The double Italian domatic number of a graph

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Abstract

A *double Italian dominating function* on a graph G with vertex set $V(G)$ is defined as a function $f : V(G) \to \{0, 1, 2, 3\}$ such that each vertex $u \in V(G)$ with $f(u) \in \{0, 1\}$ has the property that $\sum_{x \in N[u]} f(x) \geq 3$, where $N[u]$ is the closed neighborhood of u. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct double Italian dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called a *double Italian dominating family* (of functions) on G. The maximum number of functions in a double Italian dominating family on G is the *double Italian domatic number* of G, denoted by $d_{dI}(G)$. We initiate the study of the double Italian domatic number, and we present different sharp bounds on $d_{dI}(G)$. In addition, we determine the double Italian domatic number of some classes of graphs.

Keywords: Domination, Double Italian domination, Double Italian domatic number.

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1 Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [7]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers

$n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of a vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. Let K_n be the complete graph of order n, C_n the cycle of order n, P_n the path of order n, and $K_{p,q}$ the complete bipartite graph with partite sets X and Y, where $|X| = p$ and $|Y| = q$.

A set S of vertices of G is called a *dominating set* if $N[S] = \bigcup_{v \in S} N[v] = V(G)$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A domatic partition is a partition of $V(G)$ into dominating sets, and the domatic number $d(G)$ is the largest number of sets in a domatic partion. The domatic number was introduced and investigated by Cockayne and Hedetniemi [4].

In this paper we continue the study of Roman and Italian dominating functions as well as Roman and Italian domatic numbers in graphs and digraphs (see, for example, [2, 3, 5, 8, 12, 13, 14, 15, 18, 19]). A *double Roman dominating function* (DRD function) on a graph G is defined by Beeler, Haynes and Hedetniemi in [1] as a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v must have at least one neighbor u with $f(u) \geq 2$. The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on G, and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$*-function* of G.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called in [16] a *double Roman dominating family* (of functions) on G. The maximum number of functions in a double Roman dominating family (DRD family) on G is the *double Roman domatic number* of G, denoted by $d_{dR}(G)$. Further results on the double Roman domination and domatic numbers can be found in the articles [6, 9, 16, 17, 20, 21].

A *double Italian dominating function* on a graph G is defined in [10] as a function $f : V(G) \rightarrow \{0, 1, 2, 3\}$ such that each vertex $u \in V(G)$ with $f(u) \in \{0, 1\}$ has the property that $\sum_{x \in N[u]} f(x) \geq 3$. The *double Italian domination number* $\gamma_{dI}(G)$ equals the minimum weight of a double Italian dominating function on G, and a double Italian dominating function of G with weight $\gamma_{dI}(G)$ is called a $\gamma_{dI}(G)$*-function* of G.

A set $\{f_1, f_2, \dots, f_d\}$ of distinct double Italian dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called a *double Italian dominating family* (of functions) on G. The maximum number of functions in a double Italian dominating family (DID family) on G is the *double Italian domatic number* of G, denoted by $d_{dI}(G)$. According the definitions, we observe that $\gamma_{dI}(G) \leq \gamma_{dR}(G)$ and $d_{dR}(G) \leq d_{dI}(G)$.

Our purpose in this work is to initiate the study of the double Italian domatic number. We present basic properties and sharp bounds for the double Italian domatic number of a graph. In particular, for each graph G of order n with $\delta(G) \geq 2$, we prove that $\gamma_{dI}(G) + d_{dI}(G) \leq n + 3$. In addition, we determine the double Italian domatic number of some classes of graphs.

2 Properties of the double Italian domatic number

In this section we present basic properties and bounds on the double Italian domatic number.

Theorem 1. If G is a graph of order n, then

$$\gamma_{dI}(G) \cdot d_{dI}(G) \leq 3n.$$

Moreover, if we have the equality $\gamma_{dI}(G) \cdot d_{dI}(G) = 3n$, then for each DID family $\{f_1, f_2, \dots, f_d\}$ on G with $d = d_{dI}(G)$, each f_i is $\gamma_{dI}(G)$-function and $\sum_{i=1}^{d} f_i(v) = 3$ for all $v \in V(G)$.

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a DID family on G with $d = d_{dI}(G)$, and let $v \in V(G)$. Then

$$\begin{aligned} d \cdot \gamma_{dI}(G) &= \sum_{i=1}^{d} \gamma_{dI}(G) \leq \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) \\ &= \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} 3 = 3n. \end{aligned}$$

If $\gamma_{dI}(G) \cdot d_{dI}(G) = 3n$, then the two inequalities occuring in the proof become equalities. Hence for the DID family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v \in V(G)} f_i(v) = \gamma_{dI}(G)$. Thus each f_i is a $\gamma_{dI}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = 3$ for each $v \in V(G)$. □

Let $A_1 \cup A_2 \cup \ldots \cup A_d$ be a domatic partition of $V(G)$ into dominating sets such that $d = d(G)$. Then the set of functions $\{f_1, f_2, \ldots, f_d\}$ with $f_i(v) = 3$ for $v \in A_i$ and $f_i(v) = 0$ otherwise for $1 \leq i \leq d$ is a DID family on G. This shows that $d(G) \leq d_{dI}(G)$ for every graph G. Since the definition of the double Italian domination number shows easily that $\gamma_{dI}(G) \geq 3$ for each graph of order $n \geq 2$, Theorem 1 implies that $d_{dI}(G) \leq n$. In [4], the authors note that $d(K_n) = n$, and therefore we obtain the following result.

Example 2. If K_n is the complete graph, then $d_{dI}(K_n) = n$.

The next observations shows that the double Italian domatic number is mostly less or equal $(3n)/4$.

Corollary 3. If G is a graph of order $n \geq 2$ with $\Delta(G) \leq n-2$, then $d_{dI}(G) \leq (3n)/4$.

Proof. Since $\Delta(G) \leq n-2$, it follows from [10] that $\gamma_{dI}(G) \geq 4$. Therefore Theorem 1 implies

$$d_{dI}(G) \leq \frac{3n}{\gamma_{dI}(G)} \leq \frac{3n}{4}.$$

□

If G is a graph of order n, then we obtain by Zelinka [22] and $d(G) \leq d_{dI}(G)$ the upper bound

$$\left\lfloor \frac{n}{n - \delta(G)} \right\rfloor \leq d(G) \leq d_{dI}(G).$$

In [16], one can find the following propositions.

Proposition 4. Let G be a graph of order $n \geq 2$. If G has $1 \leq p \leq n-1$ vertices of degree $n-1$, then $d_{dR}(G) \geq p+1$.

Proposition 5. If G is a graph without isolated vertices, then $d_{dR}(G) \geq 2$.

Using the fact that $d_{dR}(G) \leq d_{dI}(G)$, Propositions 4 and 5 imply the next two results immediately.

Proposition 6. Let G be a graph of order $n \geq 2$. If G has $1 \leq p \leq n-1$ vertices of degree $n-1$, then $d_{dI}(G) \geq p+1$.

Proposition 7. If G is a graph without isolated vertices, then $d_{dI}(G) \geq 2$.

Theorem 8. If G is a graph, then $d_{dI}(G) \leq \delta(G)+1$.

Proof. If $d_{dI}(G)=1$, then clearly $d_{dI}(G) \leq \delta(G)+1$. Assume next that $d_{dI}(G) \geq 2$, and let $\{f_1, f_2, \ldots f_d\}$ be a DID family on G such that $d = d_{dI}(G)$. Assume that v is a vertex of minimum degree. Since $\sum_{x \in N[v]} f_i(x) = 2$ holds for at most one index $i \in \{1, 2, \ldots, d\}$, we deduce that

$$3d-1 \leq \sum_{i=1}^{d} \sum_{x \in N[v]} f_i(x) = \sum_{x \in N[v]} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N[v]} 3 = 3(\delta(G)+1).$$

This implies $d \leq \delta(G) + 4/3$ and thus $d_{dI}(G) \leq \delta(G)+1$. □

Proposition 7 and Theorem 8 imply the next result immediately.

Corollary 9. If G is a graph with $\delta(G)=1$, then $d_{dI}(G)=2$.

Corollary 10. Let G be a graph of order $n \geq 2$. Then $d_{dI}(G)=n$ if and only if G is isomorphic to the complete graph.

Proof. If G is the complete graph, then $d_{dI}(G)=n$ by Example 2.

Conversely, assume that $d_{dI}(G)=n$. If G is not complete, then $\delta(G) \leq n-2$, and Theorem 8 leads to the contradiction $n = d_{dI}(G) \leq n-1$. □

Example 11. If C_n is a cycle of length n, then $d_{dI}(C_n) = 3$.

Proof. According to Theorem 8, we have $d_{dI}(C_n) \leq 3$.

Assume now that $n = 2p$ with an integer $p \geq 2$ and let $C_n = x_1x_2 \dots x_{2p}x_1$. Define $f_1(x) = 1$ for each $x \in V(C_n)$, $f_2(x_{2i-1}) = 2$ and $f_2(x_{2i}) = 0$ and $f_3(x_{2i-1}) = 0$ and $f_3(x_{2i}) = 2$ for $i \in \{1, 2, \dots, p\}$. Then $\{f_1, f_2, f_3\}$ is a DID family on C_n with $f_1(x) + f_2(x) + f_3(x) = 3$ for each $x \in V(C_n)$ and thus $d_{dI}(C_n) = 3$ in this case.

Assume next that $n = 2p + 1$ with an integer $p \geq 1$ and let $C_n = x_1x_2 \dots x_{2p+1}x_1$. Define $f_1(x) = 1$ for each $x \in V(C_n)$, $f_2(x_{2i-1}) = 2$, $f_2(x_{2i}) = 0$ and $f_2(x_{2p+1}) = 1$ and $f_3(x_{2i-1}) = 0$, $f_3(x_{2i}) = 2$ and $f_3(x_{2p+1}) = 1$ for $i \in \{1, 2, \dots, p\}$. Then $\{f_1, f_2, f_3\}$ is a DID family on C_n with $f_1(x) + f_2(x) + f_3(x) = 3$ for each $x \in V(C_n)$ and thus $d_{dI}(C_n) = 3$ in that case. □

Example 12. Let K_n be the complete graph with $n \geq 3$ and vertex set $\{v_1, v_2, \dots, v_n\}$, and let k be an integer with $1 \leq k \leq n-2$. Define the graph $G = K_n - \{v_nv_{n-1}, v_nv_{n-2}, \dots, v_nv_{n-k}\}$. Then $\delta(G) = n - k - 1$, and thus it follows from Theorem 8 that $d_{dI}(G) \leq n - k$. Since $v_1, v_2, \dots, v_{n-k-1}$ are vertices of degree $n - 1$, we deduce from Proposition 6 that $d_{dI}(G) \geq n-k$ and thus $d_{dI}(G) = n-k = \delta(G)+1$.

Examples 11 and 12 show that Theorem 8 is sharp. Example 2 demonstrates that Theorems 1 and 8 are sharp.

Example 13. If $K_{p,p}$ is the complete bipartite graph with $p \geq 3$, then $d_{dI}(K_{p,p}) = p$.

Proof. If $p = 3$, then $\gamma_{dI}(K_{3,3}) = 5$. Thus Theorem 1 implies that

$$d_{dI}(K_{3,3}) \leq \left\lfloor \frac{3n(K_{3,3})}{\gamma_{dI}(K_{3,3})} \right\rfloor = \left\lfloor \frac{18}{5} \right\rfloor = 3.$$

If $p \geq 4$, then it is straightforward to verify that $\gamma_{dI}(K_{p,p}) = 6$, and therefore Theorem 1 implies that $d_{dI}(K_{p,p}) \leq p$.

Let now $X = \{u_1, u_2, \dots, u_p\}$ and $Y = \{v_1, u_2, \dots, v_p\}$ be a bipartition of $K_{p,p}$. Define $f_i : V(K_{p,p}) \to \{0, 1, 2, 3\}$ by $f_i(u_i) = f_i(v_i) = 3$ and $f_i(u_j) = f_i(v_j) = 0$ for $1 \leq i, j \leq p$ and $i \neq j$. Then f_i is a DID function on $K_{p,p}$ for $1 \leq i \leq p$ such that $f_1(x)+f_2(x)+\dots+f_p(x) = 3$

for each $x \in V(K_{p,p})$. Therefore $\{f_1, f_2, \dots, f_p\}$ is a double Italian dominating family on $K_{p,p}$ and thus $d_{dI}(K_{p,p}) \geq p$. This yields to $d_{dI}(K_{p,p}) = p$. □

Also Example 13 shows that Theorems 1 is sharp.

Example 14. Let $G = K_{n_1,n_2,\dots,n_r}$ be the complete r-partite graph with $r \geq 2$ and $n_1 = n_2 = \dots = n_r = 2$. Then $d_{dI}(G) = \left\lfloor \frac{3n(G)}{4} \right\rfloor = \left\lfloor \frac{3r}{2} \right\rfloor$.

Proof. Let $X_1, X_2, \dots, X_r$ be the partite sets of G. If $r = 2$, then G is a cycle of length 4, and the result follows from Example 11.

Let now $r \geq 3$. Corollary 3 implies $d_{dI}(G) \leq \left\lfloor \frac{3n(G)}{4} \right\rfloor = \left\lfloor \frac{3r}{2} \right\rfloor$.

Define now $f_i(x) = 2$ for $x \in X_i$ and $f_i(x) = 0$ for $x \in V(G) \setminus X_i$ for $1 \leq i \leq r$ as well as $f_{r+j}(x) = 1$ for $x \in X_{2j-1} \cup X_{2j}$ and $f_{r+j}(x) = 0$ for $x \in V(G) \setminus (X_{2j-1} \cup X_{2j})$ for $1 \leq j \leq \lfloor \frac{r}{2} \rfloor$. Then $\{f_1, f_2, \dots, f_{r+\lfloor r/2 \rfloor}\}$ is a DID family on G with $\sum_{i=1}^{r+\lfloor r/2 \rfloor} f_i(x) \leq 3$ for each $x \in V(G)$ and thus $d_{dI}(G) \geq r + \lfloor \frac{r}{2} \rfloor = \lfloor \frac{3r}{2} \rfloor$. This yields the desired result, and the proof is complete. □

Example 14 demostrates that Corollary 3 is sharp. If $\delta(G) \geq 1$, then $d_{dI}(G) \geq 2$, by Proposition 7. Next we prove that $d_{dI}(G) \geq 3$ when $\delta(G) \geq 3$.

Theorem 15. If G is a graph of minimum degree $\delta \geq 3$, then $d_{dI}(G) \geq 3$.

Proof. Let u and v be two different vertices of G, Define $f_1(x) = 1$ for each $x \in V(G)$, $f_2(u) = 2$, $f_2(v) = 0$ and $f_2(x) = 1$ for $x \in V(G) \setminus \{u, v\}$ and $f_3(u) = 0$, $f_3(v) = 2$ and $f_3(x) = 1$ for $x \in V(G) \setminus \{u, v\}$. Since $\delta \geq 3$, we observe that $\{f_1, f_2, f_3\}$ is a DID family on G with $f_1(x) + f_2(x) + f_3(x) = 3$ for each $x \in V(G)$ and thus $d_{dI}(G) \geq 3$. □

3 Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [11], Nordhaus and Gaddum discussed this problem for the chromatic number. We establish such inequalities for the double Italian domatic number.

Theorem 16. If G is a graph of order n, then

$$d_{dI}(G) + d_{dI}(\overline{G}) \leq n + 1.$$

If $d_{dI}(G) + d_{dI}(\overline{G}) = n + 1$, then G is regular.

Proof. Theorem 8 implies that

$$\begin{aligned} d_{dI}(G) + d_{dI}(\overline{G}) &\leq (\delta(G) + 1) + (\delta(\overline{G}) + 1) \\ &= \delta(G) + 1 + (n - \Delta(G) - 1) + 1 \leq n + 1, \end{aligned}$$

and this is the desired bound. If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the inequality chain above leads to the better bound $d_{dI}(G) + d_{dI}(\overline{G}) \leq n$. □

If $G = K_n$, then Example 2 leads to $d_{dI}(K_n) + d_{dI}(\overline{K_n}) = n + 1$, and therefore equality in the inequality of this theorem. According to Corollary 9 and Example 11, we deduce that $d_{dI}(C_4) + d_{dI}(\overline{C_4}) = 3 + 2 = 5$ and $d_{dI}(C_5) + d_{dI}(\overline{C_5}) = 3 + 3 = 6$. All these examples demonstrate that Theorem 16 is sharp.

Corollary 17. Let G be a graph of order n. If G is not regular, then $d_{dI}(G) + d_{dI}(\overline{G}) \leq n$.

If $G = K_n - e$ for $n \geq 3$, where e is an arbitray edge of K_n, then it follows from Example 12 that $d_{dI}(G) + d_{dI}(\overline{G}) = n - 1 + 1 = n$. This example shows that Corollary 17 is sharp.

For some regular graphs we can improve Theorem 16 slightly.

Theorem 18. Let G be a δ-regular graph of order n. If $1 \leq \delta < \frac{n}{4}$ or $\frac{3n}{4} - 1 < \delta \leq n - 2$, then

$$d_{dI}(G) + d_{dI}(\overline{G}) \leq n.$$

Proof. Assume first that $1 \leq \delta < \frac{n}{4}$. Then $\overline{G}$ is $(n - \delta - 1)$-regular with $n - \delta - 1 \leq n - 2$. Thus it follows from Corollary 3 and Theorem 8 that

$$d_{dI}(G) + d_{dI}(\overline{G}) \leq \delta + 1 + \frac{3n}{4} < \frac{n}{4} + 1 + \frac{3n}{4} = n + 1.$$

Second assume that $\frac{3n}{4} - 1 < \delta \le n - 2$. Then $\overline{G}$ is $(n-\delta-1)$-regular, and we deduce from from Corollary 3 and Theorem 8 that

$$\begin{aligned} d_{dI}(G) + d_{dI}(\overline{G}) &\le \frac{3n}{4} + n - \delta - 1 + 1 = \frac{3n}{4} + n - \delta \\ &< \frac{3n}{4} + n - \frac{3n}{4} + 1 = n + 1. \end{aligned}$$

Therefore $d_{dI}(G) + d_{dI}(\overline{G}) \le n$ in both cases, and the proof is complete. □

4 Bounds on $\gamma_{dI}(G) + d_{dI}(G)$

The upper bound on the product $\gamma_{dI}(G) \cdot d_{dI}(G) \le 3n$ in Theorem 1 leads to upper bounds on the sum of these two parameters.

Theorem 19. If G is a graph of order n without isolated vertices, then

$$5 \le \gamma_{dI}(G) + d_{dI}(G) \le \frac{3n}{2} + 2.$$

Proof. Proposition 7 and the fact that $\gamma_{dI}(G) \ge 3$, imply the lower bound immediately.

It follows from Theorem 1 that

$$\gamma_{dI}(G) + d_{dI}(G) \le \frac{3n}{d_{dI}(G)} + d_{dI}(G).$$

According to Proposition 7 and Theorem 8, we have $2 \le d_{dI}(G) \le n$. Using these bounds and the fact that the function $g(x) = x + (3n)/x$ is decreasing for $2 \le x \le \sqrt{3n}$ and increasing for $\sqrt{3n} \le x \le n$, the inequality above leads to

$$\begin{aligned} \gamma_{dI}(G) + d_{dI}(G) &\le \frac{3n}{d_{dI}(G)} + d_{dI}(G) \\ &\le \max\left\{\frac{3n}{2} + 2, 3 + n\right\} = \frac{3n}{2} + 2, \end{aligned}$$

and the upper bound is proved. □

Example 20. If H is isomorphic to pK_2 with an integer $p \geq 2$, then $\gamma_{dI}(H) = 3p = (3n(H))/2$ and $d_{dI}(H) = 2$. Thus $\gamma_{dI}(H)+d_{dI}(H) = (3n(H))/2+2$. This example shows that the upper bound in Theorem 19 is sharp.

If $K_{1,n-1}$ is the star with $n \geq 2$, then $\gamma_{dI}(K_{1,n-1}) = 3$, and we conclude from Corollary 9 that $d_{dI}(K_{1,n-1}) = 2$. This yields to $\gamma_{dI}(K_{1,n-1}) + d_{dI}(K_{1,n-1}) = 5$, and hence equality in the left inequality of Theorem 19.

Next we improve Theorem 19 for connected graphs. We use the following result given in [1].

Theorem 21. If G is a connected graph of order $n \geq 3$, then $\gamma_{dR}(G) \leq (5n)/4$.

Theorem 22. If G is a connected graph of order $n \geq 3$, then

$$\gamma_{dI}(G) + d_{dI}(G) \leq \frac{5n}{4} + 2,$$

with exception of the case that $G = K_3$, in which case $\gamma_{dI}(K_3) + d_{dI}(K_3) = 6$.

Proof. It follows from Theorem 1 that

$$\gamma_{dI}(G) + d_{dI}(G) \leq \gamma_{dI}(G) + \frac{3n}{\gamma_{dI}(G)}.$$

According to Theorem 21, we have $3 \leq \gamma_{dI}(G) \leq \gamma_{dR}(G) \leq (5n)/4$. Using these bounds and the fact that the function $g(x) = x+(3n)/x$ is decreasing for $3 \leq x \leq \sqrt{3n}$ and increasing for $\sqrt{3n} \leq x \leq (5n)/4$, the inequality above leads to

$$\begin{aligned}\gamma_{dI}(G) + d_{dI}(G) &\leq \gamma_{dI}(G) + \frac{3n}{\gamma_{dI}(G)} \\ &\leq \max\left\{3+n, \frac{5n}{4} + \frac{12}{5}\right\} = \frac{5n}{4} + \frac{12}{5}\end{aligned}$$

and therefore

$$\gamma_{dI}(G) + d_{dI}(G) \leq \left\lfloor \frac{5n}{4} + \frac{12}{5} \right\rfloor.$$

It is straightforward to verify that

$$\left\lfloor \frac{5n}{4} + \frac{12}{5} \right\rfloor \leq \frac{5n}{4} + 2$$

when $n \not\equiv 3 \pmod 4$. Assume next that $n \equiv 3 \pmod 4$, say $n = 4p+3$ with an integer $p \geq 0$. If $p = 0$, then G is isomorphic to K_3 or P_3. Using Example 2 and Corollary 9, we observe that $\gamma_{dI}(K_3) + d_{dI}(K_3) = 6$ and $\gamma_{dI}(P_3) + d_{dI}(P_3) = 5 \leq (15)/4 + 2$. If $p \geq 1$, then Theorem 21 implies $\gamma_{dI}(G) \leq \gamma_{dR}(G) \leq 5p + 3$, and hence it follows as above that

$$\begin{aligned} \gamma_{dI}(G) + d_{dI}(G) &\leq \gamma_{dI}(G) + \frac{3n}{\gamma_{dI}(G)} \\ &\leq \max\left\{4p + 6, 5p + 3 + \frac{12p+9}{5p+3}\right\} \\ &= 5p + 3 + \frac{12p+9}{5p+3} \end{aligned}$$

and thus

$$\gamma_{dI}(G) + d_{dI}(G) \leq \left\lfloor 5p + 3 + \frac{12p+9}{5p+3} \right\rfloor = 5p + 3 + 2 \leq \frac{5n}{4} + 2.$$

This completes the proof. □

In [1], the authors show that the following family $\mathcal{F}$ of trees attain the bound in Theorem 21. A tree T in $\mathcal{F}$ can be built from k copies of P_4 by adding $k - 1$ edges incident to support vertices of the kP_4 to connect the graph. Now it is easy to see that $\gamma_{dI}(T) = (5n)/4$ for each $T \in \mathcal{F}$. Since Corollary 9 implies that $d_{dI}(T) = 2$, we deduce that $\gamma_{dI}(T) + d_{dI}(T) = \frac{5n}{4} + 2$ for $T \in \mathcal{F}$. Therefore Theorem 22 is sharp too.

Finally, we improve Theorems 19 and 22 for graphs with minimum degree at least 2. We use the following result given in [20].

Theorem 23. If D is a digraph with $\delta^-(D) \geq 2$, then $\gamma_{dI}(D) \leq |V(D)| + 2 - \delta^-(D)$.

Corollary 24. If G is a graph with $\delta(G) \geq 2$, then $\gamma_{dI}(G) \leq |V(G)| + 2 - \delta(G) \leq n$.

Theorem 25. If G is a graph of order n with $\delta(G) \geq 2$, then

$$\gamma_{dI}(G) + d_{dI}(G) \leq n + 3.$$

Proof. According to Corollary 24, we have $3 \leq \gamma_{dI}(G) \leq n$. Hence it follows from Theorem 1 that

$$\begin{aligned} \gamma_{dI}(G) + d_{dI}(G) &\leq \gamma_{dI}(G) + \frac{3n}{\gamma_{dI}(G)} \\ &\leq \max\left\{3 + \frac{3n}{3}, n + \frac{3n}{n}\right\} = n + 3, \end{aligned}$$

and the proof is complete. □

Example 26. If H is isomorphic to pK_3 with an integer $p \geq 1$, then $\gamma_{dI}(H) = 3p = n(H)$ and $d_{dI}(H) = 3$ by Example 11. Thus $\gamma_{dI}(H) + d_{dI}(H) = n(H) + 3$. This example shows that Theorem 25 is sharp. An another example which shows the sharpness of Theorem 25 is the complete graph, since $\gamma_{dI}(K_n) + d_{dI}(K_n) = n + 3$ for $n \geq 2$ by Example 2.

Theorem 27. Let G be a graph of order n with minimum degre δ and maximum degree Δ. If $3 \leq \delta \leq \Delta \leq n - 2$ and $n > 4\delta - 8$, then

$$\gamma_{dI}(G) + d_{dI}(G) \leq n + 5 - \delta.$$

Proof. Since $\Delta \leq n - 2$, it follows from [10] that $\gamma_{dI}(G) \geq 4$. In addition, Corollary 24 implies $\gamma_{dI}(G)) \leq n + 2 - \delta$, and hence we have $4 \leq \gamma_{dI}(G) \leq n + 2 - \delta$. Therefore we deduce from Theorem 1 that

$$\begin{aligned} \gamma_{dI}(G) + d_{dI}(G) &\leq \gamma_{dI}(G) + \frac{3n}{\gamma_{dI}(G)} \\ &\leq \max\left\{4 + \frac{3n}{4}, n + 2 - \delta + \frac{3n}{n + 2 - \delta}\right\}. \end{aligned}$$

Using the condition $n > 4\delta - 8$, it is straightforward to verify that

$$\max\left\{4 + \frac{3n}{4}, n + 2 - \delta + \frac{3n}{n + 2 - \delta}\right\} < n + 6 - \delta,$$

and thus we obtain $\gamma_{dI}(G) + d_{dI}(G) \leq n + 5 - \delta$. □

By Example 13, we observe that $\gamma_{dI}(K_{3,3}) + d_{dI}(K_{3,3}) = 8 = n(K_{3,3}) + 5 - \delta(K_{3,3})$ and therefore equality in the inequality of Theorem 27. On the other hand, $\gamma_{dI}(K_{4,4}) + d_{dI}(K_{4,4}) = 10 > 9 = n(K_{4,4}) + 5 - \delta(K_{4,4})$ and hence we see that the condition $n > 4\delta - 8$ in Theorem 27 is important.

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