by Proposition 1, $\cap\{M : M \text{ in } \mathcal{M}\}$ has nonempty interior, too, the desired result.

Finally, we observe that $\cap\{M : M \text{ in } \mathcal{M}\} = Ker\, S$. To see that $\cap\{M : M \text{ in } \mathcal{M}\} \subseteq Ker\, S$, let p belong to $\cap\{M : M \text{in } \mathcal{M}\} \subseteq S$ and let s belong to S. For each countable intersection J_α of members of $\mathcal{K}$, x belongs to M_α. That is, x sees point s via staircase paths in J_α. Since every countable intersection J_α of members of $\mathcal{K}$ contains a staircase path from p to s, by Corollary 1, $\cap\{K : K \text{ in } \mathcal{K}\}$ contains such a staircase, too. This holds for every s in S, so $p \in Ker\, S$. It is easy to see that the reverse inclusion holds as well, and the sets are equal. We conclude that $Ker\, S$ is nonempty and contains a convex set of dimension d, finishing the proof. □

Replacing staircase paths with staircase n-paths for some fixed n, $n \geq 1$, we have the following analogue of Theorem 1.

Theorem 2. Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$, and let n be a fixed integer, $n \geq 1$. For each countable subfamily $\{K_m : m \geq 1\}$ of $\mathcal{K}$, assume that $\cap\{K_m : m \geq 1\}$ is consistent relative to staircase paths, that $\cap\{K_m : m \geq 1\}$ is starshaped via staircase n-paths, and that the corresponding staircase kernel contains a convex set of dimension d. Then $\cap\{K : K \text{ in } \mathcal{K}\}$ has these properties as well.

Proof. The proof follows the argument in Theorem 1, using Lemma 1 instead of Corollary 1. □

Furthermore, it is easy to see that Theorems 1 and 2 yield analogous results for sets whose intersections are convex via staircase paths and for sets whose intersections are convex via staircase n-paths.

Finally, [6, Example 2] demonstrates that we cannot replace *countable* with *finite* in Theorem 1, even when finite intersections of members of $\mathcal{K}$ are staircase convex orthogonal polygons.

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