Signed total Italian domination in graphs

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Abstract

A *signed total Italian dominating function* (STIDF) of a graph G with vertex set $V(G)$ is defined as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that (i) $\sum_{x \in N(v)} f(x) \geq 1$ for each $v \in V(G)$, where $N(v)$ is the neighborhood of v, and (ii) every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$ or adjacent to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIDF is the sum of its function values over all vertices. The *signed total Italian domination number* of G, denoted by $\gamma_{stI}(G)$, is the minimum weight of an STIDF in G. We initiate the study of the signed total Italian domination number, and we present different sharp bounds on $\gamma_{stI}(G)$. In addition, we determine the signed total Italian domination number of some classes of graphs.

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1 Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. A graph G is *regular* or *r-regular* if $\delta(G) = \Delta(G) = r$. For a subset $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph of G induced by X. Let K_n be the complete graph of order n, C_n the cycle of order n, P_n the path of order n, and $K_{p,q}$ the complete bipartite graph with partite sets X and Y, where $|X| = p$ and $|Y| = q$. Let $S(r, s)$ be the *double star* with exactly two adjacent vertices u and v that are not leaves such that u is adjacent to $r \geq 1$ leaves and v is adjacent to $s \geq 1$ leaves.

A set D of vertices of G is called by Cockayne, Dawes and Hedetniemi [2] a *total dominating set* if each vertex in $V(G)$ is adjacent to some vertex of D. The *total domination number* $\gamma_t(G)$ equals the minimum cardinality of a total dominating set in G. We note that this parameter is only defined for graphs without isolated vertices. Total domination is very well studied in the literature. For more details on total domination, the reader is referred to the two domination books by Haynes, Hedetniemi and Slater [4, 5], the survey article on total domination by Henning [6] and the book on total domination by Henning and Yeo [9].

A *signed total Roman dominating function* (STRDF) on a graph G is defined in [10] as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that $f(N(v)) = \sum_{x \in N(v)} f(x) \geq 1$ for each $v \in V(G)$ and if $f(u) = -1$, then the vertex u must have a neighbor w with $f(w) = 2$. The weight of a signed total Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *signed total Roman domination number* $\gamma_{stR}(G)$ is the minimum weight of a signed total Roman dominating function on G.

A *signed total Italian dominating function* (STIDF) of a graph G is defined as a function $f : V(G) \to \{-1, 1, 2\}$ having the property that (i) $f(N(v)) \geq 1$ for each $v \in V(G)$ and (ii) every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$ or adjacent to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIDF f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed total Italian domination number* of G, denoted by $\gamma_{stI}(G)$, is the minimum weight of an STIDF in G. A $\gamma_{stI}(G)$-function is an STIDF of weight $\gamma_{stI}(G)$. For an STIDF f on G, let $V_i = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. An STIDF f can be represented by the ordered partition $f = (V_{-1}, V_1, V_2)$.

The signed total Roman and signed total Italian domination numbers are well-defined for graphs G without isolated vertices, since the function $f : V(G) \to \{-1, 1, 2\}$ with $f(x) = 1$ for each $x \in V(G)$ is an STRDF as well as an STIDF. Thus we assume throughout this paper that $\delta(G) \geq 1$. The definitions lead to $\gamma_{stI}(G) \leq \gamma_{stR}(G) \leq n(G)$. Therefore each lower bound of $\gamma_{stI}(G)$ is also a lower bound of $\gamma_{stR}(G)$.

In this paper we continue the study of signed (total) Roman (Italian) domination in graphs (see, for example, [1, 3, 7, 8, 10, 11, 12]). Our purpose in this work is to initiate the study of the signed total Italian domination number. We present basic properties and sharp bounds for the signed total Italian domination number of a graph. In particular, we show that many lower bounds on $\gamma_{stR}(G)$ are also valid for $\gamma_{stI}(G)$. In addition, we prove $\gamma_{stI}(G) \geq (11n-12m)/4$ for connected graphs G of order n and size m, and we characterize the graphs achieving equality. Furthermore, we show that the difference $\gamma_{stR}(G) - \gamma_{stI}(G)$ can be arbitrarily large, and we determine the signed total Italian domination number of some classes of graphs.

2 Preliminary results and first bounds

In this section we present basic properties and some first bounds on the signed total Italian domination number.

Observation 1. If $f = (V_{-1}, V_1, V_2)$ is an STIDF of a graph G of order n with $\delta(G) \geq 1$, then the following holds.

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) $V_1 \cup V_2$ is a total dominating set of G.

Proof. Since (a) and (b) are immediate, we only prove (c). By the definition, each vertex of V_{-1} is adjacent to a vertex of $V_1 \cup V_2$. Suppose that $G[V_1 \cup V_2]$ has an isolated vertex v. As $\delta(G) \geq 1$, the vertex v is adjacent to a vertex in V_{-1} and all its neighbors are in V_{-1}. This leads to the contradiction $f(N(v)) \leq -1$. Therefore $G[V_1 \cup V_2]$ does not contain an isolated vertex and hence $V_1 \cup V_2$ is a total dominating set of G. □

Proposition 2. If G is graph of order n with minimum degree $\delta \geq 3$, then $\gamma_{stI}(G) \leq n - 2\lfloor(\delta-1)/2\rfloor$.

Proof. Let $t = \lfloor(\delta-1)/2\rfloor$, and let $A = \{v_1, v_2, \ldots, v_t\}$ be a set of t vertices of G. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(x) = -1$ for $x \in A$ and $f(x) = 1$ for $x \in V(G) \setminus A$. Then

$$f(N(w)) \geq -t + (\delta - t) = \delta - 2t = \delta - 2\lfloor(\delta-1)/2\rfloor \geq 1$$

for each $w \in V(G)$. Since $\delta \geq 3$, we observe that every vertex is adjacent to at least two vertices of weight 1. Therefore f is an STIDF on G of weight $n - 2t$ and thus $\gamma_{stI}(G) \leq n - 2t$. □

Proposition 3. If G is graph of order n with $\delta(G) \geq 1$, then

$$\gamma_{stI}(G) \geq \max\{\Delta + 1 - n, \delta(G) + 3 - n\}.$$

Proof. Let let f be a $\gamma_{stI}(G)$-function. If $f(x) = 1$ for all $x \in V(G)$, then $\gamma_{stI}(G) = n \geq \max\{\Delta+1-n, \delta(G)+3-n\}$. Now assume that there exists a vertex u with $f(u) = -1$. Then u has a neighbor w with $f(w) \geq 1$, and it follows that

$$\begin{aligned}\gamma_{stI}(G) &= \sum_{x\in V(G)} f(x) = f(w) + \sum_{x\in N(w)} f(x) + \sum_{x\in V(G)\setminus N[w]} f(x)\\ &\geq 1+1+\sum_{x\in V(G)\setminus N[w]} f(x) \geq 2-(n-(d(w)+1))\\ &\geq 3+\delta(G)-n.\end{aligned}$$

If w is a vertex of maximum degree, then we have

$$\begin{aligned}\gamma_{stI}(G) &= \sum_{x\in V(G)} f(x) = f(w) + \sum_{x\in N(w)} f(x) + \sum_{x\in V(G)\setminus N[w]} f(x)\\ &\geq -1+1+\sum_{x\in V(G)\setminus N[w]} f(x) \geq -(n-(\Delta(G)+1))\\ &= \Delta(G)+1-n,\end{aligned}$$

and the proof is complete. □

Theorem 4. If G is graph of order n with $\delta(G) \geq 1$, then $\gamma_{stI}(G) \geq 2\gamma_t(G) - n$, with equality if and only if $G = sK_2$ for an integer $s \geq 1$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}(G)$-function. Then it follows from Observation 1 that

$$\gamma_{stI}(G) = |V_1|+2|V_2|-|V_{-1}| = 2|V_1|+3|V_2|-n \geq 2|V_1\cup V_2|-n \geq 2\gamma_t(G)-n, \quad (1)$$

and the disired inequality is proved. Clearly, if $G = sK_2$ for an integer $s \geq 1$, then $\gamma_t(G) = n$ and $\gamma_{stI}(G) = n$ and so $\gamma_{stI}(G) = 2\gamma_t(G) - n$.

If $|V_2| \geq 1$, then it follows from (1) that

$$\gamma_{stI}(G) = 2|V_1| + 3|V_2| - n > 2|V_1 \cup V_2| - n \geq 2\gamma_t(G) - n.$$

Therefore assume now that $|V_2| = 0$. If $|V_{-1}| = 0$, then $V_1 = V(G)$ and thus $\gamma_{stI}(G) = n$. Clearly, $\gamma_t(G) \leq n$ and $\gamma_t(G) = n$ if and only if $G = sK_2$ for an integer $s \geq 1$ (see [2]). Consequently, $\gamma_{stI}(G) = 2\gamma_t(G) - n$ if and only if $G = sK_2$ for an integer $s \geq 1$ in this case.

Finally assume that $|V_{-1}| \geq 1$. Let $u \in V_{-1}$, and let $x, y \in V_1$ be two neighbors of u. The condition $f(N(x)) \geq 1$ shows that x has at least two neighbors in $V_1 \setminus \{x\}$. If $V_1 \setminus \{x\}$ does not have an isolated vertex, then $V_1 \setminus \{x\}$ is a total dominating set of G, and (1) implies $\gamma_{stI}(G) = 2|V_1| - n > 2\gamma_t(G) - n$. Next assume that $V_1 \setminus \{x\}$ has an isolated vertex z. If z has a neighbor in V_{-1}, then we obtain the contradiction $f(N(z)) \leq 0$. Hence z is a leaf in G, and we observe that $V_1 \setminus \{z\}$ is a total dominating set of G. Again (1) leads to $\gamma_{stI}(G) = 2|V_1| - n > 2\gamma_t(G) - n$. □

The proof of the next proposition is identically with the proof of Proposition 8 in [10] and is therefore omitted.

Proposition 5. Let $f = (V_{-1}, V_1, V_2)$ be an STIDF of a graph G of order n, $\Delta = \Delta(G)$ and $\delta = \delta(G) \geq 1$. Then the following holds.

(a) $(2\Delta - 1)|V_2| + (\Delta - 1)|V_1| \geq (\delta + 1)|V_{-1}|$.

(b) $(2\Delta + \delta)|V_2| + (\Delta + \delta)|V_1| \geq (\delta + 1)n$.

(c) $(\Delta + \delta)\omega(f) \geq (\delta - \Delta + 2)n + (\delta - \Delta)|V_2|$.

(d) $\omega(f) \geq (\delta - 2\Delta + 2)n/(2\Delta + \delta) + |V_2|$.

As an immediate consequence of Proposition 5 (c), we obtain a lower bound on the signed total Italian domination number of regular graphs.

Corollary 6. If G is an r-regular graph of order n with $r \geq 1$, then $\gamma_{stI}(G) \geq \lceil n/r \rceil$.

In the case that G is not regular, Proposition 5 (c) and (d) lead to the following lower bound.

Corollary 7. Let G be a graph of order n, maximum degree Δ and minimum degree $\delta \geq 1$. If $\delta < \Delta$, then

$$\gamma_{stI}(G) \geq \frac{-2\Delta + 2\delta + 3}{2\Delta + \delta} n.$$

Proof. Multiplying both sides of the inequality in Proposition 5 (d) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 5 (c), we yield the desired lower bound. □

The next example shows that Corollary 7 is sharp.

Example 8. Let $p \geq 3$ be an integer, and let $v_1, v_2, \ldots, v_p$ be the vertex set of the complete graph K_p. Now let H be the graph consisting of K_p and the $p(2p-4)$ new vertices $w_i^1, w_i^2, \ldots, w_i^{2p-4}$ for $1 \leq i \leq p$ such that v_i is adjacent to the vertices $w_i^1, w_i^2, \ldots, w_i^{2p-4}$ for $1 \leq i \leq p$ and $w_i^1 w_i^2, w_i^3 w_i^4, \ldots, w_i^{2p-5} w_i^{2p-4} \in E(H)$ for $1 \leq i \leq p$. Now define $f : V(H) \to \{-1, 1, 2\}$ by $f(v_i) = 2$ for $1 \leq i \leq p$ and $f(x) = -1$ otherwise. Then f is an STIDF on H of weight $6p - 2p^2$ and thus $\gamma_{stI}(H) \leq 6p - 2p^2$.

Since $n(H) = (2p-3)p$, $\Delta(H) = 3p-5$ and $\delta(H) = 2$, it follows from Corollary 7 that

$$\begin{aligned}\gamma_{stI}(H) &\geq \left\lceil \frac{-2\Delta(H) + 2\delta(H) + 3}{2\Delta(H) + \delta(H)} n(H) \right\rceil \\ &= \left\lceil \frac{(17-6p)p(2p-3)}{6p-8} \right\rceil = 6p - 2p^2.\end{aligned}$$

3 Special classes of graphs

In this section, we determine the signed total italian domination number for special classes of graphs.

Proposition 9. If $n \geq 2$, then $\gamma_{stI}(K_n) = 2$ when n is even and $\gamma_{stI}(K_n) = 3$ when n is odd.

Proof. According to Proposition 3, we have $\gamma_{stI}(K_n) \geq 2$. If n is even, then assign to $(n+2)/2$ vertices the weight 1 and to the remaining $(n-2)/2$ vertices the weight -1. This is an STIDF of weight 2 and so $\gamma_{stI}(K_n) \leq 2$. We deduce that $\gamma_{stI}(K_n) = 2$ when n is even.

Let now $n = 2p+1$ with $p \geq 1$ odd, and let f be a $\gamma_{stI}(K_n)$-function. If $f(x) = 1$ for all $x \in V(K_n)$, then $\omega(f) = n \geq 3$. Let now $f(w) = -1$ for at least one vertex $w \in V(K_n)$. If there exist a vertex u with $f(u) = 2$, then $\omega(f) = f(u) + f(N(u)) \geq 3$. Next assume that $f(x) = 1$ or $f(x) = -1$ for each $x \in V(K_n)$, and let $f(u) = 1$. Since $f(N(u)) \geq 1$ and $|N(u)|$ is even, we deduce that $f(N(u))$ is even and therefore $\omega(f) = f(u) + f(N(u)) \geq 1 + 2 = 3$. Conversely, assign to $p+2$ vertices the weight 1 and to the remaining $p-1$ vertices the weight -1. This is an STIDF of weight 3 and so $\gamma_{stI}(K_n) \leq 3$. We deduce that $\gamma_{stI}(K_n) = 3$ when n is odd. □

For even n, Proposition 9 shows that Proposition 3 is sharp.

Proposition 10. If $n \geq 3$, then $\gamma_{stI}(K_{1,n-1}) = 3$.

Proof. Let $G = K_{1,n-1}$, and let f be a $\gamma_{stI}(G)$-function. If w is the central vertex of the star G, then clearly $f(w) \geq 1$. If $f(w) = 1$, then $\gamma_{stI}(G) = n \geq 3$. If $f(w) = 2$, then $\gamma_{stI}(G) = f(w) + f(N(w)) \geq 3$. In addition, it follows from Example 1 in [10] that $\gamma_{stI}(G) \leq \gamma_{stR}(G) = 3$ and thus $\gamma_{stI}(K_{1,n-1}) = 3$. □

Proposition 11. If $p, q \geq 2$ are integers, then $\gamma_{stI}(K_{p,q}) = 2$.

Proof. Let $G = K_{p,q}$, and let f be a $\gamma_{stI}(G)$-function. In addition, let X, Y be a bipartition of G. If $x \in X$ and $y \in Y$, then $\gamma_{stI}(G) = f(V(G)) = f(N(x)) + f(N(y)) \geq 2$.

Conversely, assume that $p = |X|$ and $q = |Y|$ are even. Assign to $p/2$ vertices the weight -1, to one vertex the weight 2 and to the remaining $(p-2)/2$ vertices of X the weight 1. In addition, assign to $q/2$ vertices the weight -1, to one vertex the weight 2 and to the remaiming $(q-2)/2$ vertices of Y the weight 1. This produces an STIDF on G of weight 2 and thus $\gamma_{stI}(G) = 2$ in this case.

Next assume that $|X| = 2t+1$ and $|Y| = 2s+1$ are odd. Assign to t vertices the weight -1 and to the remaiming $t+1$ vertices of X the weight 1. In addition, assign to s vertices the weight -1 and to the remaiming $s+1$ vertices of Y the weight 1. This produces an STIDF on G of weight 2 and thus $\gamma_{stI}(G) = 2$ in this case.

The cases p even and q odd as well as p odd and q even are analogously, and are therefore omitted. □

Proposition 12. If $S(r,s)$ is the double star such that $r,s \geq 3$, then $\gamma_{stI}(S(r,s)) = 2$.

Proof. Let u and v be two adjacent vertices of $S(r,s)$ such that u is adjacent to r leaves and v is adjacent to s leaves. If g is a $\gamma_{stI}(S(r,s))$-function, then the definition implies $\gamma_{stI}(S(r,s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 2$.

Conversely, let $r = 2p+1$ and $s = 2q+1$ be odd. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight -1 to $p+1$ leaves of u, the weight 1 to p leaves of u, the weight -1 to $q+1$ leaves of v and the weight 1 to q leaves of v. Then f is an STIDF on $S(r,s)$ of weight 2 and thus $\gamma_{stI}(S(r,s)) \leq 2$. Therefore $\gamma_{stI}(S(r,s)) = 2$ in this case.

Let $r = 2p$ and $s = 2q$ be even with $p,q \geq 2$. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight -1 to $p+1$ leaves of u, the weight 1 to $p-2$ leaves of u, the weight 2 to one leaf of u, the weight -1 to $q+1$ leaves of v, the weight 1 to $q-2$ leaves of v and the weight 2 to one leaf of v. Then f is an STIDF on $S(r,s)$ of weight 2 and thus $\gamma_{stI}(S(r,s)) \leq 2$. Therefore $\gamma_{stI}(S(r,s)) = 2$ in this case.

The cases r even and s odd or r odd and s even are similar to the cases above and are therefore omitted. □

Similar to the proof of Proposition 12, one can show that $\gamma_{stI}(S(1,s)) = 2$ for $s = 1$ or $s \geq 3$.

Proposition 13. If $S(2,s)$ is the double star such that $s \geq 3$, then $\gamma_{stI}(S(2,s)) = 3$.

Proof. Let u and v be two adjacent vertices of $S(2,s)$ such that u is adjacent to the leaves x and y, and v is adjacent to s leaves. If g is a $\gamma_{stI}(S(r,s))$-function, then we observe first that $g(u), g(v) \geq 1$. If $g(u) = 1$, then $g(x), g(y) \geq 1$, and we obtain $\gamma_{stI}(S(2,s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 3+1 = 4$. If $g(v) = 1$, then $g(w) \geq 1$ for all leaves adjacent to v, and we obtain $\gamma_{stI}(S(2,s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 1 + s + 1 \geq 5$. Assume now that $g(u) = g(v) = 2$. Then we observe that $g(x) + g(y) \geq 0$ and thus $g(N(u)) \geq 2$, and so $\gamma_{stI}(S(2,s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 2+1 = 3$.

Conversely, let $s = 2q+1$ be odd. Define f by $f(u) = f(v) = 2$, $f(x) = 1$ and $f(y) = -1$. In addition, we assign the weight -1 to $q+1$ leaves of v, and the weight 1 to q leaves of v. Then f is an STIDF on $S(2,s)$ of weight 3 and thus $\gamma_{stI}(S(2,s)) \leq 3$. Therefore $\gamma_{stI}(S(2,s)) = 3$ in this case.

Let $s = 2q$ be even with $q \geq 2$. Define f by $f(u) = f(v) = 2$, $f(x) = 1$ and $f(y) = -1$. In addition, we assign the weight -1 to $q+1$ leaves of v, the weight 1 to $q-2$ leaves of v and the weight 2 to one leaf of v. Then f is an STIDF on $S(2,s)$ of weight 3 and thus $\gamma_{stI}(S(2,s)) \leq 3$. Therefore $\gamma_{stI}(S(2,s)) = 3$ in this case. □

The proof of Propositions 12 and 13 demonstrate that $\gamma_{stR}(S(r,s)) = 2$ for $r,s \geq 3$, $\gamma_{stR}(S(1,s)) = 2$ for $s = 1$ or $s \geq 3$ and $\gamma_{stR}(S(2,s)) = 3$ for $s \geq 3$. For the sake of completeness, we note that $\gamma_{stI}(S(1,2)) = \gamma_{stR}(S(1,2)) = 3$ and $\gamma_{stI}(S(2,2)) = \gamma_{stR}(S(2,2)) = 4$.

The next lemma is easy to prove but useful.

Lemma 14. Let G be a graph without isolated vertices, and let f be an STIDF on G. If $v_1v_2v_3v_4$ is a path of G with $d(v_2) = d(v_3) = 2$, then $f(v_1) + f(v_2) + f(v_3) + f(v_4) \geq 2$.

Proof. Since f is an STIDF on G and $d(v_2) = d(v_3) = 2$, we observe that $f(v_1) + f(v_2) + f(v_3) + f(v_4) = f(N(v_2)) + f(N(v_3)) \geq 2$. □

Proposition 15. If C_n is a cycle of length $n \geq 3$, then $\gamma_{stI}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stI}(C_n) = (n+3)/2$ when $n \equiv 1,3 \pmod 4$ and $\gamma_{stI}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.

Proof. Let $C_n = v_1v_2\ldots v_nv_1$, and let f be a $\gamma_{stI}(C_n)$-function.

Assume first that $n \equiv 0 \pmod 4$. Applying Corollary 6, we observe that $\gamma_{stI}(C_n) \geq n/2$. Conversely, it follows from [10] that $\gamma_{stI}(C_n) \leq \gamma_{stR}(C_n) = n/2$ and thus $\gamma_{stI}(C_n) = n/2$ in this case.

Assume second that $n \equiv 1 \pmod 4$. Let $n = 4t+1$ for an integer $t \geq 1$. If $f(v_i) \geq 1$ for all $1 \leq i \leq n$, then $\gamma_{stI}(C_n) \geq n \geq (n+3)/2$. Hence assume now, without loss of generality, that $f(v_2) = -1$. It follows that $f(v_{4t+1}) = 2$. Using Lemma 14, we obtain

$$\begin{aligned}\gamma_{stI}(C_n) &= f(v_{4t+1}) \\ &+ \sum_{i=0}^{t-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) \\ &\geq 2 + 2t = \frac{n+3}{2}.\end{aligned}$$

Conversely, we deduce from [10] that $\gamma_{stI}(C_n) \le \gamma_{stR}(C_n) = (n+3)/2$ and thus $\gamma_{stI}(C_n) = (n+3)/2$ in this case.

Assume third that $n \equiv 2 \pmod 4$. Let $n = 4t+2$ for an integer $t \ge 1$. If $f(v_i) \ge 1$ for all $1 \le i \le n$, then $\gamma_{stI}(C_n) \ge n \ge (n+6)/2$. Hence assume now, without loss of generality, that $f(v_2) = -1$. It follows that $f(v_{4t+2}) = f(v_4) = 2$. If $f(v_1) = 2$ or $f(v_{4t+1}) = 2$, say $f(v_{4t+1}) = 2$, then we deduce from Lemma 14 that

$$\begin{aligned}
\gamma_{stI}(C_n) &= f(v_{4t+2}) + f(v_{4t+1}) \\
&+ \sum_{i=0}^{t-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) \\
&\ge 4 + 2t = \frac{n+6}{2}.
\end{aligned}$$

Hence assume that $f(v_1), f(v_{4t+1}) \le 1$. This implies $f(v_1) = f(v_{4t+1}) = 1$ and $f(v_3) \ge 1$. Therefore Lemma 14 yields to

$$\begin{aligned}
\gamma_{stI}(C_n) &= f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_{4t+2}) + f(v_{4t+1}) \\
&+ \sum_{i=1}^{t-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) \\
&\ge 6 + 2(t-1) = \frac{n+6}{2}.
\end{aligned}$$

Otherwise, we deduce from [10] that $\gamma_{stI}(C_n) \le \gamma_{stR}(C_n) = (n+6)/2$ and thus $\gamma_{stI}(C_n) = (n+6)/2$ in this case.

Finally, assume that $n \equiv 3 \pmod 4$. Let $n = 4t+3$ for an integer $t \ge 0$. If $f(v_i) \ge 1$ for all $1 \le i \le n$, then $\gamma_{stI}(C_n) \ge n \ge (n+3)/2$. Hence assume now, without loss of generality, that $f(v_2) = -1$. It follows that $f(v_{4t+3}) = 2$. If $f(v_1) = 2$ or $f(v_{4t+2}) = 2$, say $f(v_{4t+2}) = 2$, then we deduce from Lemma 14 that

$$\begin{aligned}
\gamma_{stI}(C_n) &= f(v_{4t+3}) + f(v_{4t+2}) + f(v_{4t+1}) \\
&+ \sum_{i=0}^{t-1}(f(v_{4i+1}) + f(v_{4i+2}) + f(v_{4i+3}) + f(v_{4i+4})) \\
&\ge 3 + 2t = \frac{n+3}{2}.
\end{aligned}$$

Hence assume next that $f(v_1), f(v_{4t+2}) \le 1$. Then $f(v_1) = f(v_{4t+2}) = 1$, and therefore Lemma 14 leads to

$$\begin{aligned}\gamma_{stI}(C_n) &= f(v_1)+f(v_{4t+3})+f(v_{4t+2})\\ &+ \sum_{i=1}^{t}(f(v_{4i-2})+f(v_{4i-1})+f(v_{4i})+f(v_{4i+1}))\\ &\geq 4+2t=\frac{n+5}{2}.\end{aligned}$$

In addition, it follows from [10] that $\gamma_{stI}(C_n) \leq \gamma_{stR}(C_n) = (n+3)/2$ and thus $\gamma_{stI}(C_n) = (n+3)/2$ in this case. This completes the proof. □

Analogously to the proof of Example 6 in [10], one can determine the signed total Italian domination number of paths.

Proposition 16. Let P_n be a path of order $n \geq 3$. Then $\gamma_{stI}(P_n) = n/2$ when $n \equiv 0 \pmod 4$ and $\gamma_{stI}(P_n) = \lceil (n+3)/2 \rceil$ otherwise.

If G is 1-regular of order n, then $\gamma_{stI}(G) = n$. Corollary 6 implies $\gamma_{stI}(G) \geq \lceil n/2 \rceil$ when G is 2-regular, and it follows from Proposition 15 that $\gamma_{stI}(C_n) = n/2$ when $n \equiv 0 \pmod 4$. Therefore Corollary 6 is tight if $r = 1, 2$. By Proposition 9, the lower bound of Corollary 6 is also tight if $r = n-1$. Proposition 11 implies $\gamma_{stI}(K_{p,p}) = 2$, and thus Corollary 6 is tight for $r = n/2$. Next we will show that Corollary 6 is tight for $r \geq n/2$.

Example 17. Let H be the complete k-partite graph with $k \geq 2$ and the partite sets $X_1, X_2, \ldots, X_k$ such that $|X_i| = s \geq 2$ for $1 \leq i \leq k$. Then H is an $(n-s)$-regular graph of order $n = ks$. Corollary 6 implies $\gamma_{stI}(H) \geq \lceil n/(n-s) \rceil = 2$.

First let $s = 2p$ be even. We assign the weight 2 to one vertex, the weight -1 to p vertices and the weight 1 to the remaining $p-1$ vertices of X_1 and X_2. In addition, we assign the weight -1 to p vertices and the weight 1 to the remaining p vertices of X_i for $3 \leq i \leq k$. This is an STIDF on H of weight 2 and thus $\gamma_{stI}(H) \leq 2$ and so $\gamma_{stI}(H) = 2$ in this case.

Second let $s = 2p+1$ be odd. We assign the weight -1 to p vertices and the weight 1 to the remaining $p+1$ vertices of X_1 and X_2. In addition, we assign the weight 2 to one vertex, the weight -1 to $p+1$ vertices and the weight 1 to the remaining $p-1$ vertices of X_i for $3 \leq i \leq k$. This is an STIDF on H of weight 2 and thus $\gamma_{stI}(H) \leq 2$ and so $\gamma_{stI}(H) = 2$ in the odd case.

The next example will demonstrate that the difference $\gamma_{stR}(G) - \gamma_{stI}(G)$ can be arbitrarily large.

Example 18. Let F be an arbitrary graph of order $t \geq 1$, and for each vertex $v \in V(F)$ add a vertex-disjoint copy of a complete graph K_s with

$s \geq 6$ even and identify the vertex v with one vertex of the added complete graph. Let H be the resulting graph. Furthermore, let $H_1, H_2, \ldots, H_t$ be the added copies of K_s. For $i = 1, 2, \ldots, t$, let v_i be the vertex of H_i that is identified with a vertex of F.

First we construct an STIDF on H as follows. For each $i = 1, 2, \ldots, t$, let $f_i : V(H_i) \to \{-1, 1, 2\}$ be the STIDF on the complete graph defined as in Proposition 9 such that $f_i(v_i) \geq 1$. As shown in Proposition 9, we have $\omega(f_i) = 2$. Now let $f : V(H) \to \{-1, 1, 2\}$ be the function defined by $f(v) = f_i(v)$ for each $v \in V(H_i)$. Then f is an STIDF of H of weight $2t$ and hence $\gamma_{stI}(H) \leq 2t$.

Now let g be a $\gamma_{stR}(H)$-function. We show that $g(V(H_i)) \geq 3$ for each $1 \leq i \leq t$. If $g(x) = -1$ for at most one $x \in V(H_i)$, then $g(V(H_i)) \geq s - 2 \geq 4$. Hence assume that there exist at least two vertices $x, y \in V(H_i)$ such that $g(x) = g(y) = -1$. This implies that there exixsts a vertex $w \in V(H_i)$ with $g(w) = 2$. If $w \neq v_i$, then we deduce that $g(V(H_i)) = g(w) + g(N(w)) \geq 3$. Next assume that $w = v_i$ and $g(x) = 1$ or $g(x) = -1$ for $x \in V(H_i) \setminus \{w\}$. If z is a vertex in $V(H_i)$ with $g(z) = 1$, then assume that z has exactly j neighbors of weight 1 and $s - j - 2$ neighbors of weight -1. We deduce that $g(N(z)) = 2 + j - (s - j - 2) = 4 + 2j - s \geq 1$, and since s is even, it follows that $g(N(z)) = 4 + 2j - s \geq 2$. Thus $g(V(H_i)) = g(z) + g(N(z)) \geq 3$, and we obtain $\gamma_{stR}(H) = g(V(H)) = \sum_{i=1}^{t} g(V(H_i)) \geq 3t$. Consequently, we see that $\gamma_{stR}(H) - \gamma_{stI}(H) \geq 3t - 2t = t$.

4 A lower bound in terms of order and size

For a subset $S \subseteq V(G)$, we let $d_S(v)$ denote the number of vertices in S that are adjacent to the vertex v. For disjoint subsets U and W of vertices, we let $[U, W]$ denote the set of edges between U and W. Now let $f = (V_{-1}, V_1, V_2)$ be an STIDF. For notational convenience, we let $V_{12} = V_1 \cup V_2$, $|V_{12}| = n_{12}$, $|V_1| = n_1$ and $|V_2| = n_2$. Furthermore, let $|V_{-1}| = n_{-1}$ and so $n_{-1} = n - n_{12}$. Let $G_{12} = G[V_{12}]$ be the subgraph induced by V_{12} and let G_{12} have size m_{12}. For $i = 1, 2$, if $V_i \neq \emptyset$, let $G_i = G[V_i]$ be the subgraph induced by V_i and let G_i have size m_i. Hence $m_{12} = m_1 + m_2 + |[V_1, V_2]|$.

For $k \geq 2$, let L_k be the graph obtained from a connected graph H of order k by adding $2d_H(v) - 1$ pendant edges to each v of H. Let $\mathcal{F} = \{L_k \mid k \geq 2\}$.

Theorem 19. If G is a connected graph of order $n \geq 3$ and size m, then

$$\gamma_{stI}(G) \geq \frac{11n - 12m}{4}$$

with equality if and only if $G \in \mathcal{F}$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}(G)$-function, and let $V_{-1}^2 \subseteq V_{-1}$ be the maximum set such that each vertex $v \in V_{-1}^2$ has at least one neighbor in V_2. In addition, let $V_{-1}^1 = V_{-1} \setminus V_{-1}^2$. Since each vertex of V_{-1}^1 has at least two neighbors in V_1, we observe that

$$2|V_{-1}^1| \leq |[V_{-1}^1, V_1]| = \sum_{v \in V_1} d_{V_{-1}^1}(v).$$

For each $v \in V_1$, we have $1 \leq f(N(v)) = 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v)$ and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v) - 1$. Hence we obtain

$$\begin{aligned} 2|V_{-1}^1| &\leq \sum_{v \in V_1} d_{V_{-1}^1}(v) \leq \sum_{v \in V_1} d_{V_{-1}}(v) \\ &\leq \sum_{v \in V_1} (2d_{V_2}(v) + d_{V_1}(v) - 1) = 2|[V_1, V_2]| + 2m_1 - n_1. \end{aligned}$$

Since each vertex of V_{-1}^2 has at least one neighbor in V_2, we have

$$|V_{-1}^2| \leq |[V_{-1}^2, V_2]| = \sum_{v \in V_2} d_{V_{-1}^2}(v).$$

For each $v \in V_2$, we have $1 \leq f(N(v)) = 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v)$ and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v) - 1$. This leads to

$$\begin{aligned} |V_{-1}^2| &\leq \sum_{v \in V_2} d_{V_{-1}^2}(v) \leq \sum_{v \in V_2} d_{V_{-1}}(v) \\ &\leq \sum_{v \in V_2} (2d_{V_2}(v) + d_{V_1}(v) - 1) = 4m_2 + |[V_1, V_2]| - n_2 \\ &= 4m_{12} - n_2 - 4m_1 - 3|[V_1, V_2]|. \end{aligned}$$

Combining the corresponding inequalities, we obtain

$$\begin{aligned} n_{-1} &= |V_{-1}^1| + |V_{-1}^2| \\ &\leq |[V_1, V_2]| + m_1 - \frac{n_1}{2} + 4m_{12} - n_2 - 4m_1 - 3|[V_1, V_2]| \\ &= 4m_{12} - 3m_1 - 2|[V_1, V_2]| - n_2 - \frac{n_1}{2} \end{aligned}$$

and so $m_{12} \geq (n_{-1} + 3m_1 + 2|[V_1, V_2]| + n_2 + n_1/2)/4$. Hence we deduce that

$$\begin{aligned} m &\geq m_{12} + |[V_{-1}, V_{12}]| \\ &\geq \frac{1}{4}(n_{-1} + n_2 + \frac{n_1}{2} + 3m_1 + 2|[V_1, V_2]|) + n_{-1} \\ &= \frac{1}{4}(5n_{-1} + n_{12} - \frac{n_1}{2} + 3m_1 + 2|[V_1, V_2]|) \\ &= \frac{1}{4}(5n - 4n_{12} - \frac{n_1}{2} + 3m_1 + 2|[V_1, V_2]|). \end{aligned}$$

This yields

$$n_{12} \geq \frac{1}{4}(5n - 4m + 3m_1 + 2|[V_1, V_2]| - \frac{n_1}{2})$$

and thus

$$\begin{aligned} \gamma_{stI}(G) &= 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n = 3n_{12} - n - n_1 \\ &\geq \frac{3}{4}(5n - 4m + 3m_1 + 2|[V_1, V_2]| - \frac{n_1}{2}) - n - n_1 \\ &= \frac{3}{4}(5n - 4m - \frac{4n}{3}) + \frac{3}{4}(3m_1 + 2|[V_1, V_2]| - \frac{11}{6}n_1) \\ &= \frac{11n - 12m}{4} + \frac{3}{4}(3m_1 + 2|[V_1, V_2]| - \frac{11}{6}n_1). \end{aligned}$$

Let $\phi(n_1) = 3m_1 + 2|[V_1, V_2]| - \frac{11}{6}n_1$. It suffices to show that $\phi(n_1) \geq 0$, since then $\gamma_{stI}(G) \geq (11n - 12m)/4$, which is the desired bound. If $n_1 = 0$, then $\phi(n_1) = 0$, and we are done. Assume now that $n_1 \geq 1$. Let $H_1, H_2, \ldots, H_t$ be the components of the induced subgraph $G[V_1]$ of order $h_1, h_2, \ldots, h_t$ and size $p_1, p_2, \ldots, p_t$. Since G is connected, each component H_i contains a vertex adjacent to a vertex of V_2 or to a vertex of V_{-1} for $1 \leq i \leq t$. Assume that $H_1, H_2, \ldots, H_s$ are the components which does not contain a vertex adjacent to a vertex of V_2 and that $H_{s+1}, H_{s+2}, \ldots, H_t$ are the components which contain a vertex adjacent to a vertex in V_2. Let $n_1^1 = h_1 + h_2 + \ldots + h_s$, $n_1^2 = n_1 - n_1^1$, $m_1^1 = p_1 + p_2 + \ldots + p_s$ and $m_1^2 = m_1 - m_1^1$. We observe that $h_i \geq 3$ for $1 \leq i \leq s$ and thus $n_1^1 \geq 3s$. This leads to

$$m_1^1 = p_1 + p_2 + \ldots + p_s \geq (h_1 - 1) + (h_2 - 1) + \ldots + (h_s - 1) = n_1^1 - s \geq \frac{2}{3}n_1^1. \quad (2)$$

In addition, we observe that

$$m_1^2 + |[V_1, V_2]| \geq (h_{s+1} - 1) + (h_{s+2} - 1) + \ldots + (h_t - 1) + (t - s) = n_1^2. \quad (3)$$

Combining the inequalities (2) and (3), we obtain

$$\begin{aligned}\phi(n_1) &= 3m_1 + 2\|[V_1, V_2]\| - \frac{11}{6}n_1 \\ &> 3m_1^1 + 3m_1^2 + 2\|[V_1, V_2]\| - 2n_1^1 - 2n_1^2 \\ &\geq 3m_1^1 - 2n_1^1 + 2m_1^2 + 2\|[V_1, V_2]\| - 2n_1^2 \geq 0,\end{aligned}$$

and so $\gamma_{stI}(G) > (11n - 12m)/4$ when $n_1 \geq 1$.

Suppose that $\gamma_{stI}(G) = (11n - 12m)/4$. Then all the inequalities above must be equalities. In particular, $V_{-1}^2 = V_{-1}$, $n_1 = 0$, $n_{12} = n_2$ and so $V_{12} = V_2$ and $V(G) = V_2 \cup V_{-1}$. Furthermore, $m_{12} = m_2$, $m = m_2 + \|[V_{-1}, V_2]\|$ and $\|[V_{-1}, V_2]\| = n_{-1}$. This implies that for each vertex $v \in V_{-1}$, we have $d_{V_2}(v) = 1$ and thus $d_{V_{-1}}(v) = 0$. Hence each vertex of V_{-1} is a leaf in G. Moreover, the identity $n_{-1} = |V_{-1}^2| = |V_{-1}| = \sum_{v \in V_2}(2d_{V_2}(v) - 1)$ shows that $d_{V_{-1}}(v) = 2d_{V_2}(v) - 1$ for each $v \in V_2$ and therefore $G \in \mathcal{F}$.

Conversely, assume that $G \in \mathcal{F}$. Then $G = L_k$ for some $k \geq 2$. Thus G is obtained from a connected graph H of order k by adding $2d_H(v) - 1$ pendant edges to each vertex v of of H. Then

$$n(G) = n(H) + \sum_{v \in V(H)} (2d_H(v) - 1) = 4m(H)$$

and

$$m(G) = m(H) + \sum_{v \in V(H)} (2d_H(v) - 1) = 5m(H) - n(H).$$

Assigning to every vertex in $V(H)$ the weight 2 and to every vertex in $V(G) \setminus V(H)$ the weight -1 produces an STIDF f of weight

$$\begin{aligned}\omega(f) &= 2n(H) - \sum_{v \in V(H)} (2d_H(v) - 1) \\ &= 3n(H) - 4m(H) = \frac{1}{4}(11n(G) - 12m(G)).\end{aligned}$$

Hence $\gamma_{stI}(G) \leq \omega(f) = (11n(G) - 12m(G))/4$ and consequently $\gamma_{stI}(G) = (11n(G) - 12m(G))/4$. □

Corollary 20. If G is a connected graph of order $n \geq 3$ and size m, then

$$\gamma_{stR}(G) \geq \frac{11n - 12m}{4}$$

with equality if and only if $G \in \mathcal{F}$.

Proof. Since $\gamma_{stI}(G) \leq \gamma_{stR}(G)$, Theorem 19 leads to the desired bound immediately. If $\gamma_{stR}(G) = (11n - 12m)/4$, then it follows from Theorem 19 that

$$\frac{11n - 12m}{4} = \gamma_{stR}(G) \geq \gamma_{stI}(G) \geq \frac{11n - 12m}{4}.$$

Therefore equality in Theorem 19 and thus $G \in \mathcal{F}$. In addition, the proof of Theorem 19 shows that $\gamma_{stR}(G) = (11n(G) - 12m(G))/4$ when $G \in \mathcal{F}$, and the proof is complete. □

The lower bound in Corollary 20 can be found in [10].

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