Construction of trees with unique minimum semipaired dominating sets

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Abstract

Let G be a graph with vertex set V and no isolated vertices. A subset $S \subseteq V$ is a semipaired dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S and S can be partitioned into two element subsets such that the vertices in each subset are at most distance two apart. We present a method of building trees having a unique minimum semipaired dominating set.

Keywords: Paired-domination; Semipaired domination number
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Dedicated to Gary MacGillivray
on the special occasion of his 60th birthday
to honour his many contributions to the graph theory community

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1 Introduction

Paired domination was introduced in [8, 9] as a model for security applications involving backups for police officers. To model a backup, each vertex in the paired dominating set must be partnered with an adjacent vertex in the set. A relaxed version of paired domination, called semipaired domination, was introduced in [4] and studied, for example, in [5, 6, 10, 11, 12]. Semipaired domination in trees is the subject of this paper. We first give some definitions.

A set S of vertices in a graph G is a *dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. Further, a dominating set S is a *paired dominating set* of G if the subgraph induced by S, denoted $G[S]$, contains a perfect matching. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G and the *paired domination number* $\gamma_{\rm pr}(G)$ is the minimum cardinality of a paired dominating set of G. For a survey of paired domination, see [2].

The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. A *semi-matching* M in a graph G is a set of pairs of vertices such that every vertex of G belongs to at most one pair in M and for every pair $\{u,v\} \in M$, either u and v are adjacent in G or u and v are at distance 2 apart in G. Further, if $\{u,v\} \in M$ and $d_G(u,v) = 1$, then we call $\{u,v\}$ a *1-pair* in M, while if $\{u,v\} \in M$ and $d_G(u,v) = 2$, then we call $\{u,v\}$ a *2-pair* in M.

A set S of vertices in a graph G with no isolated vertices is a *semipaired dominating set*, abbreviated semi-PD-set, of G if S is a dominating set of G and every vertex in S is paired with exactly one other vertex in S that is within distance 2 from it. In other words, the vertices in the dominating set S can be partitioned into 2-sets such that if $\{u,v\}$ is a 2-set, then $uv \in E(G)$ or the distance between u and v is 2. We say that u and v are *paired*, and that u and v are *partners* with respect to the resulting semi-matching consisting of the pairings of vertices of S. The *semipaired domination number*, denoted by $\gamma_{\rm pr2}(G)$, is the minimum cardinality of a semi-PD-set of G. A semi-PD-set of G of cardinality $\gamma_{\rm pr2}(G)$ is called a $\gamma_{\rm pr2}$*-set* of G. The semipaired domination number is squeezed between the domination number and the paired domination number.

Observation 1 *If G is a graph with no isolated vertices, then $\gamma(G) \le \gamma_{\rm pr2}(G) \le \gamma_{\rm pr}(G)$.*

Gunther, Hartnell, Markus and Rall [3] characterized the trees having unique minimum dominating sets, and trees having unique paired dominating sets are characterized in [1]. Graphs having unique $\gamma_{\rm pr2}$-sets are called USPD-graphs, and USPD-trees are characterized in [7]. For an example of a USPD-tree, consider the path P_5 given by $u_1u_2u_3u_4u_5$, where the set $\{u_2, u_4\}$ is the unique $\gamma_{\rm pr2}$-set of P_5. In this paper, we give a method of building USPD-trees from two smaller USPD-trees.

In Section 3, we give our construction and state our main result, but first in Section 1.1 we discuss the graph theory notation and terminology we use, and thereafter in Section 2, we present some useful known results and more terminology. In Section 4, we prove our main result.

1.1 Notation and Terminology

For notation and graph theory terminology, we in general follow [13]. Specifically, the *order* of a graph G with vertex set $V(G)$ and edge set $E(G)$ is denoted by $n(G) = |V(G)|$ and its *size* by $m(G) = |E(G)|$. If the graph G is clear from the context, we simply write $V = V(G)$ and $E = E(G)$. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \cup_{v\in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. The *degree* of a vertex v in G is $d_G(v) = |N(v)|$. If the graph G is clear from context, we simply write n, m, $N(v)$, $N[v]$, and $d(v)$ rather than $n(G)$, $m(G)$, $N_G(v)$, $N_G[v]$, and $d_G(v)$, respectively.

For a set S of vertices in a graph G, the subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. A *leaf* of a tree T is a vertex of degree 1 in G, while a *support vertex* of T is a vertex adjacent to a leaf. A *star* is the graph $K_{1,k}$, where $k \geq 1$; that is, a star is a tree with at most one vertex that is not a leaf. A *double star* $S(r, s)$ for $1 \leq r \leq s$ is the tree having exactly two non-leaf vertices, one of which is adjacent to r leaves and the other to s leaves. We denote the path and cycle on n vertices by P_n and C_n, respectively.

For a subset S of vertices of G, the *S-private neighborhood* of the vertex v in S is the set $\text{pn}(v, S) = \{w \in V(G) \mid N_G[w] \cap S = \{v\}\}$, while the *external S-private neighborhood* of v is $\text{epn}(v, S) = \text{pn}(v, S) \setminus S$. An *$S$-external private neighbor* of v is a vertex in $\text{epn}(v, S)$.

2 Known Results and Terminology

The following observations from [7] determine the USPD-trees with diameter at most 3.

Observation 2 ([7]) *The path P_n for $n \geq 2$ is a USPD-tree if and only if $n = 2$ or $n \equiv 0 \pmod 5$.*

We note that the double star $S(1,1)$, that is, the path P_4: $u_1u_2u_3u_4$, is not a USPD-tree since each of $\{u_2, u_3\}$, $\{u_2, u_4\}$, and $\{u_1, u_3\}$ is a $\gamma_{\rm pr2}$-set of P_4. However, if $2 \leq r \leq s$, then the set containing the two non-leaf vertices of the double star $S(r,s)$ is its unique $\gamma_{\rm pr2}$-set. Note also, that a star is a USPD-tree if and only if it has order 2. This is stated formally as follows.

Observation 3 ([7]) *A nontrivial tree T of diameter at most 3 is a USPD-tree if and only if $T = P_2$ or T is a double star $S(r,s)$ where $r, s \geq 2$.*

To state the characterization of USPD-trees given in [7], we need some additional notation. For a given $\gamma_{\rm pr2}$-set S and semi-matching M of a graph G, we say that the set S has properties $\mathcal{P}_1$ and $\mathcal{P}_2$ if the following hold.

(a) Property $\mathcal{P}_1$ if for every 1-pair $\{u, v\}$ in M, we have $|{\rm epn}(u, S)| \geq 2$ and $|{\rm epn}(v, S)| \geq 2$.
(b) Property $\mathcal{P}_2$ if for every 2-pair $\{u, v\}$ in M, we have $|{\rm epn}(u, S)| \geq 1$ and $|{\rm epn}(v, S)| \geq 1$.

Further, a $\gamma_{\rm pr2}$-set S in the graph G has property $\mathcal{P}$ if every possible semi-matching in $G[S]$ has both Property $\mathcal{P}_1$ and Property $\mathcal{P}_2$. We are now in a position to present the characterization of USPD-trees given in [7].

Theorem 1 ([7]) *If T is a tree of order at least 3, then T is a USPD-tree if and only if T has a $\gamma_{\rm pr2}$-set with Property $\mathcal{P}$.*

3 Constructing USPD-Trees

Our main goal is to present a method of constructing a USPD-tree by combining two USPD-trees. Let T be a USPD-tree of order $n \geq 3$ with the unique $\gamma_{\rm pr2}$-set S and an associated semi-matching M. By Theorem 1, the tree T has Property $\mathcal{P}$, and so every vertex in a 1-pair of M has at least two S-external private neighbors and every vertex in a 2-pair of M has at least one S-external private neighbor.

To aid in the construction, let the *label* or *status* of a vertex v, denoted $\mathrm{sta}(v)$ be a letter $\{A_1, A_2, A, B, C\}$ and let $X(T)$ be the set of all vertices of T labeled X for $X \in \{A_1, A_2, A, B, C\}$. A *labeled graph* is simply one where each vertex is labeled with either A_1, A_2, B, or C. Let $A_i(T)$ be the set of vertices of S that are in an i-pair of M for $i \in [2]$. We form the set $B_1(T)$ by selecting two private neighbors from $V \setminus S$ for each vertex in $A_1(T)$, and we form the set $B_2(T)$ by selecting one private neighbor from $V \setminus S$ for each vertex in $A_2(T)$. Let $A(T) = A_1(T) \cup A_2(T)$ and let $B(T) = B_1(T) \cup B_2(T)$.

We assign labels to the vertices of T as follows.

$$\mathrm{sta}(v) = \begin{cases} A & \text{if } v \in A(T) \\ B & \text{if } v \in B(T) \\ C & \text{if } v \in V \setminus (A(T) \cup B(T)) \end{cases}$$

We also say that

$$\mathrm{sta}(v) = A_i \text{ if } v \in A_i(T) \text{ for } i \in [2].$$

For example, consider the following two special labeled trees. The first tree H_1 is the double star $S(2,2)$ shown in Figure 1(a), where each center is in $A_1(H_1)$ and has status A, and each leaf has status B. The second tree H_2 is the path P_5, where the center is assigned status C, each support vertex is in $A_2(H_2)$ and has status A, and each leaf has status B, as shown in Figure 1(b). We note that the tree H_i is the smallest USPD-tree of order $n \geq 3$ with an i-pair in M and $A(T_i)$ is the unique γ_{pr2}-set of H_i for $i \in [2]$.

(a) Labeled H_1 (b) Labeled H_2

Figure 1: The labeled trees H_1 and H_2

We make the following observation concerning labeled trees.

Observation 4 *If T is a labeled USPD-tree of order $n \geq 3$ with unique γ_{pr2}-set S and an associated matching M, then the following holds.*

(a) $S = A(T)$.
(b) $V \setminus S = B(T) \cup C(T)$.
(c) *For every* 1*-pair* $\{u, v\} \in M$, *each of* u *and* v *has exactly two* S*-external private neighbors in* $B_1(T)$.
(d) *For every* 2*-pair* $\{u, v\} \in M$, *each of* u *and* v *has exactly one* S*-external private neighbor in* $B_2(T)$.

We now define a construction to build a family $\mathcal{T}$ of trees T from two labeled USPD-trees T_1 and T_2, each of order at least 3. We define $T \in \mathcal{T}$ if T is a tree obtained from $T_1 \cup T_2$ by adding an edge u_1u_2, where u_1 is a vertex of T_1 and u_2 is a vertex of T_2. Let S_i along with the associated semi-matching M_i be the unique $\gamma_{\rm pr2}$-set of T_i for $i \in [2]$. Further, let $A(T_i) = A_1(T_i) \cup A_2(T_i)$ and $B(T_i) = B_1(T_i) \cup B_2(T_i)$. Now for every 1-pair $\{u, v\} \in M_i$, each of u and v has exactly two S_i-external private neighbors in $B_1(T_i)$, and for every 2-pair $\{u, v\} \in M_i$, each of u and v has exactly one S_i-external private neighbor in $B_2(T_i)$. We note that the vertices from these pairs may have additional S_i-external private neighbors in $C(T_i)$ for $i \in [2]$. If a vertex in S_i has S_i-private neighbors in both $B(T_i)$ and $C(T_i)$, it is possible to relabel these S_i-private neighbors. That is, for a vertex v with S_i-external private neighbors x and y, where $x \in B(T_i)$ and $y \in C(T_i)$, we define a (B, C)-*swap* to be a relabeling that assigns status C to x and B to y.

If one of u_1 and u_2, say u_1, has status A and u_2 has status B, then if possible we do a (B, C)-swap in T_2 to change the label of u_2 to C. Let $A(T) = A(T_1) \cup A(T_2)$, $B(T) = B(T_1) \cup B(T_2)$, and $C(T) = C(T_1) \cup C(T_2)$. Any tree built in this manner belongs to the family $\mathcal{T}$. Further, we say that $T \in \mathcal{T}$ was obtained using the following operations depending on the status of u_1 and u_2.

- **Operation $\mathcal{L}_1$.** $T \in \mathcal{T}$ and $u_i \in B(T_i) \cup C(T_i)$ for $i \in [2]$.
- **Operation $\mathcal{L}_2$.** $T \in \mathcal{T}$ and $u_i \in A(T_i) \cup C(T_i)$ for $i \in [2]$. Note that in this case, there might have been a (B, C)-swap.

For example, two illustrations of Operation $\mathcal{L}_1$ applied to the labeled trees H_1 and H_2 are given in Figure 2, while two illustrations of Operation $\mathcal{L}_2$ applied to the labeled trees H_1 and H_2 are given in Figure 3. Also, an example of either Operation $\mathcal{L}_1$ or Operation $\mathcal{L}_2$ applied to two labeled H_2 trees is given in Figure 4.

We are now ready to state our main result, which characterizes the USPD-trees in $\mathcal{T}$. We shall prove the following theorem in Section 4.

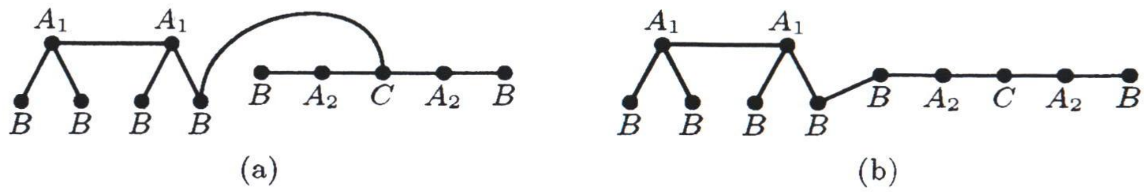

Figure 2: Two illustrations of Operation $\mathcal{L}_1$

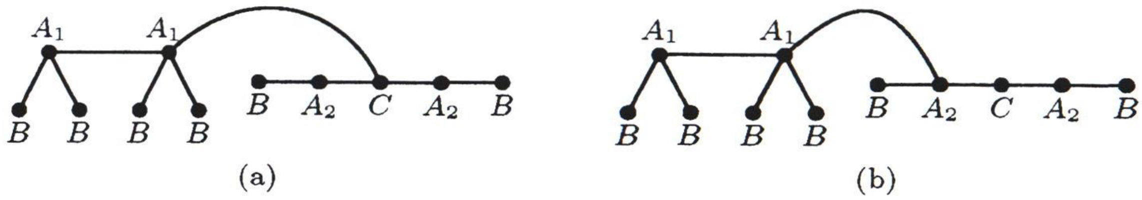

Figure 3: Two illustrations of Operation $\mathcal{L}_2$

Theorem 2 *Let $T \in \mathcal{T}$. Then T is a USPD-tree if and only if T is obtained using Operation $\mathcal{L}_1$ or $\mathcal{L}_2$.*

4 Proof of Theorem 2

We prove Theorem 2 by proving three lemmas using the construction and notation defined in Section 3.

Lemma 1 *If $T \in \mathcal{T}$, then $A(T)$ is a γ_{pr2}-set of T.*

Proof. Let $T \in \mathcal{T}$ be obtained from the labeled USPD-trees T_1 and T_2 by adding the edge u_1u_2, where $u_i \in V(T_i)$. Then T is a labeled tree with $A(T) = A(T_1) \cup A(T_2)$, $B(T) = B(T_1) \cup B(T_2)$, and $C(T) = C(T_1) \cup C(T_2)$.

Let S_i be the unique γ_{pr2}-set of T_i and let M_i be an associated semi-matching for $i \in [2]$. By Observation 4, $S_i = A(T_i)$ and $V(T_i) \setminus S_i = B(T_i) \cup C(T_i)$ for $i \in [2]$. Moreover, for every 1-pair $\{u, v\} \in M_i$, each of u and v has exactly two S-external private neighbors in $B_1(T_i)$, and for

Figure 4: An illustration of Operation $\mathcal{L}_1$ or Operation $\mathcal{L}_2$

every 2-pair $\{u, v\} \in M_i$, each of u and v has exactly one S-external private neighbor in $B_2(T_i)$. We note that the vertices from these pairs may have additional S-external private neighbors in $C(T_i)$ for $i \in [2]$.

Clearly, $A(T) = S_1 \cup S_2$ with semi-matching $M = M_1 \cup M_2$ is a semi-PD-set of T, and so $\gamma_{\rm pr2}(T) \le |S_1| + |S_2| = \gamma_{\rm pr2}(T_1) + \gamma_{\rm pr2}(T_2)$. Let D be a $\gamma_{\rm pr2}$-set of T, and let D_i be the restriction of D to T_i, and so $D_i = D \cap V(T_i)$ for $i \in [2]$. Thus, $\gamma_{\rm pr2}(T) = |D| = |D_1| + |D_2| \le |S| = |S_1| + |S_2|$. To show that $A(T)$ is a $\gamma_{\rm pr2}$-set of T, it suffices to show that $|S_1| + |S_2| \le |D_1| + |D_2|$.

For a semi-matching X associated with D, let X_i be the pairs of the vertices of D_i in X for $i \in [2]$. Note that X may contain pairs that are not in $X_1 \cup X_2$, that is, pairs that contain one vertex from D_1 and one vertex from D_2. We call such a pair a *cross pair*. Among all semi-matchings of D, let X be one with the fewest cross pairs.

Note that if neither u_1 nor u_2 is in D, then the distance between a vertex of D_1 and a vertex of D_2 is at least 3 in T. In this case, the set D_i with semi-matching X_i is a semi-PD-set of T_i for $i \in [2]$, and so $\gamma_{\rm pr2}(T_i) = |S_i| \le |D_i|$ for $i \in [2]$, as desired. Henceforth, we may assume that at least one of u_1 and u_2, say u_1, is in D for otherwise the desired result holds. Let x be the vertex paired with u_1 in X, and if $u_2 \in D$, let y be the vertex paired with u_2 in X. Note that x could be u_2. Further, we note that the set D_1 dominates T_1. Recall that the semi-matching X associated with the $\gamma_{\rm pr2}$-set D of T was chosen to contain the fewest cross pairs. We proceed further with the following series of claims.

Claim 1 *The semi-matching X has at most one cross pair.*

Proof. Suppose, to the contrary, that X has two or more cross pairs. This implies that X has exactly two cross pairs, namely $\{u_1, x\}$ and $\{u_2, y\}$, where $x \ne u_2$ and $y \ne u_1$. In this case, we note that $x \in D_2$ and x is a neighbor of u_2 in T_2, while $y \in D_1$ and y is a neighbor of u_1 in T_1. Hence, the set D with semi-matching $(X \setminus \{\{u_1, x\}, \{u_2, y\}\}) \cup \{\{u_1, y\}, \{u_2, x\}\}$ is a $\gamma_{\rm pr2}$-set of T having no cross pairs, contradicting our choice of X. ($\Box$)

Claim 2 *If the semi-matching X has no cross pairs, then $|S_1| + |S_2| \le |D_1| + |D_2|$.*

Proof. Suppose that X has no cross pairs. Thus, the vertex $x \in D_1$, and if $u_2 \in D_2$, then the vertex $y \in D_2$. That is, every vertex in D_1 is paired with a vertex of D_1 and every vertex of D_2 is paired with a vertex of D_2. Thus, D_1 with semi-matching X_1 is a semi-PD-set of T_1, and so $|S_1| \le |D_1|$. If D_2 dominates T_2, then D_2 with semi-matching X_2 is a semi-PD-set of T_2, and so $|S_2| = \gamma_{\rm pr2}(T_2) \le |D_2|$ and the result holds. Hence, we may assume that D_2 does not dominate T_2.

By assumption, the set D_2 dominates $V(T_2) \setminus \{u_2\}$ and no vertex in $N[u_2]$ is in D_2. In this case, the set $D_2' = D_2 \cup \{u_2, z\}$, where $z \in N(u_2) \setminus \{u_1\}$, with semi-matching $X_2' = X_2 \cup \{\{u_2, z\}\}$ is a semi-PD-set of T_2. Thus, $|S_2| = \gamma_{\rm pr2}(T_2) \le |D_2'|$. If D_2' is a $\gamma_{\rm pr2}$-set of T_2, then since S_2 is the unique $\gamma_{\rm pr2}$-set of T_2, we have $D_2' = S_2$. But then $z \in D_2'$ and z has no D_2'-external private neighbor, implying that $D_2' = S_2$ with semi-matching X_2' does not have Property $\mathcal{P}$. However since $D_2' = S_2$ is the unique $\gamma_{\rm pr2}$-set of T_2, by Theorem 1, D_2' has Property $\mathcal{P}$, a contradiction. Thus, D_2' is not a minimum semi-PD-set of T_2, that is, $D_2' \ne S_2$, and so $|S_2| = \gamma_{\rm pr2}(T_2) < |D_2'|$. Furthermore, since both $|S_2|$ and $|D_2'|$ are even, we have $|S_2| = \gamma_{\rm pr2}(T_2) \le |D_2'| - 2 = |D_2|$. Hence in both cases, we have $|S_2| \le |D_2|$. As observed earlier, $|S_1| \le |D_1|$. Thus, $|S_1| + |S_2| \le |D_1| + |D_2|$. (□)

By Claim 1, the semi-matching X has at most one cross pair. By Claim 2, we may assume that X has exactly one cross pair, for otherwise the desired result $|S_1| + |S_2| \le |D_1| + |D_2|$ follows. Thus, either $\{u_1, x\}$ or $\{u_2, y\}$ is the cross pair of X. Relabeling u_1 and u_2, if necessary, we may assume that the cross pair of X is $\{u_1, x\}$. Thus, $x \in D_2$, that is, $x = u_2$ or x is a neighbor of u_2 in T_2. It follows that every vertex in $D_1 \setminus \{u_1\}$ is paired in X_1. Our next two claims show that $|S_1| \le |D_1| - 1$ and $|S_2| \le |D_2| + 1$, giving $|S_1| + |S_2| \le |D_1| + |D_2|$, as desired.

Claim 3 $\gamma_{\rm pr2}(T_1) = |S_1| \le |D_1| - 1$.

Proof. We note that $D_1 \setminus \{u_1\}$ dominates the tree $T_1 - N[u_1]$. If $D_1 \setminus \{u_1\}$ with semi-matching X_1 is a semi-PD-set of T_1, then $\gamma_{\rm pr2}(T_1) \le |D_1| - 1$. Hence, we may assume that $D_1 \setminus \{u_1\}$ with semi-matching X_1 is not a semi-PD-set of T_1, for otherwise the desired result of the claim follows. This implies that some vertex in the closed neighborhood $N[u_1]$ of u_1 in T_1 is not dominated by $D_1 \setminus \{u_1\}$. This in turn implies that there is a vertex $z_1 \in N(u_1) \cap (V(T_1) \setminus D_1)$. Hence, $D_1' = D_1 \cup \{z_1\}$ with semi-matching $X_1' = X_1 \cup \{\{u_1, z_1\}\}$ is a semi-PD-set of T_1, and so $|S_1| = \gamma_{\rm pr2}(T_1) \le |D_1'| = |D_1| + 1$. If $|S_1| = |D_1'|$, then D_1' is a $\gamma_{\rm pr2}$-set of T_1. Since S_1 is the unique $\gamma_{\rm pr2}$-set of T_1, this implies that $S_1 = D_1'$. But then $z \in S_1$ and z has no S_1-external private neighbor. Hence, $D_1' = S_1$ is the unique $\gamma_{\rm pr2}$-set of T_1 and D_1' with matching X_1' does not have Property $\mathcal{P}$, contradicting Theorem 1. Thus, $|S_1| < |D_1'|$. Since each of $|S_1|$ and $|D_1'|$ is even, we therefore have that $|S_1| \le |D_1'| - 2$, and so $|S_1| \le |D_1| - 1$. (□)

Claim 4 $\gamma_{\rm pr2}(T_2) = |S_2| \le |D_2| + 1$.

Proof. If $D_2 \setminus \{x\}$ with semi-matching X_2 is a semi-PD-set of T_2, then $|S_2| = \gamma_{\rm pr2}(T_2) \le |D_2| - 1$. Hence, we may assume that $D_2 \setminus \{x\}$ with semi-matching X_2 is not a semi-PD-set of T_2, for otherwise the desired result of the claim follows. Since $\{u_1, x\}$ is the unique cross pair of X, we note that every vertex in $D_2 \setminus \{x\}$ is paired in X with a vertex of $D_2 \setminus \{x\}$. Thus since $D_2 \setminus \{x\}$ is not a semi-PD-set of T_2, this implies that at least one neighbor, say z, of x in T_2 does not belong to the set D_2. But then $D_2 \cup \{z\}$ with semi-matching $X_2 \cup \{\{x, z\}\}$ is a semi-PD-set of T_2. Hence, $\gamma_{\rm pr2}(T_2) = |S_2| \le |D_2 \cup \{z\}| = |D_2| + 1$. (□)

By Claim 3, we have $|S_1| \le |D_1| - 1$. By Claim 4, we have $|S_2| \le |D_2| + 1$. Hence, $|S_1| + |S_2| \le |D_1| + |D_2|$, completing the proof of Lemma 1. □

Lemma 2 *If $T \in \mathcal{T}$ is a tree obtained using Operation $\mathcal{L}_1$ or $\mathcal{L}_2$, then T is a labeled USPD-tree and $A(T)$ is the unique $\gamma_{\rm pr2}$-set of T.*

Proof. Let $T \in \mathcal{T}$ be obtained from labeled USPD-trees T_1 and T_2 using Operation $\mathcal{L}_1$ or $\mathcal{L}_2$ to add the edge u_1u_2, where $u_i \in V(T_i)$ for $i \in [2]$. Thus, $A(T) = A(T_1) \cup A(T_2)$, $B(T) = B(T_1) \cup B(T_2)$, and $C(T) = C(T_1) \cup C(T_2)$. Let S_i be the unique $\gamma_{\rm pr2}$-set of T_i and let M_i be an associated semi-matching for $i \in [2]$. By Observation 4, $S_i = A(T_i)$ and $V(T_i) \setminus S_i = B(T_i) \cup C(T_i)$ for $i \in [2]$. Moreover, for every 1-pair $\{u, v\} \in M_i$, each of u and v has exactly two S-external private neighbors in $B_1(T_i)$, and for every 2-pair $\{u, v\} \in M_i$, each of u and v has exactly one S-external private neighbor in $B_2(T_i)$.

By Lemma 1, $S = A(T) = A(T_1) \cup A(T_2)$ with semi-matching $M = M_1 \cup M_2$ is a $\gamma_{\rm pr2}$-set of T. If T is obtained using Operation $\mathcal{L}_1$, then $u_i \in B(T_i) \cup C(T_i)$ for $i \in [2]$. If T is obtained using Operation $\mathcal{L}_2$, then $u_i \in A(T_i) \cup C(T_i)$ for $i \in [2]$. In both cases, the S_i-external private neighbors of vertices of S_i for $i \in [2]$ are S-external private neighbors of the vertices of S. Thus, since the $\gamma_{\rm pr2}$-set S_i of T_i has Property $\mathcal{P}$ in T_i, the $\gamma_{\rm pr2}$-set S has Property $\mathcal{P}$ in T. By Theorem 1, S is the unique $\gamma_{\rm pr2}$-set of T and T is a USPD-tree. □

Lemma 3 *If $T \in \mathcal{T}$ is a USPD-tree, then T was obtained using Operation $\mathcal{L}_1$ or $\mathcal{L}_2$.*

Proof. Let $T \in \mathcal{T}$ be a USPD-tree obtained from the labeled USPD-trees T_1 and T_2 by adding the edge u_1u_2, where $u_i \in V(T_i)$ for $i \in [2]$. Then $A(T) = A(T_1) \cup A(T_2)$, $B(T) = B(T_1) \cup B(T_2)$, and $C(T) = C(T_1) \cup C(T_2)$. By Lemma 1, $A(T) = A(T_1) \cup A(T_2)$ is a $\gamma_{\rm pr2}$-set of T. Since T is a USPD-tree, it follows that $A(T)$ is the unique $\gamma_{\rm pr2}$-set of T.

Suppose, to the contrary, that neither Operation $\mathcal{L}_1$ nor $\mathcal{L}_2$ was used to build T. Renaming T_1 and T_2 if necessary, the only possibility is that $u_1 \in A(T_1)$ and $u_2 \in B(T_2)$. We note that in this case, per our construction, a (B, C)-swap would have relabeled u_2 as C, if possible. Since the swap was not possible, it follows that either u_2 is the only S_2-external private neighbor of a vertex v in a 2-pair of M_2, or u_2 is one of two S_2-external private neighbors of a vertex v in a 1-pair of M_2. But then $v \in A(T)$ and $A(T)$ does not satisfy Property $\mathcal{P}$ in T. By Theorem 1, $A(T)$ is not the unique $\gamma_{\rm pr2}$-set of T, a contradiction. □

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