Boundary Independent Broadcasts in Graphs

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This paper is dedicated to our friend and colleague,
Gary MacGillivray, on the occasion of his 60th birthday.
Thanks, Gary, for being there for your colleagues and students!

Abstract

A broadcast on a nontrivial connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \text{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$. The weight of f is $\sigma(f) = \sum_{v \in V} f(v)$. A vertex u hears f from v if $f(v) > 0$ and $d(u, v) \leq f(v)$. A broadcast f is independent, or hearing independent, if no vertex u with $f(u) > 0$ hears f from any other vertex v. We define a different type of independent broadcast, namely a boundary independent broadcast, as a broadcast f such that, if a vertex w hears f from vertices $v_1, \ldots, v_k$, $k \geq 2$, then $d(w, v_i) = f(v_i)$ for each i. The maximum weights of a hearing independent broadcast and a boundary independent broadcast are the hearing independence broadcast number $\alpha_h(G)$ and the boundary independence broadcast number $\alpha_{\text{bn}}(G)$, respectively.

We prove that $\alpha_{\text{bn}}(G) = \alpha(G)$ (the independence number) for any 2-connected bipartite graph G and that $\alpha_{\text{bn}}(G) \leq n - 1$ for all graphs G of order n, characterizing graphs for which equality holds. We compare α_{bn} and α_h and prove that although the difference $\alpha_h - \alpha_{\text{bn}}$ can be arbitrary, the ratio is bounded, namely $\alpha_h / \alpha_{\text{bn}} < 2$, which is asymptotically best possible. We deduce that $\alpha_h(G) \leq 2n - 5$ for

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all connected graphs $G \neq P_n$ of order n, which improves an existing upper bound for $\alpha_h(G)$ when $\alpha(G) \geq n/2$.

Keywords: broadcast domination; broadcast independence, hearing independence; boundary independence

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1 Introduction

In a search for the best way to generalize the concept of independent sets in graphs to independent broadcasts, there are several ways to look at an independent set X of a graph G. One way is from the point of view of the vertices in X: no two vertices are adjacent – the usual definition. Another way is from the point of view of the edges of G: no edge is incident with (or covered by) more than one vertex in X. Using the latter approach we define boundary independent broadcasts as an alternative to independent broadcasts as defined by Erwin [9], which we refer to here as hearing independent broadcasts. Among other results we show that the boundary independent broadcast number α_{bn} of any graph lies between its independence number and its hearing independent broadcast number α_h. We prove a tight upper bound for α_{bn} which leads to a new tight upper bound for α_h.

1.1 Broadcast definitions

For undefined concepts we refer the reader to [7]. The study of broadcast domination was initiated by Erwin in his doctoral dissertation [9]. A *broadcast* on a nontrivial connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \mathrm{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$. When G is disconnected, we define a broadcast on G as the union of broadcasts on its components. Define $V_f^+ = \{v \in V : f(v) > 0\}$ and partition V_f^+ into the two sets $V_f^1 = \{v \in V : f(v) = 1\}$ and $V_f^{++} = V_f^+ - V_f^1$. A vertex in V_f^+ is called a *broadcasting vertex*. A vertex u *hears* f from $v \in V_f^+$, and v *f-dominates* u, if the distance $d(u, v) \leq f(v)$. If $d(u, v) < f(v)$, we also say that say that v *overdominates* u. Denote the set of all vertices that do not hear f by U_f. A broadcast f is *dominating* if $U_f = \varnothing$. The *weight* of f is $\sigma(f) = \sum_{v \in V} f(v)$, and the *broadcast number* of G is

$$\gamma_b(G) = \min\{\sigma(f) : f \text{ is a dominating broadcast of } G\}.$$

When f and g are broadcasts on G such that $g(v) \leq f(v)$ for each $v \in V$, we write $g \leq f$. When in addition $g(v) < f(v)$ for at least one $v \in V$, we write $g < f$. A dominating broadcast f on G is a *minimal dominating broadcast* if no broadcast $g < f$ is dominating. The *upper broadcast number* of G is

$$\Gamma_b(G) = \max\{\sigma(f) : f \text{ is a minimal dominating broadcast of } G\},$$

and a dominating broadcast f of G such that $\sigma(f) = \Gamma_b(G)$ is called a *Γ_b-broadcast.* First defined by Erwin [9], the upper broadcast number was also studied by Ahmadi, Fricke, Schroeder, Hedetniemi and Laskar [1], Bouchemakh and Fergani [4], Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [8], Gemmrich and Mynhardt [10] and Mynhardt and Roux [12].

If f is a (minimal) dominating broadcast such that $V_f^+ = V_f^1$, then f is the characteristic function of a (minimal) dominating set. Hence, denoting the cardinalities of a minimum dominating set and a maximum minimal dominating set by $\gamma(G)$ and $\Gamma(G)$ (the *lower* and *upper domination numbers* of G), respectively, we see that $\gamma_b(G) \leq \gamma(G)$ and $\Gamma(G) \leq \Gamma_b(G)$ for any graph G.

We denote the independence number of G by $\alpha(G)$ and the minimum cardinality of a maximal independent set (the *independent domination number* of G) by $i(G)$. To generalize the concept of independent sets, Erwin [9] defined a broadcast f to be *independent*, or, for our purposes, *hearing independent*, if no vertex $u \in V_f^+$ hears f from any other vertex $v \in V_f^+$; that is, broadcasting vertices only hear themselves. This version of broadcast independence was also considered by, among others, Ahmane, Bouchemakh and Sopena [2], Bessy and Rautenbach [3], and Bouchemakh and Zemir [5]. We show below that other definitions of broadcast independence, which also generalize independent sets and lead to different independent broadcast numbers, are feasible.

1.2 Neighbourhoods and boundaries

Following [12], for a broadcast f on G and $v \in V_f^+$, we define the

- *f-neighbourhood* of v by $N_f(v) = \{u \in V : d(u,v) \leq f(v)\}$,
- *f-boundary* of v by $B_f(v) = \{u \in V : d(u,v) = f(v)\}$,
- *f-private neighbourhood* of v by $\mathrm{PN}_f(v) = \{u \in N_f(v) : u \notin N_f(w)$ for all $w \in V_f^+ - \{v\}\}$,

- *f-private boundary* of v by $\mathrm{PB}_f(v) = \{u \in N_f(v) : u \text{ is not dominated by } (f - \{(v, f(v))\}) \cup \{(v, f(v) - 1)\}$.

Note that if $u \in V_f^1$ and u does not hear f from any vertex $v \in V_f^+ - \{u\}$, then $u \in \mathrm{PB}_f(u)$, and if $u \in V_f^{++}$, then $\mathrm{PB}_f(u) = B_f(u) \cap \mathrm{PN}_f(u)$. If f is a broadcast such that every vertex x that hears more than one broadcasting vertex also satisfies $d(x, u) \geq f(u)$ for all $u \in V_f^+$, then the *broadcast only overlaps in boundaries*. On the other hand, if f is a dominating broadcast such that no vertex hears more than one broadcasting vertex, then f is an *efficient dominating broadcast*. When $xy \in E(G)$ and $x, y \in N_f(u)$ for some $u \in V_f^+$ such that at least one of x and y does not belong to $B_f(u)$, we say that the edge xy is *covered* in f by u. When xy is not covered by any $u \in V_f^+$, we say that xy is *uncovered by* f.

Erwin [9] determined a necessary and sufficient condition for a dominating broadcast to be minimal dominating. We restate it here in terms of private boundaries.

Proposition 1.1 [9] *A dominating broadcast f is a minimal dominating broadcast if and only if* $\mathrm{PB}_f(v) \neq \varnothing$ *for each* $v \in V_f^+$.

Ahmadi et al. [1] define a broadcast f to be *irredundant* if $\mathrm{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$. An irredundant broadcast f is *maximal irredundant* if no broadcast $g > f$ is irredundant. The *lower* and *upper broadcast irredundant numbers* of G are

$$\mathrm{ir}_b(G) = \min \{\sigma(f) : f \text{ is a maximal irredundant broadcast of } G\}$$

and

$$\mathrm{IR}_b(G) = \max \{\sigma(f) : f \text{ is an irredundant broadcast of } G\},$$

respectively. Proposition 1.1 and the above definitions imply the following two results.

Corollary 1.2 [1] *(i) Any minimal dominating broadcast is maximal irredundant.*

(ii) For any graph G,

$$\mathrm{ir}_b(G) \leq \gamma_b(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \Gamma_b(G) \leq \mathrm{IR}_b(G). \tag{1}$$

1.3 Independent broadcasts

The characteristic function of an independent set has the following features, which we generalize to obtain three different types of broadcast independence:

(a) *boundary* or bn-independent type: broadcasts overlap only in boundaries.

(b) *hearing* or h-independent type [9]: broadcasting vertices hear only themselves.

(c) *set* or s-independent type: broadcasting vertices form an independent set.

Broadcasts of type (c) were considered by Neilson [13] and found to be not very interesting. We now consider broadcasts of type (a) and define three new types of broadcast independence. Additional types can be found in [13]. If a broadcast f satisfies one of our definitions of independence and there is no broadcast g such that $g > f$ and g also meets our definition of independence, we say that f is a *maximal independent broadcast* for this type of independence. Otherwise f is not maximal independent and can be *extended* to a larger weight broadcast (for example to g) which satisfies the given definition of independence.

Definition 1.1 [13] A broadcast is *bn-independent* if it overlaps only in boundaries. The maximum (minimum) weight of a (maximal) bn-independent broadcast on G is $\alpha_{\text{bn}}(G)$ ($i_{\text{bn}}(G)$); such a broadcast is called an α_{bn}*-broadcast* (i_{bn}*-broadcast*).

Definition 1.2 [13] A broadcast is *bnr-independent* if it is bn-independent and irredundant. The maximum (minimum) weight of a (maximal) bnr-independent broadcast is $\alpha_{\text{bnr}}(G)$ ($i_{\text{bnr}}(G)$); such a broadcast is called an α_{bnr}*-broadcast* (i_{bnr}*-broadcast*).

Definition 1.3 [13] A broadcast is *bnd-independent* if it is minimal dominating and bn-independent. The maximum (minimum) weight of a bnd-independent broadcast is $\alpha_{\text{bnd}}(G)$ ($i_{\text{bnd}}(G)$); such a broadcast is called an α_{bnd}*-broadcast* (i_{bnd}*-broadcast*).

Definition 1.4 [9] The maximum (minimum) weight of a (maximal) h-independent broadcast is $\alpha_h(G)$ ($i_h(G)$); such a broadcast is called an α_h-*broadcast* (i_h*-broadcast*).

A bnd-independent broadcast, because it is minimal dominating, is maximal irredundant (Corollary 1.2), and because it is irredundant and dominating, it is minimal dominating (Proposition 1.1). The parameters $\alpha_h(G)$ and $\alpha_{\text{bn}}(G)$ are also called the *hearing* or *h-independence broadcast number* and the *boundary* or *bn-independence broadcast number*, respectively.

Since the characteristic function of an independent set is a bnd-, bnr-, bn- and h-independent broadcast, it follows from Definitions 1.1 – 1.4 that

$$\alpha(G) \leq \alpha_{\text{bnd}}(G) \leq \alpha_{\text{bnr}}(G) \leq \alpha_{\text{bn}}(G) \leq \alpha_h(G) \tag{2}$$

for any graph G.

When two parameters π and π' are incomparable, we denote this fact by $\pi \diamond \pi'$. For the path P_n, where $n \geq 4$, it is easy to see that $\Gamma_b(P_n) = \text{IR}_b(P_n) = \text{diam}(P_n) = n-1$, whereas $\alpha_h(P_n) = 2(n-2) > \Gamma_b(P_n)$. On the other hand, for the grid graph $G_{n,n} = P_n \square P_n$, if n is large enough, then $\alpha_h(G_{n,n}) = \left\lceil \frac{n^2}{2} \right\rceil$ ([5]; see Theorem 4.2 below), but Mynhardt and Roux [12] showed that $\Gamma_b(G_{n,n}) = \text{IR}_b(G_{n,n}) = n(n-1) > \alpha_h(G_{n,n})$. Therefore $\alpha_h \diamond \Gamma_b$ and $\alpha_h \diamond \text{IR}_b$, hence α_h does not fit neatly into the inequality chain (1). Our definitions of boundary independent broadcasts were partially motivated by the aim of finding a definition of broadcast independence for which the associated parameters could be inserted in (1). Neilson [13] showed that $\alpha_{\text{bn}} \diamond \Gamma_b$ and $\alpha_{\text{bnr}} \diamond \Gamma_b$, but, since a bnd-independent broadcast is minimal dominating, $\alpha_{\text{bnd}}(G) \leq \Gamma_b(G)$ (strict inequality is possible). Hence

$$\begin{aligned} \text{ir}_b(G) \leq i_{\text{bnd}}(G) \leq \gamma_b(G) \leq \gamma(G) \leq i(G) \\ \leq \alpha(G) \leq \alpha_{\text{bnd}}(G) \leq \Gamma_b(G) \leq \text{IR}_b(G) \end{aligned} \tag{3}$$

for any graph G. Therefore, with bnd-independent broadcasts we have achieved this goal.

The graph G in Figure 1 is an example of a tree T for which $\alpha_{\text{bnd}}(T) < \alpha_{\text{bnr}}(T) < \alpha_{\text{bn}}(T)$; details can be found in [13]. Broadcasting from each leaf with a strength of 5 we obtain an h-independent broadcast with a weight of 30, hence $\alpha_h(T) \geq 30 > \alpha_{\text{bn}}(T)$.

For the lower parameters i_{bn} etc., the characteristic function of a **maximal** independent set is not necessarily a **maximal** bn- or h-independent broadcast. For example, consider the path $P_6 : v_1, ..., v_6$, having maximal independent set $\{v_2, v_5\}$. This set has characteristic function f, where $f(v_2) = f(v_5) = 1$ and $f(x) = 0$ otherwise. The broadcast $g = (f - \{(v_2, 1)\}) \cup \{(v_2, 2)\}$ is bn- and h-independent and it is not difficult to verify that $i_{\text{bn}}(P_6) = i_h(P_6) = 3 > i(P_6) = 2$. On the other hand, the corona

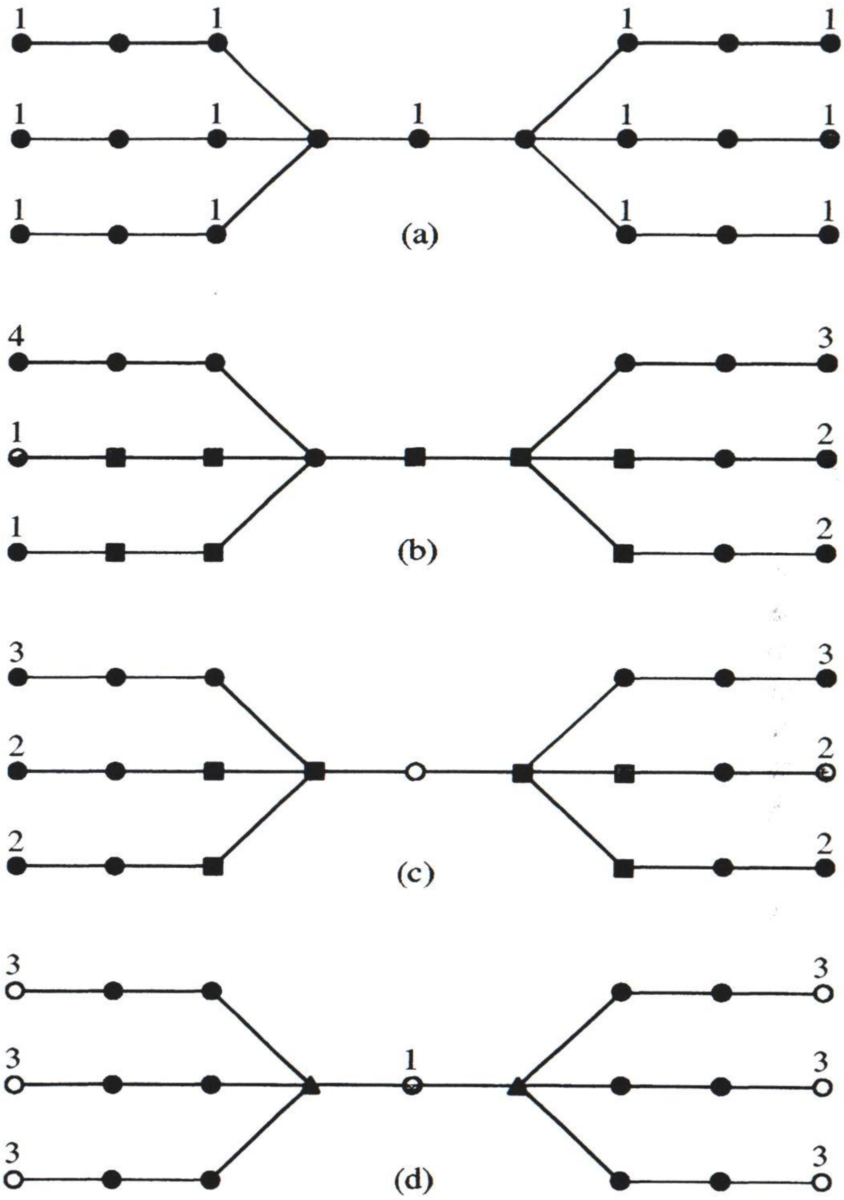

Figure 1: A tree T with $\alpha(T) = \alpha_{\text{bnd}}(T) < \alpha_{\text{bnr}}(T) < \alpha_{\text{bn}}(T)$. A maximum independent set is shown in (a), maximum bnd-broadcasts with weight 13 in (a) and (b), a maximum bnr-broadcast with weight 14 in (c), and a maximum bn-broadcast with weight 19 in (d). In (b) and (c), vertices in private boundaries of broadcasting vertices are indicated by squares, and in (d), vertices in shared boundaries by triangles.

$K_n \circ K_1$ for any complete graph K_n, $n \geq 4$, satisfies $i(K_n \circ K_1) = n \geq 4$ but $i_h(K_n \circ K_1)$, $i_{\rm bn}(K_n \circ K_1) \leq 3$. Therefore $i_h \diamond i$ and $i_{\rm bn} \diamond i$.

Dunbar et al. [8] showed that every graph has a minimum weight dominating broadcast f such that $N_f(u) \cap N_f(v) = \varnothing$ for all $u, v \in V_f^+$. Such a broadcast is maximal bnr-independent. Since any bnr-independent broadcast is irredundant by definition, it follows that

$$\mathrm{ir}_b(G) \leq i_{\rm bnd}(G) \leq i_{\rm bnr}(G) \leq \gamma_b(G) \leq \gamma(G) \leq i(G) \tag{4}$$

for any graph G. Further, although any maximal bn-independent broadcast is dominating (see Observation 2.1 below), it is not necessarily minimal dominating, hence it is possible that $i_{\rm bn} > \gamma_b$. Neilson [13] showed that $i_{\rm bn}(G) \leq \lceil \frac{4}{3}\gamma_b(G) \rceil$ for all graphs G.

We show in Section 2 that $\alpha_{\rm bn}(G) \leq n-1$ for all graphs G of order n and characterize graphs for which equality holds. In Section 3 we compare $\alpha_{\rm bn}$ and $\alpha_{\rm bnr}$ to α_h and prove that although the differences $\alpha_h - \alpha_{\rm bn}$ and $\alpha_h - \alpha_{\rm bnr}$ can be arbitrary, the ratios $\alpha_h/\alpha_{\rm bn}$ and $\alpha_h/\alpha_{\rm bnr}$ are bounded by 2 and 3, respectively, and that these ratios are asymptotically best possible. We deduce that $\alpha_h(G) \leq 2n-5$ whenever G is a connected n-vertex graph that is not a path. In Section 4 we show that $\alpha_{\rm bn}(G) = \alpha_{\rm bnr}(G) = \alpha_{\rm bnd}(G) = \alpha(G)$ for any 2-connected bipartite graph G.

2 Boundary independence

Suppose f is a bn-independent broadcast on a graph G such that $U_f \neq \varnothing$; say $u \in U_f$. Consider the broadcast $g_u = (f - \{(u,0)\}) \cup \{(u,1)\}$ and notice that if any vertex x of G hears u as well as another vertex $v \in V_f^+$, then $x \in B_{g_u}(u) \cap B_{g_u}(v)$. Therefore g_u is bn-independent and $\sigma(g_u) > \sigma(f)$, from which we deduce that f is not maximal bn-independent. When $U_{g_u} \neq \varnothing$ we can repeat this process until we obtain a maximal bn-independent broadcast g, i.e., one having $U_g = \varnothing$. We state this fact as an observation for referencing.

Observation 2.1 *Any maximal bn-independent broadcast is dominating.*

We use Observation 2.1 to prove a necessary and sufficient condition for a bn-independent broadcast to be maximal bn-independent.

Proposition 2.2 *A bn-independent broadcast f on a graph G is maximal bn-independent if and only if it is dominating, and either $V_f^+ = \{v\}$ or $B_f(v) - \mathrm{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$.*

Proof. Consider a maximal bn-independent broadcast f of G. By Observation 2.1, f is dominating. Suppose $|V_f^+| \geq 2$ and there exists a vertex $v \in V_f^+$ such that $B_f(v) - \text{PB}_f(v) = \varnothing$. Since $f(v) \leq e(v)$, the boundary $B_f(v) \neq \varnothing$. Since $|V_f^+| \geq 2$, there exists a vertex $w \in V_f^+ - \{v\}$. By the definition of bn-independence, $d(v,w) > f(v)$; this implies that $f(v) < e(v)$. Hence we may increase the strength of the broadcast from v to obtain the broadcast $f' = (f - \{(v, f(v))\}) \cup \{(v, f(v)+1)\}$. Since $B_f(v) - \text{PB}_f(v) = \varnothing$, $B_f(v) \subseteq \text{PB}_f(v)$. Hence no vertex hears f from v as well as from another vertex in V_f^+. Thus f' is a bn-independent broadcast such that $f' > f$. This contradicts the maximality of f. Hence, if $|V_f^+| \geq 2$, then $B_f(v) - \text{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$.

Conversely, suppose f is a dominating bn-independent broadcast such that either $V_f^+ = \{v\}$ or $B_f(v) - \text{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$. If $V_f^+ = \{v\}$, then, since f is dominating, $f(v) = e(v)$ and f is maximal bn-independent by definition. Hence assume $|V_f^+| \geq 2$ and $B_f(v) - \text{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$. Consider any $v \in V(G)$ and define $f' = (f - \{(v, f(v))\}) \cup \{(v, f(v)+1)\}$. If $v \in V_f^+$, then $B_f(v) - \text{PB}_f(v) \neq \varnothing$. Let $u \in B_f(v) - \text{PB}_f(v)$ and let $w \in V_f^+ - \{v\}$ be a vertex such that $u \in N_f(w)$. Since f is bn-independent, $u \in B_f(w)$. Then $u \in (N_{f'}(v) \cap N_{f'}(w)) - B_{f'}(v)$, hence f' is not bn-independent. If $f(v) = 0$, then $v \in N_f(w)$ for some $w \in V_f^+$. Then $v \notin B_{f'}(v)$ but $v \in N_{f'}(v) \cap N_{f'}(w)$. This implies that f' is not bn-independent. ■

2.1 Bounds on boundary independence

In this subsection we find an upper bound on the weight of a bn-independent broadcast on a graph G in terms of the size of G and the sum of the degrees of the broadcast vertices. When G is a tree, this bound immediately gives an upper bound on $\alpha_{\text{bn}}(G)$. Suppose f is a bn- or bnr-independent broadcast on G and an edge xy of G is covered by vertices $u, v \in V_f^+$. By the definition of covered, $\{x,y\} \nsubseteq B_f(u)$ and $\{x,y\} \subseteq N_f(u) \cap N_f(v)$. This violates the bn-independence of f. Hence we have the following observation.

Observation 2.3 *If f is a bn- or bnr-independent broadcast on a graph G, then each edge of G is covered by at most one vertex in V_f^+.*

Proposition 2.4 *Given a graph G of size m, if f is a bn-independent broadcast on G, then $\sigma(f) \leq m - \sum_{v \in V_f^+} \deg(v) + |V_f^+|$.*

Proof. By Observation 2.3, every edge of G is covered by at most one broadcast vertex. Since $f(v) \leq e(v)$ for each $v \in V_f^+$, there is at least one vertex x at distance $f(v)$ from v. The $f(v)$ edges along the $v-x$ geodesic are all covered by v, as are the remaining $\deg(v)$ edges incident with v. Therefore each broadcast vertex v covers at least $f(v)+\deg(v)-1$ edges. Counting edges we obtain

$$\sum_{v\in V_f^+}(f(v)+\deg(v)-1)\le m,$$

which simplifies to

$$\sigma(f)\le m-\sum_{v\in V_f^+}\deg(v)+|V_f^+|. \blacksquare$$

For a broadcast f on a nontrivial tree of order n, $\sum_{v\in V_f^+}\deg(v)\ge |V_f^+|$, hence the bound in Proposition 2.4 simplifies to the following bound for trees.

Corollary 2.5 *If T is a tree of order $n\ge 2$, then $\alpha_{\text{bnd}}(T)\le\alpha_{\text{bnr}}(T)\le\alpha_{\text{bn}}(T)\le n-1$.*

Broadcasting from a single leaf to the whole path, it is easy to see that $\alpha_{\text{bnd}}(P_n)=\alpha_{\text{bnr}}(P_n)=\alpha_{\text{bn}}(P_n)=n-1$ for any path P_n.

Let f be an α_{bn}-broadcast on a graph G and let T be a spanning tree of G. Removing the edges in $E(G)-E(T)$ does not affect bn-independence, hence f is also a bn-independent broadcast on T. Therefore $\alpha_{\text{bn}}(T)\ge\alpha_{\text{bn}}(G)$, and the result below follows from Corollary 2.5.

Corollary 2.6 *For any connected graph G of order $n\ge 2$,*

$$\alpha_{\text{bn}}(G)\le\min\{\alpha_{\text{bn}}(T): T \textit{ is a spanning tree of } G\}\le n-1.$$

The proof of Proposition 2.4 also shows that $\sigma(f)=n-1$ if and only if every vertex in V_f^+ is a leaf and the edge sets of the subtrees induced by the f-neighbourhoods form a partition of $E(T)$. We use this observation to characterize graphs of order n for which $\alpha_{\text{bn}}=n-1$. This characterization involves a class of trees called spiders. As we also use spiders to show in Section 3.1 that the differences $\alpha_h-\alpha_{\text{bn}}$, $\alpha_h-\alpha_{\text{bnr}}$ and $\alpha_{\text{bn}}-\alpha_{\text{bnr}}$ can be arbitrary, in Section 3.2 that the bounds for the ratios $\alpha_{\text{bn}}/\alpha_{\text{bnr}}$, $\alpha_h/\alpha_{\text{bn}}$ and $\alpha_h/\alpha_{\text{bnd}}$ are asymptotically best possible, and in Section 3.3 to prove a bound for α_h, we define these graphs and present results on their broadcast independence numbers in the next subsection.

2.2 Spiders

For $k \geq 3$ and $n_i \geq 1$, $i \in \{1, ..., k\}$, the *(generalized) spider* $\text{Sp}(n_1, ..., n_k)$ is the tree which has exactly one vertex b, called the *head*, having $\deg(b) = k$, and for which the k components of $\text{Sp}(n_1, ..., n_k) - b$ are paths of lengths $n_1 - 1, ..., n_k - 1$, respectively. The *legs* $L_1, ..., L_k$ of the spider are the paths from b to the leaves. Let t_i be the leaf of L_i, $i = 1, ..., k$. If $n_i = r$ for each i, we write $\text{Sp}(r^k)$ for $\text{Sp}(n_1, ..., n_k)$.

Corollary 2.7 *If G is a connected graph of order $n \geq 2$, then $\alpha_{\text{bn}}(G) = n - 1$ if and only if G is a path or a spider.*

Proof. Let f be a bn-independent broadcast on G and assume first that G is a tree. As shown in the proof of Proposition 2.4, $\sigma(f) = n - 1$ if and only if all edges of G are covered by f and the number of edges covered by v equals $f(v)$ for each $v \in V_f^+$. This holds if and only if

(1) each $v \in V_f^+$ is a leaf and the subgraph induced by $N_f(v)$ is a path of length $f(v)$.

Since G is connected and f is bn-independent,

(2) the subpaths induced by $N_f(v)$ for each $v \in V_f^+$ all have exactly one vertex in common, namely their non-broadcasting leaf.

This is possible if and only if G is a path or a generalized spider.

Now assume that G has a cycle and that $\alpha_{\text{bn}}(G) = n - 1$. If G has a spanning tree which is not a Hamiltonian path or a spider, then the above result for trees and Corollary 2.6 imply that $\alpha_{\text{bn}}(G) < n-1$, which is not the case. Suppose G has a Hamiltonian path $P : v_1, ..., v_n$. Since G has a cycle, $v_iv_j \in E(G)$ for some i, j such that $j \geq i+2$. Now $T = (P - v_iv_{i+1}) + v_iv_j$ is a spanning tree of G that is not a path. Since $\alpha_{\text{bn}}(G) = n - 1$, we may assume that T is a spider, otherwise we have a contradiction as above.

Assume therefore that G has a spanning spider $S = \text{Sp}(n_1, ..., n_k)$ (with notation as defined above). Consider any $\alpha_{\text{bn}}(G)$-broadcast f on G and let f' be the restriction of f to S. Then $\sigma(f') = \sigma(f) = n - 1$ and by (1) and (2), $V_{f'}^+ = V_f^+ = \{t_1, ..., t_k\}$ and $f(t_i) = n_i$ for each i. Since G has a cycle, there is an edge $uw \in E(G) - E(S)$. If u and w belong to the same leg L_i of S, then $d_G(t_i, b) < f(t_i)$, thus edges of L_j, $j \neq i$, hear f from both t_i and t_j. If u and w belong to different legs L_i, L_j, then uw hears f from both t_i and t_j. Both instances contradict f being bn-independent.

We deduce that if G is not a tree, then $\alpha_{\text{bn}}(G) \leq n-2$. ■

It follows from a result in [8] that $\alpha_h(\text{Sp}(r^k)) = k(2r-1)$. By Corollary 2.7, $\alpha_{\text{bn}}(\text{Sp}(r^k)) = kr$, and Neilson [13, special case of Proposition 2.3.8] showed that $\alpha_{\text{bnr}}(\text{Sp}(r^k)) \geq \alpha_{\text{bnd}}(\text{Sp}(r^k)) \geq kr-k+1$. Although there are spiders, for example $\text{Sp}(1, n_2, n_3)$, where $n_2, n_3 \geq 2$, whose bnd- and bnr-independence numbers exceed Neilson's general lower bound, it follows from our next proposition that $\alpha_{\text{bnr}}(\text{Sp}(r^k)) = \alpha_{\text{bnd}}(\text{Sp}(r^k)) = kr-k+1$ when $r \geq 2$ and $k \geq 3$.

Proposition 2.8 *If $S = \text{Sp}(n_1, ..., n_k)$ is a spider of order $n = \sum_{i=1}^{k} n_i+1$, where $k \geq 3$ and $n_i \geq 2$ for each $1 \leq i \leq k$, then $\alpha_{\text{bnd}}(S) = \alpha_{\text{bnr}}(\text{S}) = n-k$.*

Proof. Again we follow the notation for spiders as defined above. Define a broadcast g on S by $g(t_1) = n_1$, $g(t_i) = n_i - 1$ for $2 \leq i \leq k$, and $g(x) = 0$ otherwise. Notice that g is a dominating broadcast and $\sigma(g) = n-k$. No broadcasting vertex of g overdominates b and there is exactly one broadcasting vertex on each path L_i for $1 \leq i \leq k$. Hence g is bn-independent. Further, $\text{PB}_g(t_1) = \{b\}$ and for $2 \leq i \leq k$, the private boundary of t_i consists of the vertex adjacent to b on the path L_i. Hence g is a bnr-independent and dominating broadcast. It follows that $\alpha_{\text{bnr}}(S) \geq \alpha_{\text{bnd}}(S) \geq n-k$.

For the opposite inequality, let $\mathcal{F}$ be the set of α_{bnr}-broadcasts on S that minimize the number of non-leaf broadcasting vertices. We claim that there exists a broadcast in $\mathcal{F}$ such that b is not overdominated. Suppose this is not the case and consider any $f \in \mathcal{F}$. Since f is bn-independent and b is overdominated, b hears f from exactly one vertex $v \in V_f^+$, where possibly $v = b$. Since $f(v) \leq e(v)$, $B_f(v) \neq \varnothing$. We consider two cases, depending on whether there exists a vertex $v' \in B_f(v)$ such that v and v' belong to the same leg of S or not.

Case 1: there exists a vertex $v' \in B_f(v)$ such that v and v' belong to the same leg of S; say $v, v' \in V(L_1)$. (This includes the case where $v = b$, as b belongs to each leg.) Since v overdominates b, $d(v, b) < d(v', b)$ and $v \neq t_1$. Say $V(L_1) \cap V_f^+ = \{v, u_1, ..., u_\ell\}$. Define the broadcast f_1 by $f_1(t_1) = 2f(v) - 1 + \sum_{i=1}^{\ell} f(u_i)$, $f_1(x) = 0$ if $x \in V(L_1) - \{t_1\}$, and $f_1(x) = f(x)$ otherwise. Then $N_{f_1}(t_1) \cap V(L_i) \subsetneqq N_f(v) \cap V(L_i)$ for each i such that $2 \leq i \leq k$, which implies that f_1 is bn-independent and $\text{PN}_{f_1}(x) \neq \varnothing$ for each $x \in V_{f_1}^+$; that is, f_1 is bnr-independent. However, if $f(v) > 1$, then $\sigma(f_1) > \sigma(f)$, which is impossible. We deduce that $f(v) = 1$. Since b is overdominated, the only possibility is that $v = b$. But now f_1 is an

α_{bnr}-broadcast containing fewer non-leaf broadcasting vertices than f, a contradiction.

Case 2: no vertex in $B_f(v)$ belongs to the same leg as v; assume without loss of generality that $v \in V(L_1) - \{b\}$ and $v' \in V(L_2) - \{b\}$. Observe that $f(v) \geq 2$ and v overdominates t_1. This implies that $V(L_1) \cap V_f^+ = \{v\}$ and also that some vertex of L_i, where $i > 1$, belongs to $\text{PB}_f(v)$. We may assume that $v' \in \text{PB}_f(v)$. Say $f(v) = d(b, v) + q$, where $q > 0$. For $i = 2, 3$, let w_i be the vertex on L_i adjacent to b.

- Suppose first that $v' \neq t_2$. Then the edge e incident with v' on the $v' - t_2$-path is uncovered. Let $V(L_2) \cap V_f^+ = \{u_1, ..., u_\ell\}$ and define f_2 by $f_2(v) = f(v) - q$, $f_2(t_2) = \sum_{i=1}^{\ell} f(u_i) + q$, $f_1(x) = 0$ if $x \in V(L_2) - \{t_2\}$, and $f_2(x) = f(x)$ otherwise. Note that $\sigma(f_2) = \sigma(f)$. Since e is uncovered, $b \in \text{PB}_{f_2}(v)$ and some vertex on the $w_2 - t_2$ path belongs to $\text{PB}_{f_2}(t_2)$; furthermore, $\text{PB}_{f_2}(x) \supseteq \text{PB}_f(x)$ for all $x \in V_{f_2}^+ - (\{v\} \cup V(L_2))$. It follows that f_2 is an α_{bnr}-broadcast such that b is not overdominated, contrary to our assumption.
- Now suppose that $v' = t_2$. Then $q = n_2$. Since $n_2 \geq 2$, $q \geq 2$. Let $V(L_3) \cap V_f^+ = \{u_1, ..., u_\ell\}$ and define f_3 by $f_3(v) = f(v) - q$, $f_3(t_2) = q - 1$, $f_3(t_3) = \sum_{i=1}^{\ell} f(u_i) + 1$, $f_1(x) = 0$ if $x \in V(L_3) - \{t_3\}$, and $f_3(x) = f(x)$ otherwise. As for f_2, $\sigma(f_3) = \sigma(f)$. Clearly, $b \notin N_{f_3}(t_2)$, and since $q \geq 2$, $b \notin N_{f_3}(t_3)$. Therefore $b \in \text{PB}_{f_3}(v)$, $w_2 \in \text{PB}_{f_3}(t_2)$, some vertex on the $w_3 - t_3$ path belongs to $\text{PB}_{f_3}(t_3)$, and $\text{PB}_{f_3}(x) \supseteq \text{PB}_f(x)$ for all $x \in V_{f_3}^+ - (\{v\} \cup V(L_2) \cup V(L_3))$. As in the case of f_2, it follows that f_3 is an α_{bnr}-broadcast such that b is not overdominated, contrary to our assumption.

This completes the proof of the claim. Thus, let f be an α_{bnr}-broadcast on S such that b is not overdominated. (Possibly, b is not dominated at all.) Then $f(b) = 0$. If b is f-dominated, we may assume without loss of generality that b is dominated by a vertex $v \in V(L_1) \cap V_f^+$. Let $L_1' = L_1$ and $L_i' = L_i - \{b\}$ for each $2 \leq i \leq k$. Note that these paths from a partition of $V(S)$. Restricting f to each L_i', we obtain k separate broadcasts $f_i = f \upharpoonright L_i'$ for $1 \leq i \leq k$. Since f is bn-independent, each f_i is bn-independent. Since f is bnr-independent, $\text{PB}_f(w) \neq \varnothing$ for each $w \in V_f^+$. Also, since b is not overdominated, and by the definition of L_1', if $w \in V_f^+ \cap V(L_i)$, then $\text{PB}_f(w) \subseteq V(L_i')$. Thus $\varnothing \neq \text{PB}_{f_i}(w) \subseteq V(L_i')$. Hence the broadcasts f_i are bnr-independent. Since $\alpha_{\text{bnr}}(P) = |V(P)| - 1$ for any path P,

$$\alpha_{\text{bnd}}(S) \leq \alpha_{\text{bnr}}(S) = \sigma(f) = \sigma(f_1) + \sum_{i=2}^{k} \sigma(f_i) \leq n_1 + \sum_{i=2}^{k} (n_i - 1) = n - k. \quad \blacksquare$$

We next determine an upper bound for $\alpha_h(\mathrm{Sp}(n_1, ..., n_k))$. This result generalizes the upper bound for $\alpha_h(\mathrm{Sp}(r^k))$ in [8].

Proposition 2.9 *If S is a spider $\mathrm{Sp}(n_1, ..., n_k)$ of order n, where $k \geq 3$, then $\alpha_h(S) \leq 2n - 2 - k$.*

Proof. Assume that $n_1 \leq \cdots \leq n_k$ and note that $n = 1 + \sum_{i=1}^{k} n_i$. Let f be an α_h-broadcast on S. If $|V_f^+| = 1$, then $\sigma(f) \leq \mathrm{diam}(S) \leq n - k + 1 < 2n - 2 - k$ since $n > 3$. Hence assume $|V_f^+| \geq 2$. If the leg L_i contains a broadcast vertex other than its leaf t_i, let v be the broadcast vertex on L_i nearest to t_i. Then

$$f' = (f - \{(v, f(v)), (t_i, f(t_i))\}) \cup \{(v, 0), (t_i, f(t_i) + f(v) + 1)\}$$

is an h-independent broadcast such that $\sigma(f') > \sigma(f)$, which is impossible. Therefore $V_f^+ \subseteq \{t_1, ..., t_k\}$. If the leaves t_i and t_j are broadcasting vertices, then $\max\{f(t_i), f(t_j)\} \leq d(t_i, t_j) - 1 = n_i + n_j - 1$. Let l be the smallest index such that $t_l \in V_f^+$. Since $|V_f^+| \geq 2$, there exists an index $l' > l$ such that $t_{l'} \in V_f^+$. Since f is h-independent, $t_{l'}$ does not hear the broadcast from t_l, so t_k also does not hear the broadcast from t_l. This means that $f(t_l) \leq n_l + n_k - 1$. Moreover, $f(t_i) \leq n_l + n_i - 1$ for $i > l$. Hence

$$\sigma(f) = \sum_{i=l}^{k} f(t_i) = f(t_l) + \sum_{i=l+1}^{k} f(t_i) \leq (n_l + n_k - 1) + \sum_{i=l+1}^{k} (n_l + n_i - 1).$$

This inequality simplifies to

$$\sigma(f) \leq n_k + n_l(k - l) + \sum_{i=l}^{k} n_i - (k - l) - 1. \tag{5}$$

If $l = 1$, then, noting that $n_1 \leq n_i$, (5) becomes $\sigma(f) \leq 2\sum_{i=1}^{k} n_i - k = 2n - 2 - k$. If $l > 1$, then, noting also that $n_i \geq 1$, (5) becomes

$$\begin{aligned}
\sigma(f) &\leq 2\sum_{i=l}^{k} n_i - (k - l) - 1 = 2\sum_{i=1}^{k} n_i - 2\sum_{i=1}^{l-1} n_i - (k - l) - 1 \\
&\leq 2\sum_{i=1}^{k} n_i - 2(l - 1) - (k - l) - 1 = 2\sum_{i=1}^{k} n_i - (k + l) + 1 \\
&< 2\sum_{i=1}^{k} n_i - k = 2n - 2 - k.
\end{aligned}$$

Hence $\alpha_h(S) = \sigma(f) \leq 2n - 2 - k$ and our proof is complete. ■

3 Comparing α_{bn} and α_{bnr} to α_h

In this section we show that the differences $\alpha_h-\alpha_{\text{bn}}$, $\alpha_h-\alpha_{\text{bnr}}$ and $\alpha_{\text{bn}}-\alpha_{\text{bnr}}$ can be arbitrary, whereas the ratios $\alpha_{\text{bn}}/\alpha_{\text{bnr}}$, $\alpha_h/\alpha_{\text{bn}}$ and $\alpha_h/\alpha_{\text{bnr}}$ are bounded.

3.1 The differences

When $r \geq 2$ and $k \geq 3$, it follows from Proposition 2.8 that

$$\begin{aligned} \alpha_h(\text{Sp}(r^k)) - \alpha_{\text{bn}}(\text{Sp}(r^k)) &= k(2r-1) - kr = k\,(r-1)\,, \\ \alpha_h(\text{Sp}(r^k)) - \alpha_{\text{bnr}}(\text{Sp}(r^k)) &= k(2r-1) - (kr-k+1) = kr-1 \\ \text{and } \alpha_{\text{bn}}(\text{Sp}(r^k)) - \alpha_{\text{bnr}}(\text{Sp}(r^k)) &= kr - (kr-k+1) = k-1. \end{aligned}$$

Therefore the differences $\alpha_h - \alpha_{\text{bn}}$, $\alpha_h - \alpha_{\text{bnr}}$ and $\alpha_{\text{bn}} - \alpha_{\text{bnr}}$ can be arbitrary.

3.2 The ratios

We show next that the ratios $\alpha_{\text{bn}}/\alpha_{\text{bnr}}$, $\alpha_h/\alpha_{\text{bn}}$ and $\alpha_h/\alpha_{\text{bnr}}$ are bounded. When f is a bnr-broadcast, $\text{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$, but when f is an h- or bn-independent broadcast, it is possible that $\text{PB}_f(v) = \varnothing$ for some $v \in V_f^+$. For each of these three types of broadcasts, if $f(v) = 1$, then $v \in \text{PB}_f(v)$. Therefore we have the following observation.

Observation 3.1 *If f is an h- or a bn-independent broadcast such that $\text{PB}_f(v) = \varnothing$ for some $v \in V_f^+$, then $v \in V_f^{++}$.*

Theorem 3.2 *For any graph G, $\alpha_{\text{bn}}(G)/\alpha_{\text{bnr}}(G) < 2$, and this bound is asymptotically best possible.*

Proof. Let f be an α_{bn}-broadcast on G. If $\text{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$, then f is bnr-independent and $\alpha_{\text{bn}}(G) = \alpha_{\text{bnr}}(G)$. Hence assume $\text{PB}_f(v) = \varnothing$ for some $v \in V_f^+$. Then $|V_f^+| \geq 2$ and, by Observation 3.1, $v \in V_f^{++}$. If $V_f^1 = \varnothing$, choose an arbitrary vertex $u \in V_f^{++}$. Define the broadcast g by

$$g(x) = \begin{cases} f(x) - 1 & \text{if } V_f^1 = \varnothing \text{ and } x \in V_f^{++} - \{u\} \\ f(x) - 1 & \text{if } V_f^1 \neq \varnothing \text{ and } x \in V_f^{++} \\ f(x) & \text{otherwise.} \end{cases}$$

Then $\sigma(g) \geq \sigma(f) - |V_f^{++}| \geq \frac{1}{2}\sigma(f)$ and at least one of the inequalities is strict. Moreover, since f overlaps only in boundaries and $g(x) < f(x)$ for each $x \in V_f^{++}$ (if $V_f^1 \neq \varnothing$) or for each but one $x \in V_f^{++}$ (if $V_f^+ = V_f^{++}$), the g-neighbourhoods are pairwise disjoint. Since $g(x) \leq e(x)$ for each $x \in V_g^+$, there is at least one vertex at distance $g(x)$ from x. Hence $B_g(x) \neq \varnothing$, and since the g-neighbourhoods are pairwise disjoint, $\mathrm{PB}_g \neq \varnothing$ for each $x \in V_g^+$. Therefore g is bnr-independent and $\alpha_{\mathrm{bnr}}(G) \geq \sigma(g) > \frac{1}{2}\alpha_{\mathrm{bn}}(G)$.

To see that the bound is asymptotically best possible, consider the spiders $S = \mathrm{Sp}(2^k)$, $k \geq 3$. Since $\alpha_{\mathrm{bn}}(S) = 2k$ and $\alpha_{\mathrm{bnr}}(S) = k+1$ (Proposition 2.8), the result follows. ■

We now bound $\alpha_h/\alpha_{\mathrm{bn}}$ and $\alpha_h/\alpha_{\mathrm{bnr}}$. Since the proofs overlap, we state the results as parts of the same theorem.

Theorem 3.3 *For any graph G,*

(i) $\alpha_h(G)/\alpha_{\mathrm{bn}}(G) < 2$, *and*

(ii) $\alpha_h(G)/\alpha_{\mathrm{bnr}}(G) < 3$.

Both bounds are asymptotically best possible.

Proof. *(i)* Let f be an α_h-broadcast on G. If f is bn-independent, then $\alpha_h(G) = \alpha_{\mathrm{bn}}(G)$ and we are done, hence assume $v, w \in V_f^+$ cover the same edge, say e. Since f is h-independent, no broadcasting vertex hears any other broadcasting vertex. In particular, neither v nor w is incident with e. Hence $v, w \in V_f^{++}$. Define the broadcast f' on G by $f'(x) = \left\lceil \frac{f(x)}{2} \right\rceil$ if $x \in V_f^{++}$ and $f'(x) = f(x)$ otherwise.

We claim that for $v, w \in V_f^{++}$, if at least one of $f(v)$ and $f(w)$ is even, then no vertex of G hears f' from both v and w, while if $f(v)$ and $f(w)$ are both odd, then $N_{f'}(v) \cap N_{f'}(w) \subseteq B_{f'}(v) \cap B_{f'}(w)$. This will show that f' is bn-independent.

Suppose there exists a vertex $u \in N_{f'}(v) \cap N_{f'}(w)$ for some $v, w \in V_f^{++}$. Then $f'(v) \geq d(v,u)$, $f'(w) \geq d(w,u)$ and $d(v,w) \leq f'(v) + f'(w)$. If $f(v) \neq f(w)$, say without loss of generality $f(w) < f(v)$, then

$$d(v,w) \leq f'(v) + f'(w) = \left\lceil \frac{f(v)}{2} \right\rceil + \left\lceil \frac{f(w)}{2} \right\rceil \leq f(v).$$

But then $w \in V_f^+$ hears $v \in V_f^+ - \{w\}$, contradicting the h-independence of f. If $f(v) = f(w) \equiv 0 \pmod 2$, then

$$d(v,w) \leq \left\lceil \frac{f(v)}{2} \right\rceil + \left\lceil \frac{f(w)}{2} \right\rceil = f(v),$$

again contradicting the h-independence of f. Finally, if $f(v) = f(w) \equiv 1 \pmod 2$, then

$$d(v,w) \le \left\lceil \frac{f(v)}{2} \right\rceil + \left\lceil \frac{f(w)}{2} \right\rceil = f(v) + 1.$$

Since f is h-independent, $d(v,w) = f(v) + 1 = 2f'(v) = 2f'(w)$ and $u \in B_{f'}(v) \cap B_{f'}(w)$. It follows that f' is bn-independent.

If $f(v)$ is odd for at least one $v \in V_f^{++}$, then $\alpha_{\rm bn}(G) \ge \sigma(f') > \frac{1}{2}\sigma(f)$. If $f(v)$ is even for each $v \in V_f^{++} \ne \varnothing$, then f' is not maximal bn-independent, for at least one $f'(v)$ can be increased without any edge being covered by more than one vertex, and $\alpha_{\rm bn}(G) > \sigma(f') \ge \frac{1}{2}\sigma(f)$. If $V_f^+ = V_f^1$, then $\alpha_{\rm bn}(G) = \alpha_h(G)$. Hence $\alpha_h(G)/\alpha_{\rm bn}(G) < 2$.

(ii) If every vertex of G hears f' (as defined above) from exactly one vertex in $V_{f'}^+$, then f' is a bnr-independent broadcast and we are done, hence assume that a vertex u hears f' from two vertices v and w. Since f' is bn-independent, $u \in B_{f'}(v) \cap B_{f'}(w)$. From the analysis above, this happens if and only if $v, w \in V_f^{++}$ and $f(v) = f(w) \equiv 1 \pmod 2$. Therefore $f(v), f(w) \ge 3$. Choose any vertex $z \in V_f^{++}$ such that $f(z)$ is odd. Define the broadcast f'' by

$$f''(x) = \begin{cases} \left\lceil \frac{f(x)}{2} \right\rceil & \text{if } x = z \\ \left\lfloor \frac{f(x)}{2} \right\rfloor & \text{if } x \in V_f^{++} - \{z\} \\ f(x) & \text{otherwise.} \end{cases}$$

Then $N_{f''}(v) \cap N_{f''}(w) = \varnothing$ for all $v \in V_f^{++}$ and $w \in V_f^+$, hence f'' is bnr-independent. Moreover, $\sigma(f'') > \sigma(f) - \frac{2}{3}\sigma(f) = \frac{1}{3}\sigma(f)$. Hence $\alpha_{\rm bnr}(G) \ge \sigma(f'') > \frac{1}{3}\sigma(f) = \frac{1}{3}\alpha_h(G)$, i.e., $\alpha_h(G) < 3\alpha_{\rm bnr}(G)$.

The spiders $\mathrm{Sp}(r^k)$, which satisfy

$$\alpha_h(\mathrm{Sp}(r^k)) = k(2r-1) \text{ and } \alpha_{\rm bn}(\mathrm{Sp}(r^k)) = kr,$$

show that the ratio $\alpha_h/\alpha_{\rm bn} < 2$ is asymptotically best possible. The spiders $\mathrm{Sp}(2^k)$, which satisfy $\alpha_h(\mathrm{Sp}(2^k)) = 3k$ and $\alpha_{\rm bnr}(\mathrm{Sp}(2^k)) = k+1$, illustrate the corresponding result for the ratio $\alpha_h/\alpha_{\rm bnr} < 3$. ■

3.3 Bounds

Theorem 3.3 and any upper bounds for $\alpha_{\rm bn}$ or $\alpha_{\rm bnr}$ can be used to obtain upper bounds for α_h. Conversely, lower bounds for α_h provide lower bounds for $\alpha_{\rm bn}$ and $\alpha_{\rm bnr}$. Bessy and Rautenbach [3] obtained a general upper bound for α_h. For a broadcast f on G, define $f_{\max} = \max\{f(v) : v \in V(G)\}$.

Theorem 3.4 [3] *If G is a connected graph such that*

$$\max\{\text{diam}(G), \alpha(G)\} \geq 3,$$

and f is a maximal h-independent broadcast on G, then

$$\sigma(f) \leq 4\alpha(G) - 4\min\left\{1, \frac{2\alpha(G)}{f_{\max} + 2}\right\}.$$

Therefore $\alpha_h(G) < 4\alpha(G)$, giving the ratio $\alpha_h(G)/\alpha(G) < 4$ whenever G satisfies the conditions of Theorem 3.4. The bound on the ratio is asymptotically best possible, since $\alpha_h(P_n) = 2(n-2)$ when $n \geq 4$, whereas $\alpha(P_n) = \lceil n/2 \rceil$.

We present a sharp upper bound for $\alpha_h(G)$ in terms of the order of G as a corollary to our previous results.

Corollary 3.5 *If G is a connected graph of order n that is not a path, then $\alpha_h(G) \leq 2n - 5$.*

Proof. When G is not a spider, the result follows immediately from Corollaries 2.6 and 2.7 and Theorem 3.3(i). By Proposition 2.9, $\alpha_h(\text{Sp}(n_1, ..., n_k)) \leq 2n - 2 - k \leq 2n - 5$ when $k \geq 3$. ■

Since $\text{Sp}(r^3)$ has order $3r+1$ and $\alpha_h(\text{Sp}(r^3)) = 3(2r-1) = 2(3r+1)-5$, the bound in Corollary 3.5 is sharp. For graphs with large independence numbers, this bound is better than the bound in Theorem 3.4. If $G \neq P_n$ is a connected graph of order n such that $\alpha(G) = (1-\varepsilon)n$, where $\varepsilon \leq \frac{1}{2}$ (which is the case when G is bipartite, for example), then Corollary 3.5 gives

$$\alpha_h(G) \leq 2n - 5 = \frac{2\alpha(G)}{1-\varepsilon} - 5 < 4\alpha(G) - 4\min\left\{1, \frac{2\alpha(G)}{f_{\max} + 2}\right\}.$$

Erwin [9] noted that if a connected graph G has order $n \geq 4$, then any α_h-broadcast on G has $|V_f^+| \geq 2$. Broadcasting from two antipodal vertices v, w such that $f(v) = f(w) = \text{diam}(G) - 1$, Erwin therefore obtained that $\alpha_h(G) \geq 2(\text{diam}(G) - 1)$. Dunbar et al. [8] improved Erwin's bound as follows; note that the bound is sharp for (e.g.) $\text{Sp}(r^k)$. Let $\mu(G)$ denote the cardinality of a largest set of mutually antipodal vertices in G.

Proposition 3.6 [8] *If G is a connected graph G order at least 3, then $\alpha_h(G) \geq \mu(G)(\text{diam}(G) - 1)$, and this bound is sharp.*

Theorem 3.3 and Proposition 3.6 immediately give the following lower bounds for α_{bn} and α_{bnr}.

Corollary 3.7 *For any connected graph G of order at least 3,*

$$\alpha_{\text{bn}}(G) \geq \frac{1}{2}\mu(G)(\text{diam}(G) - 1) + 1$$

and

$$\alpha_{\text{bnr}}(G) \geq \frac{1}{3}\mu(G)(\text{diam}(G) - 1) + 1.$$

Both bounds are sharp.

For the path P_n, where $n \geq 3$, the bound for α_{bn} is

$$\alpha_{\text{bn}}(P_n) \geq \text{diam}(P_n) = n - 1,$$

which gives the exact value for $\alpha_{\text{bn}}(P_n)$, and for the spider $S = \text{Sp}(2^k)$, the bound for α_{bnr} is $\alpha_{\text{bnr}}(S) \geq k + 1$, which also gives $\alpha_{\text{bnr}}(S)$ exactly.

4 Bipartite graphs

It is well known that for the $m \times n$ grid graph $G_{m,n} = P_m \square P_n$, $\alpha(G_{m,n}) = \lceil \frac{mn}{2} \rceil$. Determining the domination number of grid graphs was a major problem in domination theory until Chang's conjecture, $\gamma(G_{m,n}) = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4$ for m, n such that $16 \leq m \leq n$ [6], was proved by Gonçalves, Pinlou, Rao and Thomassé [11]. Therefore grid graphs form an important class of graphs to consider for other domination parameters. Also, Bouchemakh and Zemir [5] considered h-independence for grids, making it one of the few classes of graphs for which any work on independent broadcasts had been done prior to the dissertation [13].

We prove a result for 2-connected bipartite graphs from which we immediately obtain $\alpha_{\text{bnr}}(G_{m,n})$ and $\alpha_{\text{bn}}(G_{m,n})$.

Theorem 4.1 *If G is a 2-connected bipartite graph, then*

$$\alpha_{\text{bn}}(G) = \alpha_{\text{bnr}}(G) = \alpha_{\text{bnd}}(G) = \alpha(G).$$

Proof. We prove that G has an $\alpha_{\rm bn}$-broadcast f such that $f(v)=1$ for each $v \in V_f^+$. Among all $\alpha_{\rm bn}$-broadcasts of G, let f be one for which $|V_f^{++}|$ is minimum. When $V_f^{++} = \varnothing$, we are done, hence assume there exists $v \in V_f^{++}$. Let $f(v)=k \geq 2$. Since $f(v) \leq e(v)$, there is a vertex u at distance k from v. Since G is 2-connected, u and v lie on a common cycle C. Suppose u is the only vertex such that $d(u,v)=k$. Then C has length $2k$. Let $C: v=v_0, v_1, ..., v_{2k}=v$. Define the broadcast g by

$$g(x)=\begin{cases} 0 & \text{if } x=v_i \text{ and } i \equiv k \pmod 2 \\ 1 & \text{if } x=v_i \text{ and } i \equiv k+1 \pmod 2 \\ f(x) & \text{otherwise.} \end{cases}$$

Suppose there is another vertex $w \neq u$ at distance k from v. Then the 2-connectivity of G implies that G contains internally disjoint $v-u$ and $v-w$ paths. Therefore there is a $u-w$ path of length $2k$ containing v, say $P: u=v_0, v_1, ..., v_k=v, ..., v_{2k}=w$. Define g by

$$g(x)=\begin{cases} 0 & \text{if } x=v_i \text{ and } i \equiv 0 \pmod 2 \\ 1 & \text{if } x=v_i \text{ and } i \equiv 1 \pmod 2 \\ f(x) & \text{otherwise.} \end{cases}$$

In either case, since G is bipartite, no two vertices v_i, v_j (on P or C) where $i \equiv j \pmod 2$ are adjacent. Also, $N_g(v_i) \subseteq N_f(v)$ for each i. Hence g is bn-independent. Notice that $\sigma(g)=\sigma(f)$. Thus either g contradicts the minimality of $|V_f^{++}|$ among the $\alpha_{\rm bn}$-broadcasts of G, or g is not maximal bn-independent and contradicts f being an $\alpha_{\rm bn}$-broadcast.

Hence G has an $\alpha_{\rm bn}$-broadcast f such that $f(v)=1$ for each $v \in V_f^+$. Then V_f^1 is an independent set, from which we deduce that $\alpha_{\rm bn}(G) \leq \alpha(G)$. The result follows from the inequalities (2). ■

Since $\alpha_{\rm bn}(\mathrm{Sp}(3^k)) = 3k$, $\alpha_{\rm bnr}(\mathrm{Sp}(3^k)) = 2k+1$ and $\alpha(\mathrm{Sp}(3^k)) = 2k$, Theorem 4.1 does not hold for bipartite graphs that are not 2-connected.

Bouchemakh and Zemir [5] determined α_h for all grid graphs, showing that when m and n are large enough, $\alpha_h(G_{m,n}) = \alpha(G_{m,n}) = \lceil \frac{mn}{2} \rceil$.

Theorem 4.2 [5] *(i) If $m,n \in \mathbb{Z}$ such that $2 \leq m \leq n$ and $m \leq 4$, then $\alpha_h(G_{m,n}) = 2(m+n-3) = 2(\mathrm{diam}(G_{m,n})-1)$.*

(ii) If $m,n \in \mathbb{Z}$ such that $5 \leq m \leq n$ and $(m,n) \notin \{(5,5),(5,6)\}$, then $\alpha_h(G_{m,n}) = \lceil \frac{mn}{2} \rceil$.

(iii) $\alpha_h(G_{5,5}) = 15$ and $\alpha_h(G_{5,6}) = 16$.

It therefore follows from the inequalities (2) that for $n \geq m \geq 5$ and $(m,n) \notin \{(5,5),(5,6)\}$,

$$\alpha(G_{m,n}) = \alpha_{\text{bnd}}(G_{m,n}) = \alpha_{\text{bnr}}(G_{m,n}) = \alpha_{\text{bn}}(G_{m,n}) = \alpha_h(G_{m,n}) = \left\lceil \frac{mn}{2} \right\rceil.$$

However, Theorem 4.1 immediately gives

$$\alpha(G_{m,n}) = \alpha_{\text{bnd}}(G_{m,n}) = \alpha_{\text{bnr}}(G_{m,n}) = \alpha_{\text{bn}}(G_{m,n}) = \left\lceil \frac{mn}{2} \right\rceil$$

whenever m and n are integers such that $2 \leq m \leq n$.

5 Future work

Although i_{bnd} and α_{bnd} fit nicely into the inequality chain (3), the definition of bnd-independence forces this to be the case. The concept is difficult to work with and not very much is known about it. For example, although the difference $\alpha_{\text{bnr}} - \alpha_{\text{bnd}}$ can be arbitrary for trees [13], the behaviour of $\alpha_{\text{bnr}}/\alpha_{\text{bnd}}$ has not been determined. It would also be interesting, for comparison, to determine $\alpha_{\text{bnd}}(G)$ for classes of graphs for which $\alpha_h(G)$, $\alpha_{\text{bn}}(G)$ or $\alpha_{\text{bnr}}(G)$ is known.

For h-independence it would be interesting to find more graphs (if they exist) for which the bound in Corollary 3.5 is sharp.

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