On the Cartesian products with crossing number three *

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Abstract: M. Klešč et al. characterized graphs G_1 and G_2 for which the crossing number of their Cartesian product $G_1 \Box G_2$ equals one or two. In this paper, their results are extended by given the necessary and sufficient conditions for all pairs of graphs G_1 and G_2 for which the crossing number of their Cartesian product $G_1 \Box G_2$ equals three, if one of the graphs G_1 and G_2 is a cycle.

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1 Introduction

All graphs considered here are simple, undirected and are also connected. A drawing of a graph $G = (V, E)$ is a mapping ϕ that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $\phi(u)$ and $\phi(v)$, not passing through the image of any other vertex. For simplicity, we impose the following conditions on a drawing: (a) no three edges have an interior point in common, (b) if two edges share an interior point p, then they cross at p, and (c) any two edges of a drawing have only a finite number of crossings (common interior points). The *crossing number*, $cr(G)$, of a graph G is the minimum number of edge crossings in any drawing of G. Let D be a drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$. It is easy to see that a drawing with minimum number of

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crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

Determining the crossing numbers of graphs is a notorious problem in Graph Theory, as in general it is quite easy to find a drawing of a sufficiently "nice" graph in which the number of crossings can hardly be decreased, but it is very difficult to prove that such a drawing indeed has the smallest possible number of crossings. In fact, Garey and Johnson [1] have proved that in general the problem of determining the crossing number of a graph is NP-complete (the reader can also refer to two results on complexity of the crossing number of graphs in [2,3], respectively). At present, exact values are known only for very restricted classes of graphs. For more about crossing number, see [4] and the references therein.

The Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \square G_2) = \{(x_1, y_1)(x_2, y_2) | x_1 = x_2$ and $y_1 y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1 x_2 \in E(G_1)\}$. Let C_n and P_n be the cycle and the path of length n, respectively, and let S_n denote the star $K_{1,n}$. Let $Q, F_1, F_2, F_3, H, J, K$ be the seven graphs depicted in Figure 1, respectively. We denote by G^{α} the subdivision of G. The length of the shortest cycle in a graph is called the girth of G and is denoted by $g(G)$. If G has no cycle, then $g(G) = \infty$. For graphical notation and terminology without explanation in this paper, we refer the reader to [5].

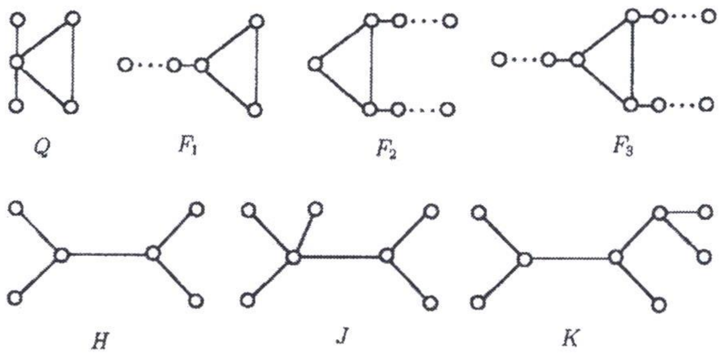

Figure 1: The special graph $Q, F_1, F_2, F_3, H, J, K$

M. Behzad and S. E. Mahmoodian [6] started to characterize graphs G_1 and G_2 for which their Cartesian product $G_1 \square G_2$ is planar. M. Klešč et al. [7-8] characterized graphs G_1 and G_2 for which the crossing number of their Cartesian product $G_1 \square G_2$ equals one or two. More precisely, they obtain the following results in [8] for value two.

Theorem A. *Let G_1 and G_2 be connected graphs and let G_1 is isomorphic to a cycle C_n, $n \geq 3$. Then $cr(G_1 \square G_2) = 2$ if and only if one of the* *following conditions holds:*

a) $G_1 = C_4$ *and* G_2 *is* S_3 *or* S_3^α,

b) $G_1 = C_3$ *and* G_2 *is one of* S_4, S_4^α, H, *and* H^α.

In this paper, the above result is extended by given the necessary and sufficient conditions for graphs G_1 and G_2 for which the crossing number of their Cartesian product $G_1 \Box G_2$ equals three, if one of the graphs G_1 and G_2 is a cycle.

2 Preliminary results

Lemma 1([9]). *Let T be a tree and $n \geq 1$. Then, for $d_v = deg_T(v)$,*

$$cr(S_n \Box T) = \sum_{v \in V(T), d_v \geq 2} cr(K_{1,d_v,n}).$$

Lemma 2. $cr(C_3 \Box F_1) = cr(C_3 \Box F_2) = 3$ *and* $cr(C_3 \Box F_3) \geq 4$.

Proof. Both F_1 and F_2 contain C_3 as a subgraph and therefore, the Cartesian products of them with the cycle C_3 contain $C_3 \Box C_3$ as a subgraph and $cr(C_3 \Box C_3) = 3$, see [10]. This implies that $cr(C_3 \Box F_1) \geq 3$ and $cr(C_3 \Box F_2) \geq 3$. The reverse inequalities follow from two drawings in Figures 2 and 3.

It is not difficult to verify that the graph $P_2 \Box F_3$ contains a subdivision of the graph $C_3 \Box C_3$ as a subgraph. This confirms that $cr(P_2 \Box F_3) \geq cr(C_3 \Box C_3) = 3$. As the graph $C_3 \Box F_3$ contains the graph $P_2 \Box F_3$ as a subgraph, its crossing number is at least three. Let C_3^i, $i = 0, 1, 2$, denote the 3-cycle in the i-th copy F_3^i of $C_3 \Box F_3$. Assume now that there is a good drawing D of $C_3 \Box F_3$ with only three crossings. Then, for any edge $e \in E(C_3 \Box F_3)$ and $e \notin E(C_3^1 \cup C_3^2 \cup C_3^3)$, there is no crossing appearing on the edge e. Otherwise, the removing of the edge e results in the drawing with at most two crossings, this contradicts the fact that $C_3 \Box F_3 - e$ contains a subgraph homeomorphic with $C_3 \Box C_3$ or $P_2 \Box F_3$ and $cr(P_2 \Box F_3) \geq cr(C_3 \Box C_3) = 3$. Therefore, $cr_D(C_3 \Box F_3) = cr_D(C_3^1 \cup C_3^2 \cup C_3^3)$. As D is a good drawing, the three edges of the 3-cycle C_3^i do not cross each other for any $i = 0, 1, 2$ and $cr_D(C_3^l, C_3^k)$ is an even number, $l, k = 0, 1, 2$. Hence, $cr_D(C_3 \Box F_3) = cr_D(C_3^1 \cup C_3^2 \cup C_3^3) \geq 4$, a contradiction. This completes the proof.

Lemma 3. $cr(C_3 \Box K) = cr(C_3 \Box K^\alpha) = 3$.

Proof. It follows from Lemma 1 and the fact that $cr(K_{1,2,3}) = cr(K_{3,3}) = 1$ that $cr(P_2 \Box K) \geq 3$. The graph $C_3 \Box K$ contains $P_2 \Box K$ as a subgraph, this implies that $cr(C_3 \Box K) \geq 3$. On the other hand, the drawing of the graph $C_3 \Box K^\alpha$ in Figure 4 shows that $cr(C_3 \Box K^\alpha) \leq 3$. Note that $C_3 \Box K^\alpha$ con-

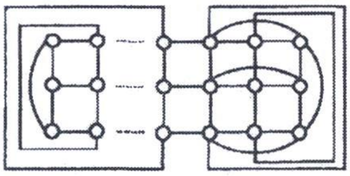

Figure 2: The graph $C_3\Box F_1$ with three crossings

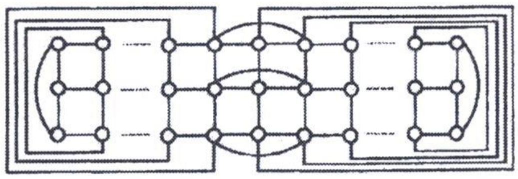

Figure 3: The graph $C_3\Box F_2$ with three crossings

tains a subgraph homeomorphic with $C_3\Box K$, it has $cr(C_3\Box K) \leq cr(C_3\Box K^\alpha)$, the proof is done.

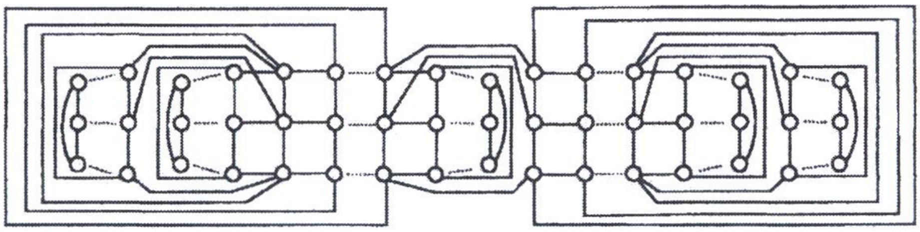

Figure 4: The graph $C_3\Box K^\alpha$ with three crossings

Lemma 4. $cr(C_3\Box J) = cr(C_3\Box J^\alpha) = 3$.

Proof. The drawing of the graph $C_3\Box J^\alpha$ in Figure 5 implies that $cr(C_3\Box J^\alpha) \leq 3$. Note that $cr(K_{1,2,4}) = cr(K_{3,4}) = 2$. The reverse inequality follows from a similar argument to that used in Lemma 3 and the details are omitted.

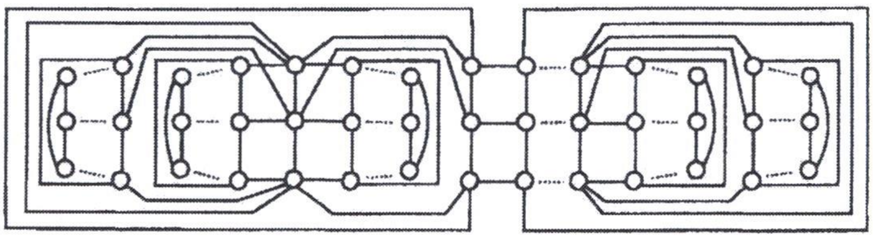

Figure 5: The graph $C_3\Box J^\alpha$ with three crossings

3 Main result

Theorem 1. *Let G_1 and G_2 be connected graphs and let G_1 is isomorphic to a cycle C_n. Then $cr(G_1\Box G_2) = 3$ if and only if $G_1 = C_3$ and G_2 is one of C_3, F_1, F_2, J, J^α, K, and K^α.*

Proof. If $G_1 = C_3$ and G_2 is one of C_3, F_1, F_2, J, J^α, K, and K^α, then it follows from [10] and Lemma 2,3,4 that $cr(G_1 \Box G_2) = 3$. In order to prove the converse we consider two cases.

Case 1. $g(G_2) < +\infty$.

Both G_1 and G_2 do not contain a cycle of length more than three, otherwise the graph $G_1 \Box G_2$ contains $C_3 \Box C_4$ or its subdivision as a subgraph and $cr(C_3 \Box C_4) = 4$, see [10]. This enforces that G_1 is C_3.

The graph G_2 contains exactly one cycle of length three. Otherwise, let C', C'' be two different cycles of length three in G_2. If $|V(C') \cap V(C'')| = 2$, then G_2 contains a cycle of length more than three, a contradiction. If $|V(C') \cap V(C'')| = 1$, then $G_1 \Box G_2$ contains two subgraphs $C_3 \Box C_3$ with exactly one common 3-cycle. Consider a good drawing, the edges of the common 3-cycle do not cross each other, thus, $cr(G_1 \Box G_2) \geq 2cr(C_3 \Box C_3) = 6$, a contradiction. If $|V(C') \cap V(C'')| = 0$, then $G_1 \Box G_2$ contains two edge-disjoint subgraphs $C_3 \Box C_3$, this implies that $cr(G_1 \Box G_2) \geq 2cr(C_3 \Box C_3) = 6$, a contradiction again.

The degree of vertices of G_2 are less than three except for that in a cycle, otherwise the graph $G_1 \Box G_2$ contains two edge-disjoint subgraphs $C_3 \Box C_3$ and $C_3 \Box S_3$. Note that $cr(C_3 \Box C_3) = 3$ and $cr(C_3 \Box S_3) = 1$, see [10,11]. Hence, $cr(G_1 \Box G_2) \geq 4$, a contradiction.

The degree of vertices in a cycle are less than four, otherwise the graph $G_1 \Box G_2$ contains $C_3 \Box Q$ as a subgraph and $cr(G_1 \Box Q) = 4$, see [12].

The previous analysis, together with Lemma 2, implies that G_2 must be one of C_3, F_1, F_2.

Case 2. $g(G_2) = \infty$.

As $cr(C_n \Box P_m) = 0$ for all $m \geq 1, n \geq 3$, the condition $cr(G_1 \Box G_2) = 3$ enforces that the graph G_2 must contain a vertex of degree more than two. Hence, the graph G_2 must be a tree other than a path. Moreover, G_1 is C_3 or C_4, because $cr(C_n \Box S_3) \geq 4$ for $n \geq 5$, see [11]. The graph G_2 does not contain a vertex of degree more than four, otherwise the graph $G_1 \Box G_2$ contains $P_2 \Box S_5$ as a subgraph and $cr(P_2 \Box S_5) = 4$, see [13].

Consider first the graph $G_1 = C_4$. The graph G_2 does not contain a vertex of degree more than three, because $cr(C_4 \Box S_4) = 4$, see [14]. The graph G_2 contains exactly one vertex of degree three, otherwise $C_4 \Box G_2$ contains a subgraph homeomorphic to the graph $C_4 \Box H$ with crossing number four, see [15]. Hence, the graph G_2 must be the graph S_3 or S_3^α. On the other hand, note also that $cr(C_4 \Box S_3) = cr(C_4 \Box S_3^\alpha) = 2$ by Theorem A. The contradiction enforces that the graph G_1 can't be C_4.

Consider now the graph $G_1 = C_3$. It is clear that $C_3 \Box G_2$ contains $P_2 \Box G_2$ as a subgraph. Let G_2 has n_i vertices of degree i, it follows from

Lemma 1 that

$$cr(P_2\Box G_2) = \sum_{i\geq 2} n_i \lfloor \frac{i}{2} \rfloor \lfloor \frac{i-1}{2} \rfloor. \tag{3.1}$$

The condition $cr(C_3\Box G_2) = 3$ enforces that $\Delta(G_2) \leq 4$ and G_2 contains at most one vertex of degree four, otherwise $cr(C_3\Box G_2) \geq cr(P_2\Box G_2) \geq 4$ by equation (3.1).

Assume first that $\Delta(G_2) = 4$. Then the graph G_2 has one and only one vertex of degree three. Because, if the degree of all vertices but the maximum degree vertex of G_2 are at most two, then G_2 is the graph S_4 or S_4^α and $cr(C_3\Box S_4) = cr(C_3\Box S_4^\alpha) = 2$ by Theorem A. If there are at least two vertices of degree three, then $cr(C_3\Box G_2) \geq cr(P_2\Box G_2) \geq 4$ by equation (3.1). Every such graph is homeomorphic to the graph J and $cr(C_3\Box G_2) = 3$, see Lemma 4.

Assume now that $\Delta(G_2) = 3$. Then the graph G_2 contains at most three vertices of degree three, otherwise $cr(C_3\Box G_2) \geq cr(P_2\Box G_2) \geq 4$ by equation (3.1). Every connected graph with three vertices of degree three is homeomorphic to the graph K and in this case $cr(C_3\Box G_2) = 3$. As every connected graph with less than three vertices of degree three is homeomorphic to the graph S_3 or H and $cr(C_3\Box G_2) \leq 2$, see [8], the proof is done.

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