A note on adaptable choosability and choosability with separation of planar graphs

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Abstract. Let F be a (possibly improper) edge-coloring of a graph G; a vertex coloring of G is *adapted to* F if no color appears at the same time on an edge and on its two endpoints. If for some integer k, a graph G is such that given any list assignment L to the vertices of G, with $|L(v)| \geq k$ for all v, and any edge-coloring F of G, G admits a coloring c adapted to F where $c(v) \in L(v)$ for all v, then G is said to be *adaptably k-choosable*. A (k,d)-*list assignment* for a graph G is a map that assigns to each vertex v a list $L(v)$ of at least k colors such that $|L(x) \cap L(y)| \leq d$ whenever x and y are adjacent. A graph is (k,d)-*choosable* if for every (k,d)-list assignment L there is an L-coloring of G. It has been conjectured that planar graphs are $(3,1)$-choosable. We give some progress on this conjecture by giving sufficient conditions for a planar graph to be adaptably 3-choosable. Since $(k,1)$-choosability is a special case of adaptable k-choosablity, this implies that a planar graph satisfying these conditions is $(3,1)$-choosable.

Keywords: Adaptable choosability, Choosability with separation, Planar graph, List coloring

1   Introduction

Given a graph G, assign to each vertex v of G a set $L(v)$ of colors (positive integers). Such an assignment L is called a *list assignment* for G and the sets $L(v)$ are referred to as *lists* or *color lists*. We then want to find a proper vertex coloring φ of G, such that $\varphi(v) \in L(v)$ for all $v \in V(G)$. If such a coloring φ exists then G is *L-colorable* and φ is called an *L-coloring*. Furthermore, G is called *k-choosable* if it is L-colorable for every k-list assignment L.

This particular variant of vertex coloring is known as *list coloring* or *choosability* of graphs and was introduced by Vizing [19] and independently by Erdős et al. [7].

A recent variation on list coloring is the so-called model of choosability with separation where we require that lists of adjacent vertices have a bounded number of common colors. A (k,d)-*list assignment* for a graph G is a map that assigns to each vertex v a list $L(v)$ of at least k colors

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such that $|L(x) \cap L(y)| \leq d$ whenever x and y are adjacent. A graph is (k, d)-*choosable* if for every (k, d)-list assignment L of G there is an L-coloring of G. Note that G is (k, k)-choosable if and only if G is k-choosable. Moreover, if G is (k, d)-choosable, then G is also (k', d')-choosable for all $k' \geq k$ and $d' \leq d$.

Choosability with separation was first considered by Kratochvil et al. [15]. Among other things they proved that every planar graph is $(4, 1)$-choosable, which is a refinement for choosability of separation of Thomassen's well-known result that planar graphs are 5-choosable [18]. By an example of Voigt [20], Thomassen's result is best possible.

Skrekovski [17] gave examples of triangle-free planar graphs that are not $(3, 2)$-choosable, and posed the following question:

Problem 1.1. Is every planar graph $(3, 1)$-choosable?

It follows from a result of Kratochvil et al. [15] that this question has a positive answer for the case of triangle-free graphs. Recently, Choi et al. [4] proved that planar graphs without 4-cycles are $(3, 1)$-choosable. This was slightly improved by Chen et al. [3] who proved that planar graphs with no adjacent 4-cycles and no adjacent 3- and 4-cycles are $(3, 1)$-choosable, where two cycles of a graph are *adjacent* if they share a common edge; two cycles are *intersecting* if they have at least one common vertex.

The main purpose of this note is to give some further progress on Problem 1.1. In particular we prove that a planar graph G is $(3, 1)$-choosable if G satisfies that

  (i) no two triangles are intersecting, and every triangle is adjacent to at most one 4-cycle, or

  (ii) no triangle is adjacent to a triangle or a 4-cycle, and every 5-cycle is adjacent to at most three triangles.

Further related results on Problem 1.1 appear in [4, 2, 13, 1].

Let G be a graph and let F be a (possibly improper) coloring of the edges of G with integers. A k-coloring $c : V(G) \rightarrow \{1, \ldots, k\}$ of the vertices of G is *adapted* to F if for every $uv \in E(G)$, $c(u) \neq c(v)$ or $c(v) \neq F(uv)$. In other words, there is no monochromatic edge i.e. an edge whose two ends are colored with the same color as the edge itself. If there is an integer k such that for any edge coloring F of G, there exists a vertex k-coloring of G adapted to F, we say that G is *adaptably k-colorable*. The smallest k such that G is adaptably k-colorable is called the *adaptable chromatic number* of G, denoted by $\chi_{ad}(G)$. The concept of adapted coloring of graphs was introduced by Hell and Zhu in [12].

Let L be a list assignment for the vertices of a graph G, and F be a (possibly improper) edge coloring of G. A coloring c of G adapted to F is an L-*coloring adapted to F* if for any vertex $v \in V(G)$, we have $c(v) \in L(v)$. If for any edge coloring F of G and any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$ there exists an L-coloring of G adapted to F, we say that G is *adaptably k-choosable*. The smallest k such that G is adaptably k-choosable is called the *adaptable choosability* (or the *adaptable choice number*) of G, denoted by $\mathrm{ch}_{ad}(G)$. The concept of adapted list coloring of graphs and hypergraphs was introduced by Kostochka and Zhu in [14].

The following observation was first made in [8].

Observation 1.2. *If G is adaptably k-choosable, then G is $(k, 1)$-choosable.*

Proof. Assume that G is adaptably k-choosable. Let L be a $(k, 1)$-list assignment for G. For any edge $e = xy$ of G we color xy with the unique element in $L(x) \cap L(y)$; let F be this edge coloring of G. Since G is adaptably k-choosable, there is a coloring of G from the lists which is adapted to F. Since any two adjacent vertices of G have at most one common color in their lists, this coloring is proper. Hence, G is $(k, 1)$-choosable. $\square$

Our results on $(3, 1)$-choosability are based on this connection; thus any planar graph satisfying (i) or (ii) is adaptably 3-choosable. In Section 2 we give some further connections between adaptable choosability and $(3, 1)$-choosability based on the edge-arboricity of a graph.

Our main results are proved in Section 3. Our proofs are based on connections between the maximum average degree of a graph and orientations of the underlying graph. A benefit of our method is that it yields rather short proofs; many results in this area are based on rather lengthy discharging arguments, or uses precoloring extension techniques based on the proof of Thomassen's celebrated theorem on 5-choosability of planar graphs [18] combined with a detailed structural analysis of the graph (cf. [4, 2, 3, 1]).

In Section 4 we note that yet another family of planar graphs are $(3, 1)$-choosable, namely the so-called Halin graphs.

2   Edge arboricity and Adaptable choosability

The *edge-arboricity* $a(G)$ of a graph G is the minimum number of forests into which its edges can be partitioned. It is well-known that if a graph has arboricity at most d, then it has an orientation with out-degree at most d (see e.g. [6]).

The following proposition demonstrates the connection between adaptable choosability (and thus $(k, 1)$-choosability) and orientations.

Proposition 2.1. *If $a(G) \leq k$, then G is adaptably $(k + 1)$-choosable, and thus $(k + 1, 1)$-choosable.*

Proof. By assumption, G has an orientation in which each vertex x_i has $d^+(x_i) \leq k$. Assume each vertex x_i is given a list $L(x_i)$ of $k + 1$ colors and F is an edge coloring of G. Let $c(x_i)$ be any color in $L(x_i)$ which does not appear on any outgoing edges of x_i. Then it is obvious that c is an L-coloring of G adapted to F. This completes the proof of Proposition 2.1. $\square$

Since the edge-arboricity of a triangle-free planar graph is at most 2 and the edge-arboricity of planar graphs is at most 3, the preceding proposition yields yet another immediate proof of the facts that every triangle-free planar graph is $(3, 1)$-choosable, and that every planar graph is $(4, 1)$-choosable [15].

As pointed out above, Choi et al. [4] proved that planar graphs without 4-cycles are $(3, 1)$-choosable, but there are planar graphs without 4-cycles which are not adaptably 3-colorable [9].

However, as follows from Proposition 2.1, every planar graph is adaptably 4-choosable. We note that this in fact holds for any graph with no K_5-minor; which was first established in in [9].

Corollary 2.2. *Every K_5-minor free (simple) graph is adaptably 4-choosable.*

As noted in [16], it is easy to prove that the edge-arboricity of a K_5-minor free (simple) graph G is at most 3 (this follows since such a graph satisfies $|E(G)| \leq 3|V(G)| - 6$); so Proposition 2.1 implies Corollary 2.2. The latter statement yields the following.

Corollary 2.3. *If G is a K_5-minor free graph, then G is $(4, 1)$-choosable.*

3 Sufficient conditions for adaptable 3-choosability and $(3,1)$-choosability of planar graphs

In this section we prove our main results on adaptable 3-choosability and $(3,1)$-choosability of planar graphs.

Given a graph G, the *maximum average degree* of G, denoted by $\mathrm{Mad}(G)$, is the maximum of the average degrees of all subgraphs of G, i.e.,

$$\mathrm{Mad}(G) = \max\{2|E(H)|/|V(H)| : H \text{ is a subgraph of } G\}.$$

We denote by $d(v)$ the degree of a vertex v in G and by $r(f)$ the degree of a face f, i.e. the number of edges incident with it. A k-face is a face of degree k, and a k^+-face (respectively, k^--face) is a face with degree at least k (respectively, at most k).

In [16] the following theorem is proved using the orientation method presented in Section 2 (see [5] for a detailed introduction to orientation and Mad).

Theorem 3.1. *For any graph G,*

$$\mathrm{ch}_{ad}(G) \leq \lceil \mathrm{Mad}(G)/2 \rceil + 1.$$

By using Obervation 1.2 we have the following.

Lemma 3.2. *Every graph G is $(\lceil \mathrm{Mad}(G)/2 \rceil + 1, 1)$-choosable.*

Now we prove the following theorem.

Theorem 3.3. *If G is a planar graph with no intersecting triangles and where every triangle is adjacent to at most one 4-cycle, then $\mathrm{Mad}(G) < 4$.*

Proof. Suppose, for a contradiction, that there is a subgraph H of G with average degree at least 4. Clearly, we may assume that H is connected. We shall use a discharging argument for proving that H has average degree less than 4, thus yielding the desired contradiction.

Let V, E, F be the sets of vertices, edges and faces of H, respectively. By Euler's formula $|V| - |E| + |F| = 2$, we have

$$4|E| - 4|V| - 4|F| = -8.$$

Rewriting this yields

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(r(f) - 4) = -8. \tag{1}$$

We now define a weight function $\omega : V \cup F \to \mathbb{R}$ by setting $w(v) = d(v) - 4$ if $v \in V$, and $w(f) = r(f) - 4$ if $f \in F$.

A 3-face that shares two edges with an adjacent 5-face is called a *bad* 3-face for this 5-face; a 3-face that shares one edge with an adjacent 5-face is called an *ordinary* 3-face for this 5-face. Note that a 5-face has at most one bad 3-face.

Our discharging procedure is simple.

(R1) A 5-face gives $\frac{1}{2}$ to each adjacent 3-face if none of them are bad; if one adjacent 3-face is bad, then the 5-face gives 1 to this bad 3-face, and nothing to any other adjacent ordinary 3-face.

(R2) A 6^+-face gives $\frac{1}{2}$ to every adjacent 3-face that it shares one edge with, and it gives 1 to every 3-face that it shares at least two edges with.

Let w' be the weight function obtained by applying (R1) and (R2) to the graph H and the function w. We shall prove that $w'(f) \geq 0$ for any face f of H.

Now, since G has no intersecting triangles and every triangle is adjacent to at most one 4-cycle, each 3-face in H is adjacent to no 3-face, and at most one 4-face (via at most one edge). We consider some cases.

- If a 3-face f is adjacent to two 5^+-faces with no bad 3-faces, then $w'(f) \geq -1 + 2 \times 1/2 = 0$.

- If a 3-face f is adjacent to a 5-face with a bad 3-face via one edge, then the other two edges of f do not lie on 4-faces or on 5-faces with bad 3-faces; because every triangle in G is adjacent to at most one 4-cycle. Thus, we have $w'(f) = -1 + 2 \times 1/2 = 0$.

- If a 3-face f is a bad 3-face for an adjacent 5-face, then it receives 1 from this 5-face. Hence, $w'(f) \geq -1 + 1 = 0$.

In conclusion, every 3-face f satisfies that $w'(f) \geq 0$.

Every 4-face f clearly satisfies $w'(f) = 0$. If a 5-face f is adjacent to a bad 3-face then it satisfies $w'(f) = 1 - 1 = 0$; if it is not adjacent to a bad 3-face, then it is adjacent to at most two distinct 3-faces, since no pair of triangles intersect in G; thus $w'(f) \geq 1 - 2 \times \frac{1}{2} = 0$.

Consider a $2k$-face f, where $k \geq 3$. The face f gives 1 to an adjacent 3-face that it shares at least two edges with, and it gives $\frac{1}{2}$ to a 3-face that it shares one edge with. Since no pair of triangles intersect, f is adjacent to 3-faces via at most $\frac{4k}{3}$ edges if f shares at most two edges with every adjacent 3-face. Since a 3-face which shares two or three edges with f receives 1 from f, it follows that $w'(f) \geq 2k - 4 - \frac{2k}{3} \geq 0$ if $k \geq 3$. A similar calculation shows that $w'(f) \geq 0$ if f is a $(2k+1)$-face, where $k \geq 3$.

By (1), we have that

$$\sum_{v \in V} w(v) + \sum_{f \in F} w(f) < 0.$$

Hence, the same holds for the weight function w' obtained by redistributing the charge. Now, since $w'(f) \geq 0$ for any face f in H, we have that $\sum_{v \in V} w'(v) < 0$, which means that $\sum_{v \in V}(d(v) - 4) < 0$, and so the average degree in H is less than 4, contrary to our assumption. $\square$

Theorem 3.1 and Lemma 3.2 yield the following.

Theorem 3.4. *If G is a planar graph with no intersecting triangles and where every triangle is adjacent to at most one 4-cycle, then $\mathrm{ch}_{ad}(G) \leq 3$, and thus G is $(3,1)$-choosable.*

Remark 3.5. *It is easily verified that the proof of Theorem 3.3 (with the exact same discharging rules and calculations) is valid if the following two conditions hold:*

(i) *every triangle satisfies that at most one of its edges lie on another cycle of length at most four;*

(ii) *every 5-, 6-, and 7-cycle satisfies that at most two, four, and six of its edges lie on triangles, respectively.*

Hence, every planar graph G satisfying these conditions is adaptably 3-choosable, and also $(3,1)$-choosable.

Another immediate consequence of the preceding theorem is the following.

Corollary 3.6. *If G is a planar graph with no intersecting triangles and no intersecting 4-cycles, then G is adaptably 3-choosable, and thus $(3,1)$-choosable.*

Using a similar argument as in the proof of Theorem 3.3 we can prove the following.

Theorem 3.7. *If G is a planar graph where no triangle of G is adjacent to a triangle or a 4-cycle, and each 5-cycle is adjacent to at most three triangles, then $\mathrm{Mad}(G) < 4$. Hence, G is adaptably 3-choosable and $(3,1)$-choosable.*

Proof. The proof of Theorem 3.7 is similar to the proof of Theorem 3.3. The only substantial difference is that instead of the discharging rules (R1) and (R2) we apply the following.

(R3) For each edge between a 5^+-face and a triangle, the 5^+-face gives $\frac{1}{3}$ to the adjacent triangle.

Similar calculations as in the proof of Theorem 3.3 then yield the desired contradiction; the details are omitted. □

We have to notice that our Theorem 3.4 gives a very short proof of the following theorem proved in [10].

Theorem 3.8. *Suppose G is a planar graph. Then G is adaptably 3-choosable if any two triangles in G have distance at least 2 and no triangle is adjacent to a 4-cycle.*

4 Halin graphs

A *Halin graph* is a planar graph constructed from a planar drawing of a tree with at least four vertices and with no vertices of degree two by connecting its leaves by a cycle that crosses none of its edges. The following lemma is well-known and easy to prove.

Lemma 4.1. *Every cycle is $(2,1)$-choosable.*

Proposition 4.2. *Every Halin graph is $(3,1)$-choosable.*

Proof. Let $G = T \cup C$ be a Halin graph, where T is the spanning tree, and C is the outer cycle. Consider a $(3,1)$-list assignment L for G. Now, any tree is trivially $(2,1)$-choosable; thus, we can pick an L-coloring φ of the tree T' obtained from T by removing all leaves. Now, for every vertex v of C we define a new list assignment L' by removing any color from $L(v)$ that is used on a neighbor of v in T'. Note that L' is $(2,1)$-list assignment for C. By Lemma 4.1, C is L'-colorable. By taking an L'-coloring of C together with the coloring φ of T', we obtain an L-coloring of G. □

Since K_4 is $(3,1)$-choosable, but not $(2,1)$-choosable, Proposition 4.2 is in fact sharp.

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