Edge-coloured graph homomorphisms, paths, and duality

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Abstract

We present a 2-edge-coloured analogue of the duality theorem for transitive tournaments and directed paths. Given a 2-edge-coloured path P whose edges alternate blue and red, we construct a 2-edge-coloured graph D so that for any 2-edge-coloured graph G

$$P \to G \Leftrightarrow G \not\to D.$$

The duals are simple to construct, in particular $|V(D)| = |V(P)| - 1$.

Dedicated to Gary MacGillivray on the occasion of his 60^{th} birthday.

1 Introduction

In this paper we study homomorphisms of edge-coloured graphs. Our main result is a 2-edge-coloured analogue of the duality theorem for transitive tournaments and directed paths. While duality theorems for finite structures are fully described by Nešetřil and Tardif [9], our contribution is to present a simple construction of small (linear) duals for 2-edge-coloured alternating paths, and use them to highlight similarities and differences between edge-coloured graph and digraph homomorphisms.

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A 2-*edge-coloured graph* G is a vertex set $V(G)$ together with 2 edge sets $E_1(G), E_2(G)$. We allow loops and multiple edges provided parallel edges have different colours. The *underlying graph of* G is the (classical) graph with vertices $V(G)$ and edge set $E_1(G) \cup E_2(G)$, i.e. the graph obtained by ignoring edge colours and deleting multiple edges. A 2-edge-coloured graph is a *path* if the underlying graph is a path. We similarly define other standard notions like cycle and walk.

As a convention colours 1 and 2 are blue and red respectively. Blue edges are depicted in diagrams as solid while red edges are dashed. A vertex is *mixed* if it is incident with both blue and red edges. A vertex is *blue only* (respectively *red only*) if it is incident only with blue (respectively red) edges. See for example Figure 1: the vertices labelled +1 are blue only and vertices labelled +2 are mixed.

Let G and H be 2-edge-coloured graphs. A *homomorphism* φ of G to H is a mapping $\varphi : V(G) \to V(H)$ such that $uv \in E_i(G)$ implies $\varphi(u)\varphi(v) \in E_i(H)$. We write $\varphi : G \to H$ to indicate the existence of a homomorphism or simply $G \to H$ when the name of the mapping is not important. Homomorphisms of digraphs and relational systems are similarly defined. If $G \to H$ and $H \to G$ we say G is *homomorphically equivalent* to H and write $G \sim H$.

Finally, we remark that the above definitions naturally generalize to k-edge coloured graphs.

The launching point for our work is the well known digraph homomorphism duality theorem for transitive tournaments. Let T_n be the *transitive tournament on n vertices* with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{ij | 1 \leq i < j \leq n\}$. Let $\vec{P}_n$ be the directed path on n vertices. Given a digraph G, then

$$G \not\to T_n \Leftrightarrow \vec{P}_{n+1} \to G$$

The digraphs $(\vec{P}_{n+1}, T_n)$ are a *duality pair.* In general, given a relational system F, the pair $(F, D(F))$ is a *duality pair* if for all relational systems G, $G \not\to D(F)$ if and only if $F \to G$. More generally, $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ and H have *finite duality* if for any G

$$G \not\to H \Leftrightarrow F_i \to G \text{ for some } i.$$

The pair $(\mathcal{F}, H)$ is a *finite homomorphism duality.* In [11], Nešetřil and Tardif prove the existence of finite homomorphism dualities if and only if $\mathcal{F}$ is a set whose underlying graphs are trees. Their construction produces exponentially large duals. In some cases these duals are homomorphically equivalent to a much small structure, but there are trees F with exponentially large dual $D(F)$ that cannot be reduced to a smaller structure [9].

The motivation for this work is to explicitly (and easily) construct the duals for 2-edge-coloured alternating paths (defined below). This is motivated by the fact that there are many parallels to the theory of 2-edge-coloured graph homomorphisms and digraph homomorphisms. Specifically, homomorphisms of 2-edge-coloured graphs to alternating paths behave like homomorphisms of digraphs to directed paths. This similarity has been observed in multiple works [1, 2, 3, 8]. In this paper we now study homomorphisms of alternating paths into 2-edge-coloured graphs with the hope that their dual 2-edge-coloured targets behave like transitive tournaments. Our work elicits similarities and differences between the two families.

2 Alternating path duality

A 2-edge-coloured graph G is an *alternating path* if G is a path whose edges successively alternate blue and red.

Definition 2.1. *The 2-edge-coloured graphs F_k^B and F_k^R are alternating paths of length k with vertices $v_0, v_1, v_2, \ldots, v_k$. When k is odd the edge $v_{\lfloor k/2 \rfloor} v_{\lceil k/2 \rceil}$ is colour blue in F_k^B and colour red in F_k^R. For even k, F_k^B and F_k^R are isomorphic alternating paths and we simply write F_k to denote either. We define the families $\mathcal{F}_k := \{F_k^R, F_k^B\}$ and $\mathcal{F} = \bigcup_{k=1}^{\infty} \mathcal{F}_k$.*

The first family of dual 2-edge-coloured graphs is defined as follows. See Figure 1 for examples.

Definition 2.2. *Let $k \geq 1$ be an integer and let $j = \lfloor k/2 \rfloor$. The 2-edge-coloured graph D_k has vertex set:*

$$V(D_k) = \begin{cases} \{\pm 1, \pm 2, \ldots, \pm j\} & \text{if } k \text{ is even} \\ \{0, \pm 1, \pm 2, \ldots, \pm j\} & \text{if } k \text{ is odd.} \end{cases}$$

There are blue loops on $\{1, 2, \ldots, j\}$, red loops on $\{-1, -2, \ldots, -j\}$, and an edge rs for all $|r| < |s|$. The edge rs is blue if $r > 0$ and red if $r < 0$. If $r = 0$, the edge rs is blue if $s > 0$ and red if $s < 0$. For odd values of k, the 2-edge-coloured graph D_k^B (respectively D_k^R) is obtained from D_k by adding a blue loop (respectively red loop) to the vertex 0. We define the families

$$\mathcal{D}_k := \begin{cases} \{D_k, D_k^B, D_k^R\} & \text{if } k \text{ is odd,} \\ \{D_k\} & \text{if } k \text{ is even,} \end{cases}$$

and $\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k$.

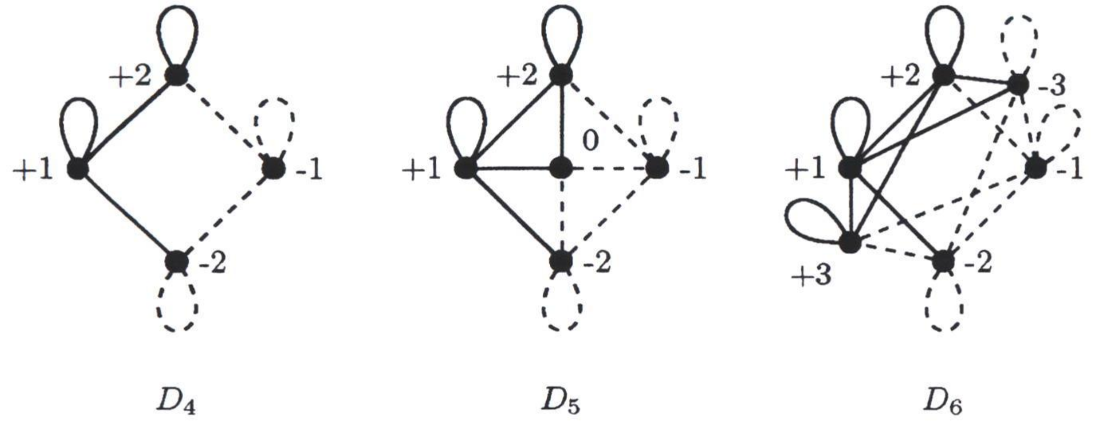

Figure 1: The targets D_4, D_5, D_6

Alternatively, one can describe the 2-edge-coloured graphs D_k with a recursive construction. The 2-edge-coloured graph D_1 is a single vertex: $V(D_1) = \{0\}$. The 2-edge-coloured graph D_2 has two vertices: $V(D_2) = \{-1, 1\}$. There is a blue loop on 1 and a red loop on -1. For $k \geq 3$, the 2-edge-coloured graph D_k is obtained from D_{k-2} by adding two vertices $\{-j, j\}$ where $j = \lfloor k/2 \rfloor$. There is a blue loop on j and red loop on $-j$. For a vertex $v \in V(D_{k-2})$ there is a blue edge to each of j and $-j$ if $v > 0$; a red edge to each if $v < 0$; and in the case k is odd there is a blue edge from j to 0 and a red edge from $-j$ to 0.

The following proposition is immediate from the definitions.

Proposition 2.3. *For $k \geq 1$, $D_{2k-1} \subseteq D^{c_0}_{2k-1} \subseteq D_{2k} \subseteq D_{2k+1}$ and $F^{c_1}_k \subseteq F^{c_2}_{k+1}$ where $c_0, c_1, c_2 \in \{B, R\}$.*

Since $G \subseteq H$ corresponds precisely to an embedding homomorphism $G \to H$, the proposition implies $D \to D'$ whenever $D \in \mathcal{D}_k$ and $D' \in \mathcal{D}_{k'}$ with $k < k'$. Also, $F^{c_1}_k \to F^{c_2}_{k'}$ for $k < k'$ and $c_1, c_2 \in \{B, R\}$. The homomorphism partial order on $\mathcal{D}$ under $\to$ is shown in Figure 2, which also includes the maps $D_k \to D^c_k$.

Our main results are the following duality theorems. The first covers duality pairs of the form $(F, D(F))$. The second gives finite duality theorems for $(\mathcal{F}_k, D_k)$, specifically for odd k there is a family of two obstructions.

Theorem 2.4. *Let G be a 2-edge-coloured graph and $k \geq 1$ be an integer. Then the following hold:*

(i) $G \to D_{2k}$ *if and only if* $F_{2k} \not\to G$,

(ii) $G \to D^B_{2k-1}$ *if and only if* $F^R_{2k-1} \not\to G$, *and*

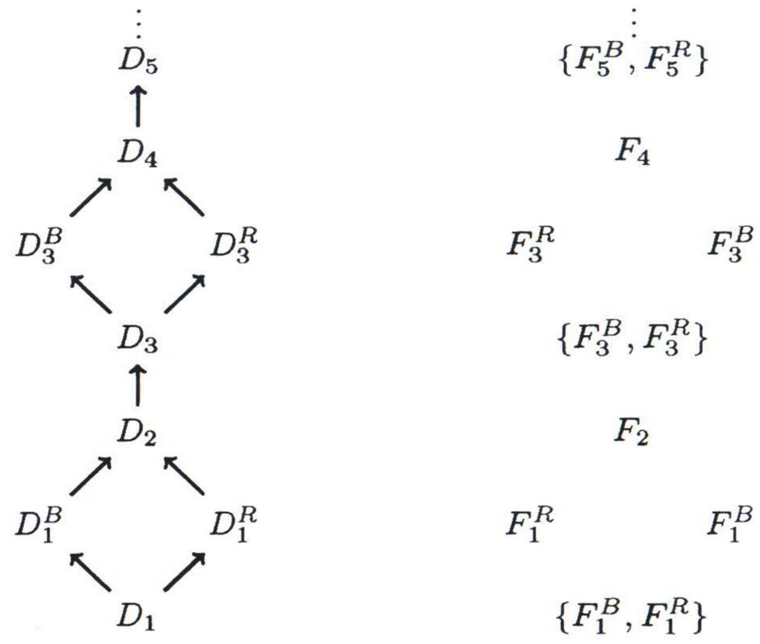

Figure 2: Homomorphism order for $\mathcal{D}_k$ and their duals. Each dual admits a homomorphism to a dual with a larger subscript.

(iii) $G \to D_{2k-1}^R$ *if and only if* $F_{2k-1}^B \not\to G$.

Theorem 2.5. *Let* G *be an 2-edge-coloured graph and* $k \geq 1$ *be an integer. Then* $G \to D_k$ *if and only if for all* $F \in \mathcal{F}_k, F \not\to G$.

If $G \to D$ for some $D \in \mathcal{D}$, then there is a (unique) minimal element $D' \in \mathcal{D}$ such that $G \to D'$. (This follows from our work below, but Figure 2 strongly suggests the result.) A consequence of the theorems above is that we can certify the minimality using homomorphisms of the form $F \to G$ for $F \in \mathcal{F}$. That is, we certify that $G \not\to D''$ for any predecessor of D'. Formally,

Definition 2.6. *Suppose* $G \to D$ *for some* $D \in \mathcal{D}_k$. A certificate of minimality *is:*

- *a homomorphism* $f : F_{k-1} \to G$ *when* $D = D_k$ *and* k *is odd;*
- *a homomorphism* $f : F_k^R \to G$ *when* $D = D_k^R$ *and* k *is odd;*
- *a homomorphism* $f : F_k^B \to G$ *when* $D = D_k^B$ *and* k *is odd; or*
- *a pair of homomorphisms* $f_R : F_{k-1}^R \to G$ *and* $f_B : F_{k-1}^B \to G$ *when* $D = D_k$ *and* k *is even.*

In the following section we give a linear time algorithm that takes as input a 2-edge-coloured graph G, determines the minimum k for which $G \to D$ for some $D \in \mathcal{D}_k$ (if it exists), and produces a certificate of minimality. (In an abuse of notation to simplify the presentation of the algorithm, we write it returns $f : F \to G$ as the certificate of minimality. In the last case listed above, the certificate is technically 2 maps, but could be equivalently encoded as a single map $f : \{F^R_{k-1} \cup F^B_{k-1}\} \to G$.)

The proof of both theorems is accomplished as follows. The following lemma proves one direction for each of the duality claims. The converse will follow from our algorithm.

Lemma 2.7. *Let $k \geq 1$ and $F \in \mathcal{F}_k$. Then $F \not\to D_k$. Further if k odd, then $F^B_k \not\to D^R_k$ and $F^R_k \not\to D^B_k$.*

Proof. We proceed by induction on k. For $k = 1$ and $k = 2$ the results are trivial. Let $k > 2$ and assume to the contrary that there exists a homomorphism $f : F_k \to D_k$. Let F' be the subpath of F induced by $\{v_1, \ldots, v_{k-1}\}$. Note that each vertex of F' is mixed in F. Thus under f each vertex of F' must map to a mixed vertex of D_k. Let D' be the subgraph of D_k induced by the mixed vertices, specifically $D' = D_k \backslash \{\pm 1\}$. However, $F' \to D'$ contradicts the inductive hypothesis as $F' \in \mathcal{F}_{k-2}$ and D' is isomorphic to D_{k-2}.

A similar argument shows for k odd, $F^B_k \not\to D^R_k$ and $F^R_k \not\to D^B_k$. □

A digraph containing a directed cycle does not admit a homomorphism to T_n for any n. Analogously, a 2-edge-coloured graph G containing a closed, alternating walk has the property that $F \to G$ for all $F \in \mathcal{F}$. Consequently G does not admit a homomorphism to D_k for any k. Thus a closed alternating walk in G certifies $G \not\to D_k$ for any k. A 2-edge-coloured graph G is *smooth* if each vertex is mixed.

Lemma 2.8. *Let G be a smooth 2-edge-coloured graph. Then G contains a closed alternating walk and $F_k \to G$ for all $k \geq 1$. Consequently, $G \not\to D_k$ for any k.*

Proof. Let G be a smooth 2-edge-coloured graph. Let P be a maximal alternating path in G. Let the vertices of the path be $p_0, p_1, \ldots, p_t$. Suppose p_0p_1 is blue. By smoothness, p_0 is incident with a red edge. By the maximality of P, if p_0u is red, then $u = p_i$ for some i. If i odd, then $p_0, p_1, \ldots, p_i, p_0$ is an alternating cycle and we are done.

Thus suppose i is even. Similarly, p_t has a neighbour p_j such that $p_{t-1}p_tp_j$ is an alternating path. In the case j and t have opposite parity, $p_j, p_{j+1}, \ldots, p_{t-1}, p_t, p_j$ is an alternating cycle. Otherwise we have $p_0, p_i, p_{i+1}, \ldots, p_t, p_j, p_{j-1}, \ldots, p_1, p_0$ is a closed alternating walk.

The claim $F_k \to G$ for all $k \geq 1$ is trivial as one can simply wrap F_k around the closed walk. By Lemma 2.7, $G \not\to D_k$ for any k. $\square$

Clearly the proof of the lemma gives an algorithm for finding a closed alternating walk in a smooth 2-edge-coloured graph. (The path P can be constructed greedily starting at an arbitrary vertex v.) This is used in Step 2.5 of our algorithm.

3 Duality Algorithm

We now present our algorithm that takes as input a 2-edge-coloured graph G, and returns either a homomorphism $g : G \to D$, $D \in \mathcal{D}$ together with a certificate of minimality for D, or a smooth subgraph of G. In the latter case the smooth subgraph certifies $G \not\to D$ for all $D \in \mathcal{D}$. (See Figure 3 for the algorithm pseudocode.)

Roughly, the algorithm works by mapping all blue (respectively red) only vertices of G to +1 (respectively -1). The mapped vertices are deleted from G and the process is repeated mapping the now blue (red) only vertices to +2 (-2). Delete the mapped vertices and iterate until the entirety of G is mapped to D_{2i} or a smooth subgraph is discovered.

The algorithm has a post-processing phase in Step 3. The goal of this step is to find the minimum $D \in \{D_{2i-1}, D^B_{2i-1}, D^R_{2i-1}, D_{2i}\}$ to which G maps. A small note on the word minimum is required here. At the termination of the algorithm in Figure 3 we know that G maps to some subset of the four targets above. The only way this subset could not have a well defined minimum (see Figure 2) is if $G \to D^B_{2i-1}$ and $G \to D^R_{2i-1}$ but $G \not\to D_{2i-1}$. However, by Corollary 4.3, this is impossible.

The final step of the algorithm is the construction of the certificate of minimality in a subroutine Build(f). We believe the subroutine is more easily described in words rather than pseudocode. The paragraph below labelled Build(f) contains its description. Note we may assume that G is connected as we can apply the algorithm on each component of G.

The correctness of the algorithm follows from some straightforward observations. Using the notation as defined in Algorithm 1 (Figure 3), let G_i be the subgraph of G induced by $\bigcup_{j=1}^{i}(B_j \cup R_j \cup I_j)$.

Proposition 3.1. *After i iterations of the loop (Step 2) the following invariants are true.*

Duality Algorithm

Input: A connected 2-edge-coloured graph G.

Output: Either $g : G \to D$ where $D \in \mathcal{D}$ together with a certificate of minimality $f : F \to G$, or a closed alternating walk W in G.

1. **Set** $i = 0$, $G_0 = G$. (G_0 *is the unmapped subgraph*)

2. **While** $(G_0 \neq \emptyset)$

   2.1. $i++$

   2.2. **Let** $B_i = \{u \in V(G_0) | u$ is blue only in $G_0\}$.

   2.3. **Let** $R_i = \{u \in V(G_0) | u$ is red only in $G_0\}$.

   2.4. **Let** $I_i = \{u \in V(G_0) | u$ is isolated in $G_0\}$.

   2.5. **If** $B_i \cup R_i \cup I_i = \emptyset$, **then** G_0 is smooth. Find a closed alternating walk W in G_0. **Return** W and **answer NO**.

   2.6. **For each** $u \in B_i \cup I_i$: **Set** $g(u) = i$.

   2.7. **For each** $u \in R_i$: **Set** $g(u) = -i$.

   2.8. $G_0 = G_0 - (B_i \cup R_i \cup I_i)$

   End while

3. **If** $B_i \cup R_i = \emptyset$, **set** $g(u) = 0$ for all $u \in I_i$, **return** $g : G \to D_{2i-1}$
   Elseif $R_i = \emptyset$, **set** $g(u) = 0$ for all $u \in B_i \cup I_i$, **return** $g : G \to D^B_{2i-1}$
   Elseif $B_i = \emptyset$, **set** $g(u) = 0$ for all $u \in R_i \cup I_i$, **return** $g : G \to D^R_{2i-1}$
   Else return $g : G \to D_{2i}$.

4. **Call** Build(f)

5. **Return** g, f and **answer YES**.

Figure 3: The algorithm

1. *The map g defines a homomorphism $G_i \to D_{2i}$.*

2. *Each vertex $v \in B_i$ (respectively $v \in R_i$) is the terminus of an alternating path of length $i-1$ whose last edge is coloured red (respectively coloured blue), whereas each vertex $v \in I_i$ is the midpoint of an alternating path of length $2i-2$.*

Proof. Invariant 2 is trivial when $i = 1$. Assume $i > 1$. Observe for $v \in B_i$, v is incident with only blue edges during iteration i, but was mixed at any previous iteration, in particular at iteration $i-1$. Hence v is adjacent to some $u \in R_{i-1}$. It follows by induction v is the terminus of an alternating path of length $i-1$ whose last edge is red. The case $v \in R_i$ is analogous. For a vertex $v \in I_i$, observe that v was mixed at iteration $i-1$, but isolated at iteration i. Hence, v is adjacent to a vertex in R_{i-1} and B_{i-1}. Thus Invariant 2 holds.

To see Invariant 1 holds, let uv be an edge of G_i. We may assume u is mapped at iteration i. Suppose $g(u) = i$ and $g(v) = j$. (The case $g(u) = -i$ is similar.) Since $u \in B_i \cup I_i$, if $i = |j|$, then it must be the case that $u, v \in B_i$ and the edge uv maps to the loop ii. Otherwise, $|j| < i$. If $j < 0$, then $v \in R_j$; otherwise $v \in B_j$. In the former case uv is red (as v is red only at iteration j). In the latter case uv is blue. The definition of D_{2i} states that ji is red for $j < 0$ and blue for $j > 0$. Thus, g is a homomorphism. □

Observe that the alternating paths described in Invariant 2 can be computed in linear time by simply storing a pointer from each vertex to a parent whose deletion in the previous iteration causes the vertex to no longer be mixed.

From Proposition 3.1, at the terminus of Step 2 we have $g : G \to D_{2i}$. The post processing at Step 3 is straightforward to analyze. We complete the proof of correctness for the algorithm with the subroutine Build(f).

Build(f) At Step 4, the algorithm calls a subroutine to build a certificate of minimality. Recall the certificate is either a homomorphism $f : F \to G$ or a pair of homomorphisms $f_1 : F_1 \to G$ and $f_2 : F_2 \to G$ to certify that $G \not\to D'$ for any $D' < D$ where $g : G \to D$ is returned by the algorithm. (Here $D' < D$ is the homomorphism order on $\mathcal{D}$. Thus, $D' \to D$ but $D \not\to D'$.)

Consider the post-processing phase of the algorithm at Step 3. If $B_i \cup R_i = \emptyset$, then let $u \in I_i$. By Proposition 3.1, u is the centre vertex of an alternating path of length $2i-2$, i.e. there exists $f : F_{2i-2} \to G$ which is the certificate of minimality for $g : G \to D_{2i-1}$. Otherwise, consider the next case: Elseif $R_i = \emptyset$. Let $u \in B_i$. Then u is incident with blue edges in the final iteration of the loop. In particular, there is a vertex $v \in B_i$ such that uv is a blue edge. Both u and v are termini of alternating paths of length $i-1$ whose final edges are red. These two paths together with the edge uv allow us to define $f : F^B_{2i-1} \to G$. This is a certificate of minimality for $g : G \to D^B_{2i-1}$. Similarly, in the next case (Elseif $B_i = \emptyset$) we can find a certificate of minimality $f : F^R_{2i-1} \to G$ for $g : G \to D^R_{2i-1}$. Finally, in the last case $R_i \neq \emptyset$ and $B_i \neq \emptyset$. Using the arguments above, we can find $f_B : F^B_{2i-1} \to G$ and $f_R : F^R_{2i-1} \to G$. This pair of maps is a certificate of minimality for $g : G \to D_{2i}$.

4 Proofs of Theorems 2.4 and 2.5

Proof of Theorem 2.4: Suppose $F_{2k} \to G$ and suppose to the contrary $G \to D_{2k}$. Then by composition, $F_{2k} \to D_{2k}$ contrary to Lemma 2.7. Similarly, if $F^R_{2k-1} \to G$, then $G \not\to D^B_{2k-1}$, and if $F^B_{2k-1} \to G$, then $G \not\to D^R_{2k-1}$.

Conversely, suppose that $F_{2k} \not\to G$. Run the algorithm in Figure 3. First observe that F_{2k} maps to any closed alternating walk. We can conclude that G does not contain a closed alternating walk. In particular, the algorithm does not terminate at Step 2.5, but rather the main loop terminates with a map $g : G \to D_{2i}$. In the post processing step we construct at least one of the following maps: $f : F_{2i-2} \to G$, $f_R : F^R_{2i-1} \to G$, or $f_B : F^B_{2i-1} \to G$. By Proposition 2.3 and the assumption $F_{2k} \not\to G$, we have $2i-2 < 2k$. Hence, $2i \leq 2k$ and $G \to D_{2i} \to D_{2k}$.

Suppose $F^R_{2k-1} \not\to G$. Again the algorithm cannot discover a smooth subgraph in G, so $g : G \to D_{2i}$ for some i. If $i < k$, then $G \to D_{2i} \to D^B_{2k-1}$ as required. The map(s) constructed in the post processing step ensures $F_{2i-2} \to G$ and thus $i \leq k$, so $i = k$. This implies the certificate of minimality from Step 3 is either $f : F_{2k-2} \to G$ (in which case $g : G \to D_{2k-1}$ is returned by the algorithm) or $f : F^B_{2k-1} \to G$ (in which case $g : G \to D^B_{2k-1}$ is returned by the algorithm). In both cases $G \to G^B_{2k-1}$ as required. The case $F^B_{2k-1} \not\to G$ is analogous. □

Proof of Theorem 2.5: Note for any even positive integer $2k$, $\mathcal{F}_{2k} = \{F_{2k}\}$. Thus $G \to D_{2k}$ if and only if for all $F \in \mathcal{F}_{2k}, F \not\to G$ by Theorem 2.4. For the odd integer $2k-1$, $\mathcal{F}_{2k-1} = \{F^R_{2k-1}, F^B_{2k-1}\}$. Suppose $F^R_{2k-1} \not\to G$ and $F^B_{2k-1} \not\to G$. At the termination of the algorithm $g : G \to D_{2i}$. If $2i < 2k-1$, then $G \to D_{2k-1}$ as required. Otherwise, it must be the case $2i = 2k$. Consequently, the algorithm must return $F_{2k-2} \to G$ as the certificate of minimality (from Case 1 of the post processing). Hence $G \to D_{2k-1}$ as required. □

Definition 4.1. *Let G and H be edge-coloured graphs. The* categorical product *$G \times H$ is the graph with vertex set $V(G \times H) = V(G) \times V(H)$, and edges $((u, u'), (v, v'))$ of colour i if and only if u' is adjacent with v' with colour i and u is adjacent with v with colour i.*

The following corollary follows from [10] and Theorem 2.5.

Theorem 4.2 (Nešetřil and Tardif [10]). *The pair $(\mathcal{F}, D)$ is a finite homomorphism duality if and only if*

$$D \sim \prod_{F \in \mathcal{F}} D(F).$$

It is well known that $G \to X \times Y$ if and only if $G \to X$ and $G \to Y$. Hence, the following corollary shows when $G \to D^R_{2k-1}$ and $G \to D^B_{2k-1}$ it must be the case that $D \to D_{2k-1}$. (This follows from our certificates of minimality.) We conclude that if $G \to D \in \mathcal{D}$ there is a well defined minimum element D' such that $G \to D'$.

Corollary 4.3. *The 2-edge-coloured D_{2k-1} can be expressed as*

$$D_{2k-1} \sim D^R_{2k-1} \times D^B_{2k-1}.$$

5 Future Work

As stated in the introduction, our aim in this paper was to find the duals of alternating paths and examine them for similarities with transitive tournaments. On the one hand we have that any 2-edge-coloured graph G without a closed alternating walk admits a homomorphism to $\mathcal{D}$. The ordering on $\mathcal{D}$ ensures that there is a well defined minimum element in $\mathcal{D}$ to which G maps. This is analogous to acyclic digraphs admitting a homomorphism to a minimum transitive tournament. That is, $\mathcal{D}$ maybe viewed as a *calibrating family* as defined by Hell and Nešetřil [6].

On the other hand, our attempts to generalize other results remain open. For example, given an undirected graph, the Gallai-Hasse-Roy-Vitaver Theorem ([4, 5, 12, 13] see also Corollary 1.21 [6]) gives the well known connection between colourings of the graph and acyclic orientations of the graph. Specifically using the duality pair $(\vec{P}_{n+1}, T_n)$ the following is immediate.

Theorem 5.1. *A graph G is n-colourable if and only if there is an orientation D of G such that $\vec{P}_{n+1} \not\to D$.*

The connection between orientations and colourings of a graph also appears in Minty's Theorem. Let D be an orientation of a graph G. Given a cycle C of D, we select a direction of traversal. Let $|C^+|$ denote the set of forward arcs of C, and $|C^-|$ denote the set of backward arcs of C with respect to the traversal direction. Define $\tau(C) = \max\{|C|/|C^-|, |C|/|C^+|\}$ to be the *imbalance* of C.

Theorem 5.2 (Minty [7]). *For any graph G,*

$$\chi(G) = \min_D\{\lceil \max\{\tau(C) : \text{C is a cycle of D}\}\rceil\}$$

where the minimum is over all acyclic orientations D of G.

We wonder if these theorems can be generalized.

Question 5.3. *Are there analogues to Theorems 5.1 and 5.2 using the family $\mathcal{D}$ for 2-edge-coloured graphs?*

On the surface the idea of assigning colours to the edges of a graph and minimizing the length of an alternating path will not work. Assigning all edges to blue produces a 2-edge-coloured with longest alternating path of length one and a homomorphism to D_1^B, independently of the chromatic number of the graph.

One may also consider the duals of other 2-edge-coloured paths. The variety (closure under products and retracts) of these duals gives the set of 2-edge-coloured graphs admitting a *near unanimity function* of order 3: NU_3.

Question 5.4. *Can we easily describe duals of other 2-edge-coloured paths?*

Finally, we can also study the case with more edge colours or even (m, n)-mixed graphs (graphs with m edge sets and n arc sets [8]). The introduction of arcs and edges changes the category under question with significant implications on duality pairs. Replacing unordered coloured edges with a pair of oppositely directed arcs (of the same colour) to move our setting to 2-edge-coloured digraphs causes our main results to no longer hold. For example, (F_1^R, D_1^B) is no longer a duality pair. A single red arc is a counterexample. In fact, in this setting F_1^R has no dual.

Question 5.5. *Can the duality results here be generalized to (m, n)-mixed graphs?*

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