Packing 5-cycles into balanced complete n-partite graphs for even n: the 1-factor leave*

Hongjuan Liu[a], Zhen Wang[a], Lidong Wang[b]
[a]Department of Computer Science and Engineering, Langfang Polytechnic Institute, Langfang 065000, P. R. China
[b]Department of Basic Courses, China People's Police University, Langfang 065000, P. R. China
email address (lidongwang@aliyun.com)

Abstract

Let $K_{g_1,g_2,\dots,g_n}$ be a complete n-partite graph with partite sets of sizes g_i for $1 \le i \le n$. A complete n-partite is balanced if each partite set has g vertices. In this paper, we will solve the problem of finding a maximum packing of the balanced complete n-partite graph, n even, with edge-disjoint 5-cycles when the leave is a 1-factor.

1 Introduction

Let H be a simple graph and $\mathcal{G}$ a set of simple graphs. An $(H, \mathcal{G})$-*packing* is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is a collection of subgraphs (called *blocks*) of H, such that each block is isomorphic to a graph of $\mathcal{G}$, and each edge of H is contained in at most one blocks of $\mathcal{B}$. If $\mathcal{G}$ contains a single graph G, we speak of an (H, G)-packing.

The *leave* L of an (H, G)-packing $(X, \mathcal{B})$ is the subgraph induced by the set of edges of H that do not occur in any block $B \in \mathcal{B}$. For fixed H and G, an (H, G)-packing $(X, \mathcal{B})$ is called *maximum* if it has the *minimum* leave L, a leave with minimum number of edges, among all (H, G)-packings. If

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the leave of an (H, G)-packing is null, then such a packing is maximum, and referred to as an (H, G)-design.

Let $K_{g_1,g_2,\ldots,g_n}$ be a complete n-partite graph with partite sets of sizes g_i for $1 \le i \le n$. A complete n-partite graph is *balanced* if each partite set has g vertices, which is denoted by $K_{n(g)}$. A $(K_{u_1,u_2,\ldots,u_t}, G)$-packing is said to be a *group divisible packing* (each partite set is called a *group*), written as a G-GDP of type $\{u_1, u_2, \ldots, u_t\}$. The multiset $\{u_1, u_2, \ldots, u_t\}$ is called the *group type* (or *type*) of the GDP. For simplicity, we will use an "exponential" notation to describe group types: type $g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$ indicates that there are n_i groups of size g_i. A $(K_{u_1,u_2,\ldots,u_t}, G)$-design is often said to be a *group divisible design*, denoted by G-GDD of type $\{u_1, u_2, \ldots, u_t\}$. We always write G-MGDP instead of maximum G-GDP.

When G is C_k, a cycle of length k, the existence problem for C_k-GDDs of type g^n has been studied for more than 40 years. The necessary and sufficient conditions have been determined for $n \in \{3, 4, 5\}$ [1, 2]. For general n, the spectrum has also been determined for $k = 3$ [12]; $k = 5$ [5]; $k \in \{4, 6, 8\}$ [9]; prime $k \ge 7$ [15]; k twice or thrice a prime [19, 20] and prime square [21, 22]. For recent progress on C_k-GDDs, the readers may refer to [7, 17].

However, if we turn to the C_k-MGDPs of type g^n, relatively little is known. The problem has been solved only for $k = 3$ [6, 13, 25], $k = 4$ [3, 4] and $k = 6$ [11]. For the case of $k = 5$, some partial solutions to this problem have been obtained. The existence problem for C_5-GDDs of type g^n has been completely solved by Billington et al. [5].

Lemma 1 *[5] There exists a C_5-GDD of type g^n if and only if $n \ge 3$, $g(n-1) \equiv 0 \pmod 2$ and $n(n-1)g^2 \equiv 0 \pmod{10}$.*

Cavenagh et al. [10] considered the existence problem for C_5-GDDs of type g^2x^1.

Lemma 2 *[10] There exists a C_5-GDD of type g^2x^1 if and only if $g/3 \le x \le 2g$, $g - x \equiv 0 \pmod 2$, and $g(g+2x) \equiv 0 \pmod 5$.*

For two graphs G and H, the notation $G \cup H$ represents the union of graphs G and H without common vertices. We give one possible leave of a C_5-MGDP of type g^n in Table 1 in which the rows and the columns are indexed by congruence classes of n and g modulo 10. Here F is a 1-factor, F_i denotes a graph on gn vertices with $gn/2 + i$ edges and each vertex has odd degree, and $2C_3$ represents two C_3 with one vertex in common.

Rosa and Znám [18] first discussed the C_5-packing problem and showed that a C_5-MGDP of type 1^n with leave given in Table 1 always exists.

Table 1: Possible leave of a C_5-MGDP of type g^n

$n \backslash g$	0	1	2	3	4	5	6	7	8	9
0	$\emptyset$	F	$\emptyset$	F	$\emptyset$	F	$\emptyset$	F	$\emptyset$	F
1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\emptyset$	F	C_4	F_1	$2C_3$	F	$2C_3$	F_2	C_4	F_2
3	$\emptyset$	C_3	$C_3 \cup C_4$	$C_3 \cup C_4$	C_3	$\emptyset$	C_3	$C_3 \cup C_4$	$C_3 \cup C_4$	C_3
4	$\emptyset$	F_4	C_4	F_3	$2C_3$	F	$2C_3$	F	C_4	F_3
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	$\emptyset$	F_2	$\emptyset$	F_1	$\emptyset$	F	$\emptyset$	F_4	$\emptyset$	F_3
7	$\emptyset$	$2C_3$	C_4	C_4	$2C_3$	$\emptyset$	$2C_3$	C_4	C_4	$2C_3$
8	$\emptyset$	F_4	$C_3 \cup C_4$	F	C_3	F	C_3	F_4	$C_3 \cup C_4$	F_2
9	$\emptyset$	$2C_3$	C_4	C_4	$2C_3$	$\emptyset$	$2C_3$	C_4	C_4	$2C_3$

Lemma 3 *[18] There exists a C_5-MGDP of type 1^n with leave L in Table 1 for each integer $n \geq 3$.*

Huang et al. [14] investigated the case of $g > 1$ and determined the leave of a C_5-MGDP of type g^n for any positive integer $n \equiv 1 \pmod 2$.

Lemma 4 *[14] There exists a C_5-MGDP of type g^n with leave L in Table 1 for each odd integer $n \geq 3$.*

This paper is a continuation of [14], and we are concerned about C_5-MGDPs of type g^n with $n \equiv 0 \pmod 2$. As the main result of this paper, we show that a C_5-MGDP of type g^n with leave F (those in Table 1) always exists, i.e., we are to prove

Theorem 5 *There exists a C_5-MGDP of type g^n with leave F if and only if one of the following conditions holds.*

(1) $n \equiv 0 \pmod{10}$, $n \geq 3$ *and* $g \equiv 1 \pmod 2$;

(2) $n \equiv 2 \pmod{10}$, $n \geq 3$ *and* $g \equiv 1, 5 \pmod{10}$;

(3) $n \equiv 4 \pmod{10}$, $n \geq 3$ *and* $g \equiv 5, 7 \pmod{10}$;

(4) $n \equiv 6 \pmod{10}$, $n \geq 3$ *and* $g \equiv 5 \pmod{10}$;

(5) $n \equiv 8 \pmod{10}$, $n \geq 3$ *and* $g \equiv 3, 5 \pmod{10}$.

The rest of this paper is structured as follows. In Sections 2 and 3, we introduce two auxiliary designs, i.e., holey group divisible design and incomplete group divisible packing respectively, to build new combinatorial constructions for C_5-MGDPs of type g^n. Finally, we obtain the main result of this paper in Section 4.

2 Holey group divisible designs

In what follows, we always assume that $I_n = \{0, 1, \ldots, n-1\}$ and denote by Z_v the additive group of integers modulo v.

Let $S_1, S_2, \ldots, S_{t+1}$ be disjoint subsets of the vertex set of the balanced complete n-partite graph $K_{n(ht+w)}$ with partite sets G_i, $1 \leq i \leq n$, satisfying $|S_j| = hn$ for $1 \leq j \leq t$, $|S_{t+1}| = wn$, and $|G_i \cap S_j| = h$ for any $1 \leq i \leq n$ and $1 \leq j \leq t$, $|G_i \cap S_{t+1}| = w$ for any $1 \leq i \leq n$. Let $K[S_j]$ be the subgraph of $K_{n(ht+w)}$ induced by S_j. A $(K_{n(ht+w)} \setminus \cup_{j=1}^{t+1} K[S_j], G)$-design is referred to as a *holey group divisible design* and each S_j is called a *hole*. Such a design is denoted by a G-HGDD of type $(n, h^t w^1)$.

The following "filling in holes" construction is straightforward.

Construction 6 *Suppose that there exists a G-HGDD of type $(n, h^t w^1)$ with the hole set $\{S_1, S_2, \ldots, S_{t+1}\}$. If there exist a G-GDP of type h^n with leave L_j for each hole S_j, $1 \le j \le t$, and a G-GDP of type w^n with leave L_{t+1} for the hole S_{t+1}, then there exists a G-GDP of type $(ht+w)^n$ with leave $\cup_{j=1}^{t+1} L_j$.*

We quote the following result for later use.

Lemma 7 *[24] There exists a C_3-HGDD of type (n, h^t) if and only if $n, t \ge 3$, $(t-1)(n-1)h \equiv 0 \pmod 2$, and $t(t-1)n(n-1)h^2 \equiv 0 \pmod 6$.*

Lemma 8 *[16] There exists a C_5-HGDD of type $(n, 1^t)$ if and only if $n, t \ge 3$, $(t-1)(n-1) \equiv 0 \pmod 2$ and $t(t-1)n(n-1) \equiv 0 \pmod{10}$.*

We need to construct some more C_5-HGDDs.

Construction 9 *If there exists a C_5-GDD of type $g^t x^1$, then there exists a C_5-HGDD of type $(n, g^t x^1)$ for any $n \ge 3$.*

Proof Suppose $\mathcal{A}$ is a set of blocks of a C_5-GDD of type $g^t x^1$ on vertex set X with group set $\{G_i : 1 \le i \le t+1\}$. Suppose $\mathcal{B}$ is a set of blocks of a C_3-HGDD of type $(3, 1^n)$ (from Lemma 7) on $I_3 \times I_n$ with group set $\{\{i\} \times I_n : i \in I_3\}$ and hole set $\{I_3 \times \{j\} : j \in I_n\}$. For each $A = \{a, b, c, d, e\} \in \mathcal{A}$ and $B = \{(0, x), (1, y), (2, z)\} \in \mathcal{B}$, let

$$A_B = ((x, a), (y, b), (z, c), (x, d), (y, e)).$$

Let $\mathcal{C} = \cup_{A \in \mathcal{A}} (\cup_{B \in \mathcal{B}} A_B)$. It is readily checked that $\mathcal{C}$ forms a set of blocks of the desired C_5-HGDD of type $(n, g^t x^1)$ on $I_n \times X$ with group set $\{\{i\} \times X : i \in I_n\}$ and hole set $\{I_n \times G_i : 1 \le i \le t+1\}$. □

Lemma 10 *There exists a C_5-GDD of type $5^t x^1$ if $x \in \{3, 7\}$ and $t \equiv 0 \pmod 2$.*

Proof When $t = 2$, a C_5-GDD of type $5^2 x^1$ exists by Lemma 2. When $(t, x) = (4, 3)$, we construct the required design on vertex set $X = I_{23}$ with group set $\{\{0, 4, 8, 12, 16\} + i : 0 \le i \le 3\} \cup \{\{20, 21, 22\}\}$. We list all the 42 blocks as follows.

$(0,18,21,9,6)$,	$(0,10,15,1,11)$,	$(0,7,16,1,19)$,	$(0,5,14,3,9)$,
$(0,13,6,20,14)$,	$(0,15,22,16,17)$,	$(0,20,8,10,21)$,	$(0,22,4,19,2)$,
$(1,4,2,7,12)$,	$(1,20,11,22,18)$,	$(0,3,12,22,1)$,	$(1,14,16,9,2)$,
$(1,21,13,11,6)$,	$(1,7,10,19,8)$,	$(2,21,16,18,20)$,	$(1,10,17,4,3)$,
$(2,11,4,18,17)$,	$(2,8,7,18,15)$,	$(2,12,5,3,16)$,	$(2,22,7,14,13)$,
$(2,3,6,16,5)$,	$(3,22,5,6,17)$,	$(3,10,9,4,13)$,	$(3,18,19,5,8)$,
$(4,14,9,19,20)$,	$(3,20,7,4,21)$,	$(4,10,22,17,15)$,	$(5,15,12,6,7)$,
$(4,6,15,20,5)$,	$(9,15,21,14,22)$,	$(6,8,22,19,21)$,	$(6,19,16,13,22)$,
$(5,10,12,17,11)$,	$(7,13,15,8,9)$,	$(7,17,14,12,21)$,	$(10,13,8,17,20)$,
$(10,16,15,14,11)$,	$(9,20,12,18,11)$,	$(5,21,11,8,18)$,	$(9,12,19,13,18)$,
$(8,21,17,19,14)$,	$(11,16,20,13,12)$.		

When $(t,x)=(4,7)$, there exists a C_5-GDD of type $10^2 4^1$ by Lemma 2. Take three infinite points and fill in the groups of size ten with the above C_5-GDD of type $5^2 3^1$ to obtain a C_5-GDD of type $5^4 7^1$. When $t \geq 6$, by Lemma 1, there exists a C_5-GDD of type $10^{t/2}$. Take x infinite points and fill in the groups with a C_5-GDD of type $5^2 x^1$ from Lemma 2 to obtain a C_5-GDD of type $5^t x^1$. □

Lemma 11 *There exists a C_5-HGDD of type $(n, 5^t x^1)$ for any $n \geq 3$, $t \equiv 0 \pmod 2$ and $x \in \{3,7\}$.*

Proof By Lemma 10, there is a C_5-GDD of type $5^t x^1$. Apply Construction 9 to complete the proof. □

Lemma 12 *Let $n \geq 3$. There exists a C_5-HGDD of type (n, g^t) for $g(t-1) \equiv 1 \pmod 2$, $t(t-1)g^2 \equiv 0 \pmod{10}$ and $t \geq 3$.*

Proof By Lemma 1, we have C_5-GDD of type g^t. Apply Construction 9 to complete the proof. □

3 Incomplete group divisible packings

A G-GDP of type $g^n(gt)^1$ is called an *incomplete group divisible packing* of type $g^{(n+t,t)}$ (briefly G-IGDP of type $g^{(n+t,t)}$), where the group of size gt is said to be the *hole*. The following construction is simple but very useful.

Construction 13 *If there exist a G-IGDP of type $g^{(n,t)}$ with leave L_1 and a G-GDP of type g^t with leave L_2, then there exists a G-GDP of type g^n with leave $L_1 \cup L_2$.*

To present a construction for IGDPs, we shall introduce a special kind of IGDP. A G-IGDP of type $g^{(n,t)}$ is said to have the "star" property, denoted by G-IGDP* of type $g^{(n,t)}$, if the leave L of this IGDP forms a 1-factor of the points outside the hole. It is readily checked that a G-IGDP* of type $g^{(n,t)}$ has $\frac{(n(n-1)-t(t-1))g^2-g(n-t)}{2|E(G)|}$ blocks. Note that a G-IGDP* of type $g^{(n,0)}$ is actually a G-GDP of type g^n with leave F.

Example 14 *There exists a C_5-IGDP* of type $g^{(n,t)}$ for $(g,n,t) \in \{(5,6,2),(3,18,8),(7,14,4)\}$.*

Proof For $(g,n,t) = (5,6,2)$, the design is constructed on $Z_{20} \cup Y$ with group set $\{\{0,4,8,12,16\}+i : 0 \le i \le 3\} \cup \{Y\}$, where $Y = \{a,b,c,d,e\} \times Z_2$. All the 68 blocks can be obtained by developing the following base blocks by +5 mod 20. For $(x,i) \in Y$, $(x,i)+5 = (x,j)$, where $j \equiv i+5 \pmod 2$. The leave F is $\{\{i, 10+i\} : 0 \le i \le 9\}$. We write x_i instead of (x,i) to save space.

$(0,c_0,14,5,6)$, $(0,11,17,4,13)$, $(0,e_1,4,18,19)$, $(1,d_0,19,17,a_1)$,
$(0,2,c_1,1,14)$, $(0,15,c_0,3,a_0)$, $(1,2,b_1,17,18)$, $(1,16,18,a_0,19)$,
$(0,a_1,2,d_1,7)$, $(0,3,12,18,e_0)$, $(1,b_0,9,14,a_0)$, $(2,c_0,19,e_1,17)$,
$(0,d_0,3,5,b_1)$, $(1,c_0,8,13,e_0)$, $(1,8,b_1,13,d_1)$, $(2,19,b_1,11,e_1)$,
$(0,d_1,9,6,17)$.

For $(g,n,t) = (3,18,8)$, we construct the required design on $Z_{30} \cup Y$ with group set $\{\{0,10,20\}+i : 0 \le i \le 9\} \cup \{Y\}$, where $Y = \{a,b,c,d\} \times Z_6$. All the 222 blocks can be obtained by developing the following base blocks (+10 mod 30 for the first fourteen base blocks and +1 mod 30 for the last six base blocks). For $x_i \in Y$, $x_i + 1 = x_j$, where $j \equiv i+1 \pmod 6$. The leave F is $\{\{i, 15+i\} : 0 \le i \le 14\}$.

$(0,1,28,6,14)$, $(0,29,7,18,19)$, $(0,22,14,5,8)$, $(0,11,19,5,13)$,
$(0,27,6,7,21)$, $(1,2,3,24,23)$, $(1,12,9,26,15)$, $(1,22,8,29,18)$,
$(2,13,10,27,16)$, $(1,14,15,7,4)$, $(5,2,15,4,18)$, $(5,26,29,20,6)$,
$(4,23,7,8,17)$, $(3,16,13,29,12)$, $(0,a_0,1,7,b_0)$, $(2,a_0,3,5,b_0)$,
$(4,a_0,5,9,b_0)$, $(0,c_0,1,13,d_0)$, $(2,c_0,3,10,d_0)$, $(5,c_0,4,9,d_0)$.

For $(g,n,t) = (7,14,4)$, we construct the required design on $Z_{70} \cup Y$ with group set $\{\{0,10,\ldots,60\}+i : 0 \le i \le 9\} \cup \{Y\}$, where $Y = \{a,b\} \times Z_{14}$. All the 826 blocks can be obtained by developing the following base blocks (+1 mod 70 for the first eleven base blocks and +10 mod 70 for the last eight base blocks). For $x_i \in Y$, $x_i + 1 = x_j$, where $j \equiv i+1 \pmod{14}$. The leave F is $\{\{i, 35+i\} : 0 \le i \le 34\}$.

$(0,56,a_0,63,9)$,	$(0,41,43,62,44)$,	$(0,59,44,12,57)$,	$(0,12,65,22,28)$,
$(0,62,a_7,25,b_0)$,	$(0,3,b_0,68,a_2)$,	$(0,65,18,b_5,66)$,	$(0,a_{13},35,a_4,24)$,
$(0,36,b_{13},51,b_7)$,	$(0,a_5,49,b_1,37)$,	$(0,a_3,8,39,b_{10})$,	$(0,48,69,62,63)$,
$(0,69,47,48,49)$,	$(1,49,56,57,50)$,	$(3,51,58,65,66)$,	$(0,22,15,64,1)$,
$(5,26,47,54,53)$,	$(4,53,32,31,52)$,	$(18,66,44,45,67)$.	

The following construction is a generalization of Construction 4.21 in [23]. It is a routine matter to check the "star" property.

Construction 15 *Suppose a G-GDD of type* $(gt_0)^1(gt_1)^{n_1}(gt_2)^{n_2}\cdots(gt_q)^{n_q}$ *exists. If there exists a G-IGDP* of type* $g^{(t_i+e,e)}$ *for each* $1 \le i \le q$, *then there exists a G-IGDP* of type* $g^{(n+e,t_0+e)}$, *where* $n = t_0+\sum_{i=1}^{q} t_i n_i$. *If further there exists a G-IGDP* of type* $g^{(t_0+e,e)}$, *then there exists a G-IGDP* of type* $g^{(n+e,e)}$.

Now we apply Construction 15 to give some existence results of C_5-IGDP*s.

Lemma 16 *There exists a* C_5*-IGDP* of type* $5^{(n,6)}$ *for any* $n \equiv 2 \pmod 4$ *and* $n \ge 14$.

Proof By Lemma 1, there exists a C_5-GDD of type $20^{(n-2)/4}$. Apply Construction 15 with a C_5-IGDP* of type $5^{(6,2)}$ (from Example 14) to obtain a C_5-IGDP* of type $5^{(n,6)}$. □

Lemma 17 *There exists a* C_5*-IGDP* of type* $3^{(n,8)}$ *for any* $n \equiv 8 \pmod{10}$ *and* $n \ge 18$.

Proof For $n = 18$, by Example 14, there exists a C_5-IGDP* of type $3^{(18,8)}$. For $n = 28$, by Lemma 2, there exists a C_5-GDD of type $30^2 24^1$. Filling in the groups of size 30 with a C_5-IGDP* of type $3^{(10+0,0)}$, which is also a C_5-GDP of type 3^{10} (from Lemma 19), we have a C_5-IGDP* of type $3^{(28,8)}$ by Construction 15 . For $n \ge 38$, by Lemma 1, there exists a C_5-GDD of type $30^{(n-8)/10}$. Filling in the groups with the above C_5-IGDP* of type $3^{(18,8)}$, we have a C_5-IGDP* of type $3^{(n,8)}$ by Construction 15. □

Lemma 18 *There exists a* C_5*-IGDP* of type* $7^{(n,4)}$ *for any* $n \equiv 4 \pmod{10}$ *and* $n \ge 14$.

Proof For $n = 14$, by Example 14, there exists a C_5-IGDP* of type $7^{(14,4)}$. For $n = 24$, by Lemma 2, there exists a C_5-GDD of type $70^2 28^1$. Filling in the groups of size 70 with a C_5-IGDP* of type $7^{(10+0,0)}$ (from Lemma 19), we have a C_5-IGDP* of type $7^{(24,4)}$ by Construction 15. For $n \geq 34$, by Lemma 1, there exists a C_5-GDD of type $70^{(n-4)/10}$. Apply Construction 15 with the above C_5-IGDP* of type $7^{(14,4)}$, we have a C_5-IGDP* of type $7^{(n,4)}$. □

Remark: The results of Lemma 19 are used in Lemmas 17 and 18, respectively. Note that the constructions of Lemma 19 only need the conclusions of Lemmas 3 and 12.

4 Main result

To avoid confusion, in the proof of the following lemmas, we write $F(g, n)$ instead of F to distinguish different F for different g and n.

Lemma 19 *There exists a C_5-MGDP of type g^n with leave F for any $g \equiv 1 \pmod 2$ and $n \equiv 0 \pmod{10}$.*

Proof For $g = 1$, the conclusion holds by Lemma 3. For $g \geq 3$, by Lemma 8, there exists a C_5-HGDD of type $(n, 1^g)$. Filling in the holes with a C_5-MGDP of type 1^n with leave $F(1, n)$ (from Lemma 3), we have a C_5-MGDP of type g^n with leave $F(g, n)$ by Construction 6. □

Lemma 20 *There exists a C_5-MGDP of type g^n with leave F for any $g \equiv 1 \pmod{10}$ and $n \equiv 2 \pmod{10}$ and $n \geq 12$.*

Proof For $g = 1$, the conclusion holds by Lemma 3. For $g \geq 11$, by Lemma 8, there exists a C_5-HGDD of type $(n, 1^g)$. Fill in the holes with a C_5-MGDP of type 1^n with leave $F(1, n)$ (from Lemma 3) to obtain a C_5-MGDP of type g^n with leave $F(g, n)$ by Construction 6. □

Lemma 21 *There exists a C_5-MGDP of type 5^n with leave F for any $n \in \{4, 6, 8\}$.*

Proof Let the vertex set be Z_{5n} and the group set $\{\{i, n+i, \ldots, 4n+i\} : 0 \leq i \leq n-1\}$. For $n = 4$, all the blocks are listed below.

$(0,1,7,14,3)$, $(0,9,4,18,19)$, $(0,2,4,5,6)$, $(0,5,7,2,11)$,
$(0,15,2,8,17)$, $(0,13,4,3,18)$, $(0,14,4,10,7)$, $(1,4,19,6,3)$,
$(1,19,2,9,6)$, $(1,16,18,5,10)$, $(1,8,13,6,11)$, $(2,12,5,3,16)$,
$(4,6,8,7,17)$, $(1,2,17,18,12)$, $(3,8,14,15,9)$, $(1,14,17,6,15)$,
$(3,10,15,16,13)$, $(2,3,12,14,13)$, $(4,7,9,8,11)$, $(5,15,17,10,11)$,
$(5,14,19,17,16)$, $(5,19,9,18,8)$, $(6,7,16,19,12)$, $(7,18,15,12,13)$,
$(8,10,19,13,15)$, $(9,16,14,11,12)$, $(9,11,18,13,10)$, $(10,16,11,17,12)$.

For $n = 6$, all the blocks can be obtained by developing the following base blocks by $+10 \bmod 30$.

$(0,1,2,3,4)$, $(0,2,4,1,3)$, $(0,5,1,6,7)$, $(0,8,1,9,10)$,
$(0,9,2,5,13)$, $(0,11,1,10,26)$, $(0,14,1,12,17)$, $(0,19,2,6,22)$,
$(0,23,1,15,25)$, $(0,27,1,17,28)$, $(1,18,2,13,22)$, $(1,24,2,15,26)$,
$(1,28,2,25,29)$, $(2,12,7,3,29)$, $(3,5,4,6,8)$, $(3,6,5,7,14)$,
$(3,13,4,7,16)$, $(3,17,4,8,19)$, $(3,26,4,14,28)$, $(4,9,6,15,29)$,
$(4,15,7,8,25)$, $(5,8,9,7,28)$, $(6,16,8,28,19)$, $(6,17,27,19,29)$.

For $n = 8$, all the blocks can be obtained by developing the following base blocks by $+5 \bmod 40$.

$(0,1,2,3,4)$, $(0,2,4,1,3)$, $(0,5,1,6,12)$, $(0,6,2,5,14)$,
$(0,7,2,8,10)$, $(0,11,1,8,13)$, $(0,15,1,10,27)$, $(0,18,1,12,19)$,
$(0,21,2,11,23)$, $(0,22,3,6,28)$, $(0,29,1,14,33)$, $(0,34,1,16,39)$,
$(1,19,2,12,27)$, $(1,28,2,13,39)$, $(2,14,3,7,29)$, $(2,33,3,9,39)$,
$(3,18,9,14,39)$.

The leave is $\{\{0,10\},\{1,18\},\{2,5\},\{3,17\},\{4,15\},\{6,16\},\{7,12\},\{8,19\},\{9,14\},\{11,13\}\}$ for $n = 4$, and the leave is $\{\{i, 5n/2+i\} : 0 \le i \le 5n/2-1\}$ for $n \in \{6,8\}$. □

Lemma 22 *There exists a C_5-MGDP of type g^n with leave F for any $g \equiv 5 \pmod{10}$, $n \equiv 0 \pmod 2$ and $n \ge 4$.*

Proof For $(g,n) \in \{(5,4),(5,6),(5,8)\}$, we construct the required C_5-MGDPs directly in Lemma 21.

For $(g,n) = (5,10)$, by Lemma 8, there exists a C_5-HGDD of type $(10, 1^5)$. Apply Construction 6 with a C_5-MGDP of type 1^{10} with leave $F(1,10)$ from Lemma 3 to obtain a C_5-MGDP of type 5^{10} with leave $F(5,10)$.

For $g = 5$, $n \equiv 0 \pmod 4$ and $n \ge 12$, start from a C_5-GDD of type $20^{n/4}$, which exists by Lemma 1. Apply Construction 15 with a C_5-IGDP* of type $5^{(4,0)}$ (which is the above C_5-MGDP of type 5^4 with leave $F(5,4)$) to obtain a C_5-MGDP of type 5^n with leave $F(5,n)$.

For $g = 5$, $n \equiv 2 \pmod 4$ and $n \geq 14$, apply Construction 13 with a C_5-IGDP* of type $5^{(n,6)}$ (from Lemma 16) and a C_5-MGDP of type 5^6 with leave $F(5,6)$ to obtain a C_5-MGDP of type 5^n with leave $F(5,n)$.

For $g \geq 15$, by Lemma 12, there exists a C_5-HGDD of type $(n, 5^{g/5})$. Filling in the holes with the above C_5-MGDP of type 5^n with leave $F(5,n)$, we have a C_5-MGDP of type g^n with leave $F(g,n)$ by Construction 6. □

Lemma 23 *There exists a C_5-MGDP of type g^n with leave F for any $g \equiv 3 \pmod{10}$ and $n \equiv 8 \pmod{10}$.*

Proof For $(g,n) = (3,8)$, we construct a C_5-MGDP of type 3^8 with leave $F(3,8)$ on Z_{24}, where the group set is $\{\{i, 8+i, 16+i\} : 0 \leq i \leq 7\}$. All the blocks can be obtained by developing $(0,1,3,6,10)$ and $(0,5,11,4,13)$ by $+1$ mod 24. Here the leave is $\{\{i, 12+i\} : 0 \leq i \leq 11\}$.

For $g = 3$, $n \equiv 8 \pmod{10}$ and $n \geq 18$, applying Construction 13 with a C_5-IGDP* of type $3^{(n,8)}$ from Lemma 17 and the above C_5-MGDP of type 3^8 with leave $F(3,8)$, we have a C_5-MGDP of type 3^n with leave $F(3,n)$.

When $g \geq 13$, let $g = 5t+3$. Then $t \equiv 0 \pmod 2$. By Lemma 11, there exists a C_5-HGDD of type $(n, 5^t 3^1)$. Filling in the holes with a C_5-MGDP of type 5^n with leave $F(5,n)$ from Lemma 22 and a C_5-MGDP of type 3^n with leave $F(3,n)$, we have a C_5-MGDP of type g^n with leave $F(g,n)$ by Construction 6. □

Lemma 24 *There exists a C_5-MGDP of type g^n with leave F for any $g \equiv 7 \pmod{10}$ and $n \equiv 4 \pmod{10}$.*

Proof For $(g,n) = (7,4)$, we construct a C_5-MGDP of type 7^4 with leave $F(7,4)$ on Z_{28}, where the group set is $\{\{0,4,\cdots,24\} + i : 0 \leq i \leq 3\}$. All the blocks can be obtained by developing the following eight base blocks by $+4$ mod 28.

$$(0,1,2,3,5), \quad (0,9,2,8,19), \quad (0,6,1,7,10), \quad (0,17,26,7,18),$$
$$(0,2,4,1,3), \quad (0,7,2,5,15), \quad (0,13,3,10,21), \quad (0,23,1,14,27).$$

Here the leave is $\{\{i, 14+i\} : 0 \leq i \leq 13\}$.

For $g = 7$, $n \equiv 4 \pmod{10}$ and $n \geq 14$, applying Construction 13 with a C_5-IGDP* of type $7^{(n,4)}$ from Lemma 18 and the above C_5-MGDP of type 7^4 with leave $F(7,4)$, we have a C_5-MGDP of type 7^n with leave $F(7,n)$.

When $g \geq 17$, let $g = 5t + 7$. Clearly, $t \equiv 0 \pmod 2$. By Lemma 11, there exists a C_5-HGDD of type $(n, 5^t 7^1)$. Filling in the holes with a C_5-MGDP of type 5^n with leave $F(5,n)$ from Lemma 22 and a C_5-MGDP of type 7^n with leave $F(7,n)$, we have a C_5-MGDP of type g^n with leave $F(g,n)$ by Construction 6. □

Proof of Theorem 5: The conclusion follows from Lemmas 19-24.

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