Extremal Problems in Royal Colorings of Graphs

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Dedicated to Gary MacGillivray
on the Occasion of his 60th Birthday

Abstract

An edge coloring c of a graph G is a royal k-edge coloring of G if the edges of G are assigned nonempty subsets of the set $\{1, 2, \ldots, k\}$ in such a way that the vertex coloring obtained by assigning the union of the colors of the incident edges of each vertex is a proper vertex coloring. If the vertex coloring is vertex-distinguishing, then c is a strong royal k-edge coloring. The minimum positive integer k for which G has a strong royal k-edge coloring is the strong royal index of G. It has been conjectured that if G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ for a positive integer k, then the strong royal index of G is either k or $k+1$. We discuss this conjecture along with other information concerning strong royal colorings of graphs. A sufficient condition for such a graph to have strong royal index $k+1$ is presented.

JCMCC 116 (2021), pp.201-218

1 Introduction

During the past several years, a number of edge colorings (or edge labelings) have been introduced that give rise to vertex colorings that are either proper or vertex-distinguishing (see [1, 2, 3, 7], for example). Many of these are discussed in the books [6, 9]. We discuss two of these colorings here. For a connected graph G of order 3 or more and a positive integer k, let $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ be an unrestricted edge coloring of G, that is, adjacent edges of G may be assigned the same color. We write $\mathcal{P}^*([k])$ for the set consisting of the $2^k - 1$ nonempty subsets of $[k]$. The edge coloring c gives rise to the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ where $c'(v)$ is the set of colors of the edges incident with v. If c' is a proper vertex coloring of G, then c is a *majestic k-edge coloring* and the minimum positive integer k for which G has a majestic k-edge coloring is the *majestic index* $\text{maj}(G)$ of G. If c' is *vertex-distinguishing* (that is, $c'(u) \neq c'(v)$ for every two distinct vertices u and v of G), then c is a *strong majestic k-edge coloring* and the minimum positive integer k for which G has a strong majestic k-edge coloring is the *strong majestic index* $\text{smaj}(G)$ of G. Majestic edge colorings were introduced by Györi, Horňák, Palmer, and Woźnick [10] under different terminology and studied further in [12, 13]. Strong majestic edge colorings were introduced by Harary and Plantholt [11] in 1985, also using different terminology, and studied further by others (see [9, 14, 15]).

While an edge coloring c of a graph G typically uses colors from the set $[k]$ for some positive integer k resulting in $c(e) = i$ for some $i \in [k]$, we might equivalently define $c(e) = \{i\}$ as well. Expressing the edge coloring c in this way results in both c and the induced vertex coloring c' assigning subsets of $[k]$ to the edges as well as the vertices of G. Furthermore, expressing c in this manner suggests the idea of studying edge colorings c where both c and its derived vertex coloring c' assign nonempty subsets of $[k]$ to the elements (edges and vertices) of a graph G such that the color assigned to an edge of G by c is not necessarily a singleton subset of $[k]$. This observation gives rise to the primary concepts of this paper, namely royal and strong royal colorings, which were introduced in [8].

For a positive integer k, let $\mathcal{P}^*([k])$ denote the collection of the $2^k - 1$ nonempty subsets of the set $[k]$. For a connected graph G of order 3 or more, an edge coloring $c : E(G) \to \mathcal{P}^*([k])$ of G is a *royal k-edge coloring* if the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ defined by $c'(v) = \bigcup_{e \in E_v} c(e)$, where E_v is the set of edges of G incident with v, is proper, that is, adjacent vertices are assigned distinct colors. The minimum positive integer k for which G has a royal k-edge coloring is the *royal index* of G, denoted by $\text{roy}(G)$. If c' is vertex-distinguishing, then c is a *strong royal k-edge coloring* of G.

The minimum positive integer k for which G has a strong royal k-edge coloring is the *strong royal index* of G, denoted by $\text{sroy}(G)$. Therefore, royal colorings are generalizations of majestic edge colorings and strong royal colorings are generalizations of strong majestic colorings. This concept was independently introduced and studied in [4, 8]. While there are many connected graphs G for which $\text{sroy}(G) \neq \text{smaj}(G)$, we know of no graph G for which $\text{roy}(G) \neq \text{maj}(G)$. Consequently, our emphasis here is on the strong royal indexes of graphs. If G is a connected graph of order $n \geq 4$, there is a unique integer $k \geq 3$ such that $2^{k-1} \leq n \leq 2^k - 1$. We now present several useful observations made in [4, 8].

Observation 1.1 *If G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$, then $\text{sroy}(G) \geq k$.*

Observation 1.2 *If G is a connected graph of order 4 or more, then*

$$\text{sroy}(G) \leq 1 + \min\{\text{sroy}(H) : \ H \text{ is a connected spanning subgraph of } G\}.$$

In particular, $\text{sroy}(G) \leq 1 + \min\{\text{sroy}(T) : \ T \text{ is a spanning tree of } G\}$.

It was shown in [4] that if G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$, then $\text{sroy}(G) \leq k + 2$. Furthermore, it was conjectured in [8] that the strong royal index of every connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ is either k or $k+1$. This gives rise to the following concepts. A connected graph G of order $n \geq 3$ where $2^{k-1} \leq n \leq 2^k - 1$ is a *royal-zero graph* if $\text{sroy}(G) = k$ and is a *royal-one graph* if $\text{sroy}(G) = k + 1$. Therefore, the conjecture on the strong royal index can be rephrased as follows.

Conjecture 1.3 *Every connected graph of order at least 4 is either royal-zero or royal-one.*

By Observation 1.2, the strong royal indexes of trees play an important role in the study of strong royal indexes of connected graphs. It was conjectured in [8] that every tree of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ has strong royal index k and consequently is royal-zero. This conjecture can therefore be rephrased in terms of royal-zero graphs.

Conjecture 1.4 *Every tree of order at least 4 is royal-zero.*

Conjecture 1.4 has been verified for trees of small order (order 10 or less), all paths, all complete binary trees, all caterpillars of diameter 4 or less as well as some specialized trees (see [4, 8]). By Observation 1.2, it follows that if Conjecture 1.4 is true, then Conjecture 1.3 is true as well. While the strong royal index of each cycle was stated in [4], we illustrate the concepts described above by providing a proof that describes in each case an appropriate edge coloring.

Theorem 1.5 *For every integer $n \geq 3$,*

$$\mathrm{sroy}(C_n) = \begin{cases} 1 + \lceil \log_2(n+1) \rceil & \text{if } n = 3, 7 \\ \lceil \log_2(n+1) \rceil & \text{if } n \neq 3, 7. \end{cases}$$

That is, if C_n is a cycle of length $n \geq 3$ where $2^{k-1} \leq n \leq 2^k - 1$ for some integer k, then $\mathrm{sroy}(C_n) = k$ unless $n = 3$ or $n = 7$, in which case, $\mathrm{sroy}(C_3) = 3$ and $\mathrm{sroy}(C_7) = 4$.

Proof. Let $k = \lceil \log_2(n+1) \rceil \geq 2$. Then $2^{k-1} \leq n \leq 2^k - 1$. We show that $\mathrm{sroy}(C_3) = 3$, $\mathrm{sroy}(C_7) = 4$, and $\mathrm{sroy}(C_n) = k$ if $n \neq 3, 7$. Figure 1 shows a strong royal 3-edge coloring of C_3 and a strong royal 4-edge coloring of C_7, which shows that $\mathrm{sroy}(C_3) \leq 3$ and $\mathrm{sroy}(C_7) \leq 4$. (For simplicity, we write the set $\{a\}$ as a, $\{a, b\}$ as ab, and $\{a, b, c\}$ as abc.) If $\mathrm{sroy}(C_3) = 2$, then because $|\mathcal{P}^*([2])| = 3$, there are vertices of C_3 colored 1 and 2, implying that two edges of C_3 are colored with each of these two colors, which is impossible. If $\mathrm{sroy}(C_7) = 3$, then because $|\mathcal{P}^*([3])| = 7$, there are vertices of C_7 colored 1, 2, and 3, implying that two edges of C_7 are colored with each of these three colors. Regardless of how the seventh edge of C_7 is colored, the resulting set of vertex colors is not $\mathcal{P}^*([3])$. Consequently, $\mathrm{sroy}(C_3) = 3$ and $\mathrm{sroy}(C_7) = 4$. By Observation 1.1, it suffices to show that C_n has a strong royal k-edge coloring if $n \neq 3, 7$. Figure 1 also shows a strong royal 3-edge coloring for each of C_4, C_5, and C_6 and so $\mathrm{sroy}(C_n) = 3$ for $n = 4, 5, 6$.

Next, suppose that $n \geq 8$, where $2^{k-1} \leq n \leq 2^k - 1$ for a unique integer $k \geq 4$. We show that C_n has a strong royal k-edge coloring by considering two cases, depending on whether n is even or n is odd. Let $P_n = (v_1, v_2, \ldots, v_n)$ where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n - 1$.

Case 1. *$n \geq 8$ is even.* Figure 2 shows a strong royal 4-edge coloring for each of C_8, C_{10}, and C_{12} and so $\mathrm{sroy}(C_n) = 4$ for $n = 8, 10, 12$.

Thus, we assume that $n = 2r \geq 14$ where $r \geq 7$ is an integer such that $2^{k-2} \leq r \leq 2^{k-1} - 1$. If $r = 7$, then $k - 1 = 3$; while if $8 \leq r \leq 15$, then $k - 1 = 4$. A strong royal $(k-1)$-edge coloring c for each path P_r $(7 \leq r \leq 15)$ is shown in Figure 3.

For $7 \leq r \leq 15$, let $P_r = (v_1, v_2, \ldots, v_r)$ and let $P_r^* = (u_1, u_1, \ldots, u_r)$. The path P_{2r} is constructed from P_r and P_r^* by adding the edge $v_r u_r$

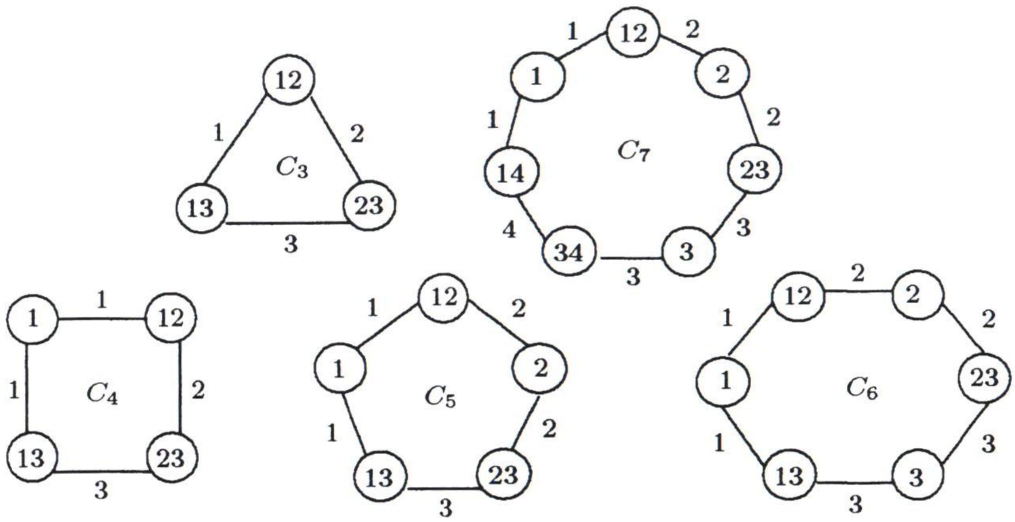

Figure 1: Strong royal colorings of C_n where $3 \le n \le 7$

and the cycle C_{2r} is constructed from P_{2r} by adding the edge v_1u_1. The edge coloring c is extended (1) to an edge coloring c of P_{2r} by defining $c(u_iu_{i+1}) = c(v_iv_{i+1}) \cup \{k\}$ (where $k = 4$ if $r = 7$ and $k = 5$ if $8 \le r \le 15$) for $1 \le i \le r-1$ and $c(v_ru_r) = c(v_{r-1}v_r)$ and (2) to an edge coloring c of C_{2r} by defining $c(v_1u_1) = c(v_1v_2)$ in addition. In this manner, no vertex of P_{2r} is colored $\{k\}$. Since this edge coloring is a strong royal k-edge coloring of C_{2r}, it follows that $\text{sroy}(P_{2r}) = \text{sroy}(C_{2r}) = k$ for $7 \le r \le 15$, where $k = 4$ if $r = 7$ and $k = 5$ if $8 \le r \le 15$. Figure 4 shows the construction of a strong royal 4-edge coloring of C_{14} from the paths P_7 and P_7^*.

For each such path P_{2r} ($7 \le r \le 15$), we construct the path P_{2r+1} by adding a new vertex u_0 and the edge u_0u_1 and coloring the edge u_0u_1 by $\{k\}$, where $k = 4$ if $r = 7$ and $k = 5$ if $8 \le r \le 15$. Then u_0 is colored $\{k\}$, resulting in a strong royal k-edge coloring of P_{2r+1} for $7 \le r \le 15$. Next, we repeat this procedure by beginning with the paths P_{14}, P_{15}, ..., P_{31}; that is, we use P_{14} to create a strong royal 5-edge coloring of C_{28} (where $r = 14$) and use $P_{15}, P_{16}, \ldots, P_{31}$ to create a strong royal 6-edge coloring of C_{2r} for $15 \le r \le 31$. Continued repetition of this procedure gives the desired result for all even cycles. Therefore, $\text{sroy}(C_n) = k$ for all even integers $n \ge 4$ with $2^{k-1} \le n \le 2^k - 1$.

Case 2. $n \ge 9$ *is odd.* Figure 5 shows a strong royal 4-edge coloring for each of C_9, C_{11}, and C_{13} and so $\text{sroy}(C_n) = 4$ for $n = 9, 11, 13$. Thus, we assume that $n = 2r + 1 \ge 15$, where $r \ge 7$.

For each path P_r, there is a subpath $Q = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$, where $3 \le i < i + 4 \le r$ such that $c'(v_{i+1}) = \{1, 2\}$, $c(v_{i+1}v_{i+2}) = \{2\}$, and $c'(v_{i+2}) =$

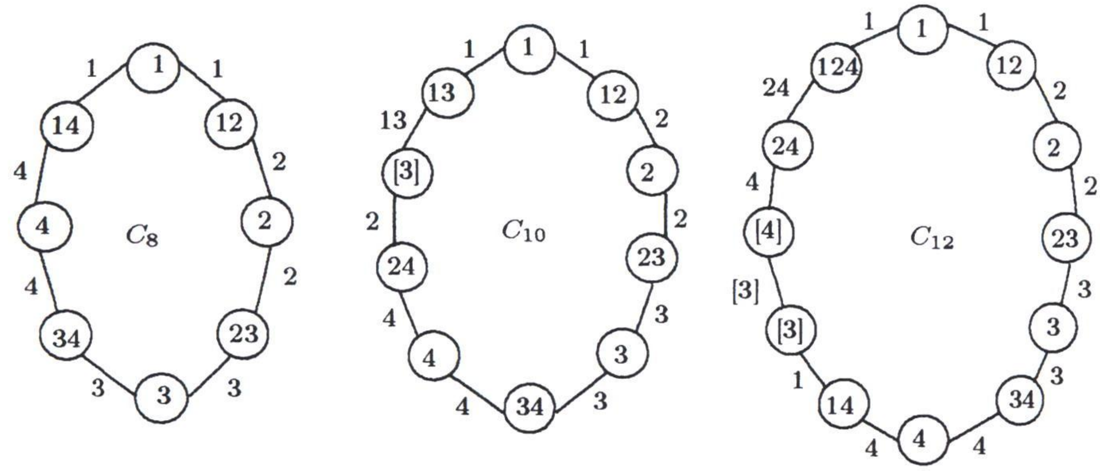

Figure 2: Strong royal 4-edge colorings of C_n for $n = 8, 10, 12$

$\{2\}$. From the manner in which each even cycle C_{2r} was constructed and a strong royal k-edge coloring c of C_{2r} was defined in Case 1, the path $Q^* = (u_i, u_{i+1}, u_{i+2}, u_{i+3})$ is a subapth in C_{2r} such that $c'(u_{i+1}) = \{1, 2, k\}$, $c(u_{i+1}u_{i+2}) = \{2, k\}$, and $c'(u_{i+2}) = \{2, k\}$. Furthermore, $c'(x) \neq \{k\}$ for each vertex x of C_{2r}. We now construct the cycle C_{2r+1} from C_{2r} by deleting the edge $u_{i+1}u_{i+2}$ from C_{2r} and adding a new vertex u along with the two new edges $u_{i+1}u$ and uu_{i+2}. We define an edge coloring c of C_{2r+1} from the strong royal k-edge coloring c of C_{2r} (as described in Case 1) by assigning the color $\{k\}$ to the edges $u_{i+1}u$ and uu_{i+2} where the colors of remaining edges of C_{2r+1} are the same as in C_{2r}. Thus, $c'(u) = \{k\}$ and $c'(x)$ is the same as in C_{2r} for all other vertices x of C_{2r+1}. Figure 6 shows the construction of such a strong royal 4-edge coloring of C_{15} from the strong royal 4-edge coloring of C_{14} of Figure 4. Since this edge coloring is a strong royal k-edge coloring of C_{2r+1}, it follows that $\text{sroy}(C_n) = k$ for all odd integers $n \geq 3$ with $2^{k-1} \leq n \leq 2^k - 1$ with the exception of $n = 3$ and $n = 7$. ∎

It is therefore a consequence of Theorem 1.5 that C_3 and C_7 are royal-one but all other cycles are royal-zero.

2 Classes of Royal-Zero & Royal-One Graphs

In this section we determine some classes of graphs that are royal-zero or royal-one. For complete graphs, the following result was obtained in [8].

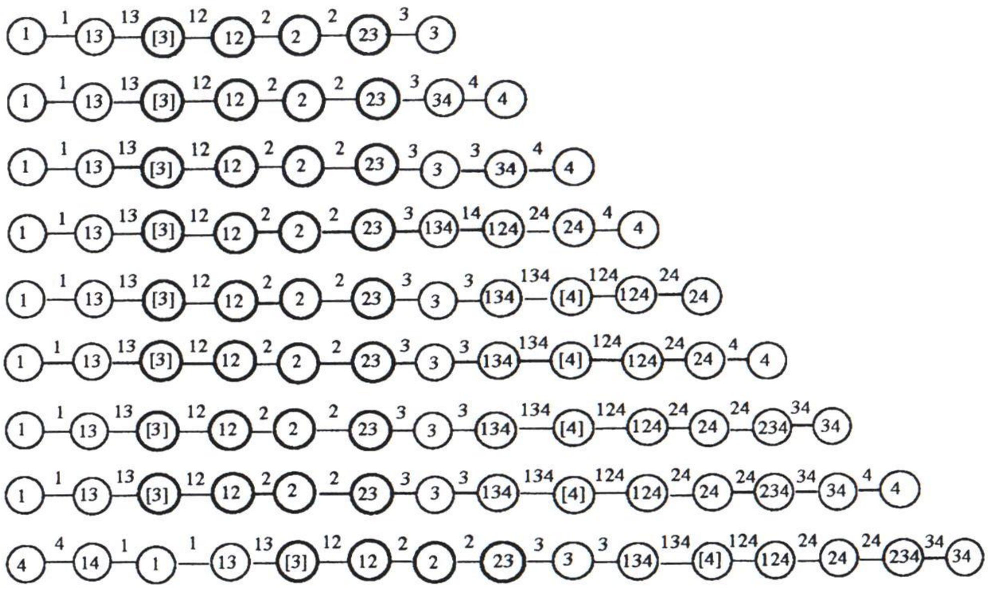

Figure 3: Strong royal $(k-1)$-edge colorings of P_r for $7 \le r \le 15$

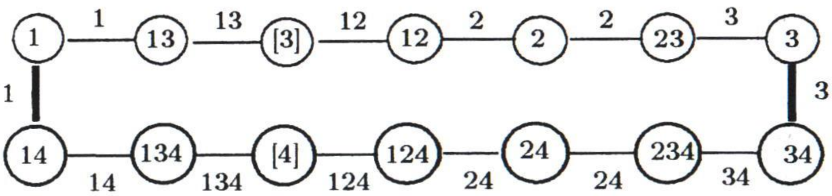

Figure 4: Constructing a strong royal 4-edge coloring of C_{14}

Proposition 2.1 *For an integer $n \ge 4$, the complete graph K_n is a royal-zero graph if n is a power of 2 and royal-one otherwise.*

We now consider the effect that certain operations can have on graphs that are royal-zero or royal-one. The *corona* $\text{cor}(G)$ of a graph G is that graph obtained from G by adding a pendant edge at each vertex of G. Thus, if the order of G is n, then the order of $\text{cor}(G)$ is $2n$. The strong royal index of $\text{cor}(G)$ never exceeds $\text{sroy}(G)$ by more than 1.

Proposition 2.2 *If G is a connected graph of order $n \ge 4$, then*

$$\text{sroy}(\text{cor}(G)) \le \text{sroy}(G) + 1.$$

Consequently, if G is a royal-zero graph, then so is $\text{cor}(G)$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and let $H = \text{cor}(G)$ be obtained from G by adding the pendant edge $u_i v_i$ at v_i for $1 \le i \le n$. Suppose that

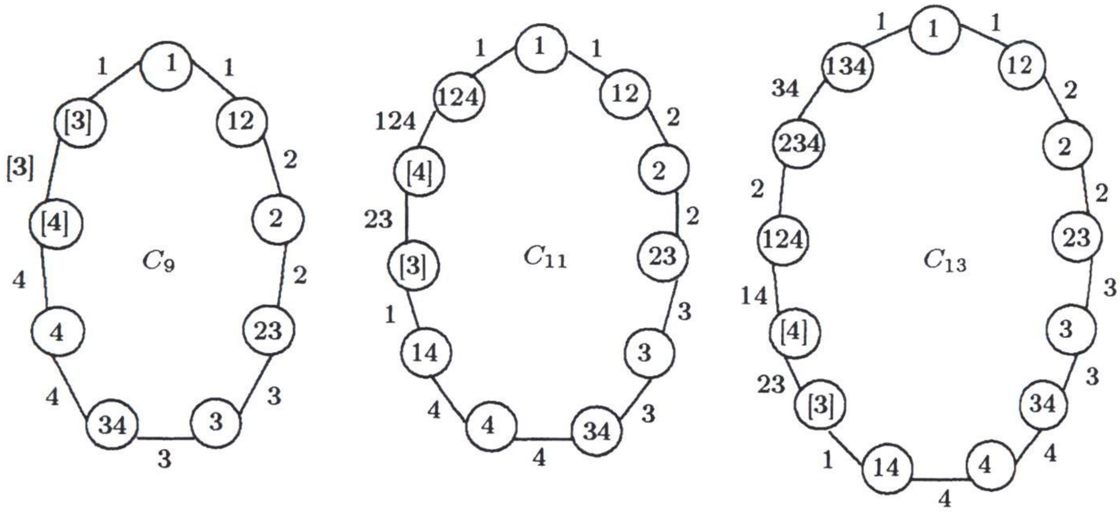

Figure 5: Strong royal 4-edge colorings of C_n for $n = 9, 11, 13$

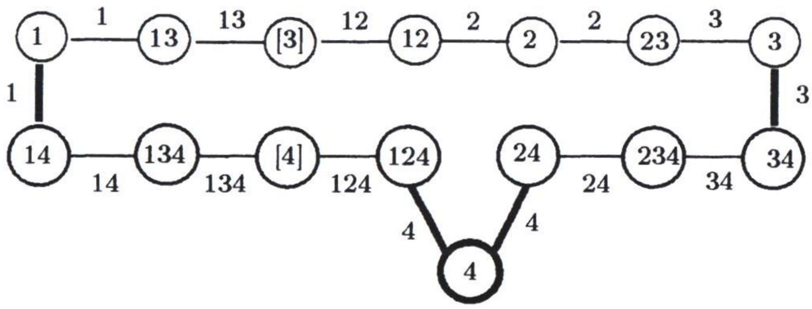

Figure 6: Constructing a strong royal 4-edge coloring of C_{15}

$\text{sroy}(G) = k$. Then there is a strong royal k-edge coloring $c_G : E(G) \to \mathcal{P}^*([k])$ of G. Define an edge coloring $c_H : E(H) \to \mathcal{P}^*([k+1])$ by

$$c_H(e) = \begin{cases} c_G(e) \cup \{k+1\} & \text{if } e \in E(G) \\ c'_G(v_i) & \text{if } e = u_i v_i \text{ for } 1 \le i \le n. \end{cases}$$

Then the induced vertex coloring c'_H is given by

$$c'_H(u_i) = c'_G(v_i) \text{ and } c'_H(v_i) = c'_G(v_i) \cup \{k+1\} \text{ for } 1 \le i \le n.$$

Since c'_H is vertex-distinguishing, it follows that c_H is a strong royal $(k+1)$-edge coloring of $\text{cor}(G)$ and so $\text{sroy}(H) \le k+1 = \text{sroy}(G) + 1$.

If G is a connected royal-zero graph of order $n \ge 4$ where $\text{sroy}(G) = k$, say, then $2^{k-1} \le n \le 2^k - 1$. Since $\text{cor}(G)$ is a connected graph of order $2n \ge 8$ where $2^k \le 2n \le 2^{k+1} - 2$, it follows that $\text{sroy}(\text{cor}(G)) \ge k+1$. On the other hand, there is a strong royal $(k+1)$-edge coloring of $\text{cor}(G)$ and so $\text{sroy}(\text{cor}(G)) = k+1$, which implies that $\text{cor}(G)$ is royal-zero as well. ■

A tree T is called *cubic* if every vertex of T that is not an end-vertex has degree 3. The following result makes use of the proof of Proposition 2.2.

Corollary 2.3 *If T is a cubic caterpillar of order at least 4, then T is royal-zero.*

Proof. Let T be a cubic caterpillar. Since the statement is true if T has four vertices, we may assume that T has six or more vertices. For an integer $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$, let $H = P_n = (v_1, v_2, \ldots, v_n)$ be a longest path in T, where then $\text{diam}(T) = n - 1 \geq 3$ and the order of T is $2n-2$. As noted earlier, it was shown in [8] that all paths of order 4 or more are royal-zero and so $\text{sroy}(H) = k$. Let u_iv_i be the pendant edges at v_i for $2 \leq i \leq n-1$. We consider two cases, according to whether $2^{k-1} < n \leq 2^k - 1$ or $n = 2^{k-1}$. In the first case, we apply the same procedure used in the proof of Proposition 2.2.

Case 1. $2^{k-1} < n \leq 2^k - 1$. Then $2^k \leq 2n - 2 \leq 2^{k+1} - 4$. Thus, it suffices to show that $\text{sroy}(T) \leq k+1$. Since $\text{sroy}(H) = k$, there is a strong royal k-edge coloring $c_H : E(H) \to \mathcal{P}^*([k])$. Define an edge coloring $c_T : E(T) \to \mathcal{P}^*([k+1])$ by

$$c_T(e) = \begin{cases} c_H(e) \cup \{k+1\} & \text{if } e \in E(H) \\ c'_H(v_i) & \text{if } e = u_iv_i \text{ for } 2 \leq i \leq n-1. \end{cases}$$

Then the induced vertex coloring c'_T is given by $c'_T(u_i) = c'_H(v_i)$ for $2 \leq i \leq n-1$ and $c'_T(v_i) = c'_H(v_i) \cup \{k+1\}$ for $1 \leq i \leq n$. Since c'_T is vertex-distinguishing, it follows that c_T is a strong royal $(k+1)$-edge coloring of T and $\text{sroy}(T) \leq k+1$. Thus, T is royal-zero.

Case 2. $n = 2^{k-1}$. Then $2n-2 = 2^k - 2$. Here, we show that $\text{sroy}(T) = \text{sroy}(H) = k$. First, we consider the case where $n = 4$ and $k = 3$. A strong royal 3-edge coloring c of $H = P_4 = (v_1, v_2, v_3, v_4)$ is shown in Figure 7, namely $c(v_1v_2) = 1$, $c(v_2v_3) = \{1,2\}$, and $c(v_3v_4) = \{1,3\}$. Observe that the induced vertex colors of the vertices of H are all subsets of [3] containing 1 and $c'(v_1) = \{1\}$. The tree T is constructed from H by attaching the pendant edges u_2v_2 and u_3v_3 to v_2 and v_3, respectively. The colors of u_iv_i, $i = 2, 3$, are defined by $c(u_iv_i) = c'(v_i) - \{1\}$, which results in a strong royal 3-edge coloring of T. In the case where $n = 8$ and $k = 4$, we begin with the path $H = P_8 = (v_1, v_2, \ldots, v_8)$, where the edges v_1v_2, v_2v_3, v_3v_4 of P_8 are colored as in the case when $n = 4$, and define

$c(v_4v_5) = c'(v_4)$ and $c(v_iv_{i+1}) = c(v_{8-i}v_{9-i}) \cup \{4\}$ for $i = 5, 6, 7$. Here too, each edge color and induced vertex color contains 1 and $c'(v_1) = \{1\}$. The tree T in this case is constructed from H by attaching the pendant edges u_iv_i for $2 \leq i \leq 7$. The color of u_iv_i is defined by $c(u_iv_i) = c'(v_i) - \{1\}$ for $2 \leq i \leq 7$, which results in a strong royal 4-edge coloring of T. This is illustrated in Figure 7. Continuing in this manner gives the desired result. ■

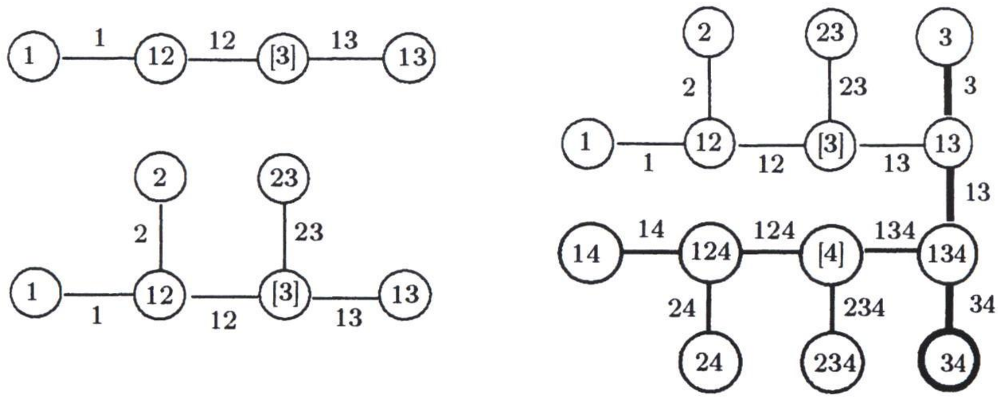

Figure 7: Constructing strong royal colorings of the cubic caterpillars

As stated in Proposition 2.2, if G is a connected graph of order 4 or more, then $\text{sroy}(\text{cor}(G)) \leq \text{sroy}(G) + 1$, which implies that if G is a royal-zero graph, then $\text{cor}(G)$ is a royal-zero graph. On the other hand, it is possible that G is a royal-one graph and $\text{cor}(G)$ is a royal-zero graph. By Proposition 2.1, every complete graph K_n where n is not a power of 2 is a royal-one graph. Thus, if $2^{k-1} + 1 \leq n \leq 2^k - 1$ for some integer $k \geq 3$, then $\text{sroy}(K_n) = k + 1$. If one were to assign distinct nonempty subsets of $[k]$ to the n pendant edges of $\text{cor}(K_n)$ and the color $\{k + 1\}$ to the remaining $\binom{n}{2}$ edges of $\text{cor}(K_n)$, then we have a strong royal $(k + 1)$-edge coloring of $\text{cor}(K_n)$ and so $\text{sroy}(\text{cor}(K_n)) = k + 1$. Therefore, $\text{cor}(K_n)$ is a royal-zero graph for each integer $n \geq 5$ where n is not a power of 2. For a more interesting example, Figure 8 shows a strong royal 4-edge coloring of $\text{cor}(C_7)$ and so $\text{sroy}(\text{cor}(C_7)) = \text{sroy}(C_7) = 4$ (by Theorem 1.5). Thus, C_7 is royal-one, while $\text{cor}(C_7)$ is royal-zero.

A graph operation somewhat related to the corona of a graph G is the *Cartesian product* of G with K_2. In fact, we have the following result that corresponds to Proposition 2.2.

Proposition 2.4 *If G is a connected graph of order $n \geq 4$, then*

$$\text{sroy}(G \,\square\, K_2) \leq \text{sroy}(G) + 1.$$

Consequently, if G is a royal-zero graph, then $G \,\square\, K_2$ is a royal-zero graph.

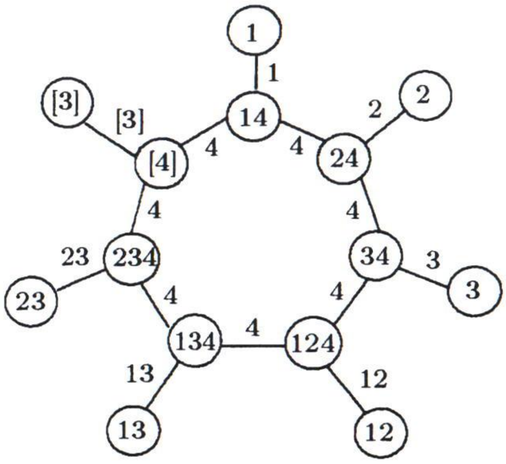

Figure 8: A strong royal 4-edge coloring of $\text{cor}(C_7)$

Proof. Let G be a connected graph of order $n \geq 4$ where $\text{sroy}(G) = k$ for some positive integer k. Let $H = G \square K_2$ where G_1 and G_2 are the two copies of G. Suppose that $V(G_1) = \{u_1, u_2, \ldots u_n\}$ where u_i is labeled v_i in G_2. Thus, $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ and $E(H) = E(G_1) \cup E(G_2) \cup \{u_i v_i : 1 \leq i \leq n\}$. Since $\text{sroy}(G) = k$, there is a strong royal k-edge coloring $c_{G_1} : E(G_1) \to \mathcal{P}^*([k])$ of G_1. Define an edge coloring $c_H : E(H) \to \mathcal{P}^*([k+1])$ by

$$c_H(e) = \begin{cases} c_{G_1}(e) & \text{if } e \in E(G_1) \\ c_{G_1}(u_i u_j) \cup \{k+1\} & \text{if } e = v_i v_j \in E(G_2) \text{ for } 1 \leq i \neq j \leq n \\ c'_{G_1}(u_i) & \text{if } e = u_i v_i \text{ for } 1 \leq i \leq n. \end{cases}$$

The induced coloring $c'_H : V(H) \to \mathcal{P}^*([k+1])$ is then given by $c'_H(u_i) = c'_{G_1}(u_i)$ and $c'_H(v_i) = c'_{G_1}(u_i) \cup \{k+1\}$. Since c'_H is vertex-distinguishing, it follows that c'_H is a strong royal $(k+1)$-edge coloring of H. Thus, $\text{sroy}(H) \leq k + 1 = \text{sroy}(G) + 1$. Therefore, if G is a royal-zero graph, then $G \square K_2$ is a royal-zero graph. ■

The *hypercube* Q_k is K_2 if $k = 1$, while for $k \geq 2$, Q_k is defined recursively as the Cartesian product $Q_{k-1} \square K_2$ of Q_{k-1} and K_2. Since $Q_2 = C_4$ is royal-zero by Theorem 1.5, the following is a consequence of Proposition 2.4.

Corollary 2.5 *For each integer $k \geq 2$, the hypercube Q_k is a royal-zero graph.*

As stated in Proposition 2.4, if G is a royal-zero graph, then $G \square K_2$ is a royal-zero graph. On the other hand, it is possible that G is a royal-one graph and $G \square K_2$ is a royal-zero graph. To see an example of this, we return to the 7-cycle C_7, which we saw (in Theorem 1.5) is a royal-one graph. Figure 9 shows a strong royal 4-edge coloring of $C_7 \square K_2$ and so $\text{sroy}(C_7) = \text{sroy}(C_7 \square K_2) = 4$. Thus, C_7 is royal-one, while $C_7 \square K_2$ is royal-zero. As mentioned in Proposition 2.1, the complete graphs K_5 and K_6 are royal-one graphs. For these two graphs G, the graphs $G \square K_2$ are royal-zero; that is, $\text{sroy}(K_5 \square K_2) = \text{sroy}(K_6 \square K_2) = 4$. A strong royal 4-edge coloring c of $H = K_6 \square K_2$ can be defined as follows. Let H_1 and H_2 be two copies of K_6 in H, where $V(H_1) = \{u_1, u_2, \ldots u_6\}$ and $V(H_2) = \{v_1, v_2, \ldots v_6\}$ such that $u_i v_i \in E(H)$. First, we define the vertex-distinguishing coloring $c' : V(H) \to \mathcal{P}^*([4])$ by

$$c'(u_1) = \{1,4\},\ c'(u_2) = \{1\},\ c'(u_3) = \{1,2,4\},$$
$$c'(u_4) = \{1,2,3\},\ c'(u_5) = \{1,3\},\ c'(u_6) = \{1,2\},$$
$$c'(v_1) = \{4\},\ c'(v_2) = \{1,3,4\},\ c'(v_3) = [4],$$
$$c'(v_4) = \{2,4\},\ c'(v_5) = \{3,4\},\ c'(v_6) = \{2,3,4\}.$$

The edge coloring $c : V(H) \to \mathcal{P}^*([4])$ is then defined by $c(xy) = c'(x) \cap c'(y)$ for each edge $xy \in E(H)$. Since c' is the induced vertex coloring of c, it follows that c is a strong royal 4-edge coloring of H. Thus, H is royal-zero.

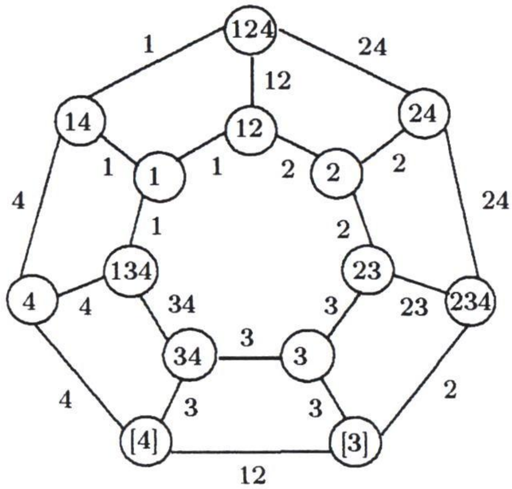

Figure 9: A strong royal 4-edge coloring of $C_7 \square K_2$

As noted in Proposition 2.1, the complete graph K_7 is also a royal-one graph. However, $H = K_7 \square K_2$ is royal-one as well. That there is a strong royal 5-edge coloring of H is straightforward. To show that $\text{sroy}(K_7 \square K_2) = 5$, however, it is necessary to show that there is no strong royal 4-edge coloring of H, for assume that such an edge coloring c of H

exists. Since the order of H is 14, the induced vertex colors of H must consist of 14 elements of $\mathcal{P}^*([4])$. In particular, at least three of the four singleton subsets of [4] must be vertex colors of H. Suppose that H_1 and H_2 are the two copies of K_7 in the construction of H. Therefore, at least one of H_1 and H_2 has at least two singleton subsets as its vertex colors, say $c'(u_1) = \{1\}$ and $c'(u_2) = \{2\}$ where $u_1, u_2 \in V(H_1)$, which is impossible since u_1 and u_2 are adjacent. Hence, $\text{sroy}(K_7 \,\square\, K_2) = 5$.

3 Conditions for Royal-One Graphs

We have seen that many graphs are royal-zero graphs. We now present a sufficient condition for a connected graph G of order $n \geq 4$ to be a royal-one graph. Let k be the unique integer such that $2^{k-1} \leq n \leq 2^k - 1$. A graph G_k of order $2^k - 1$ is now constructed as follows. The vertices of G_k are labeled with the $2^k - 1$ distinct elements of $\mathcal{P}^*([k])$. For each vertex v of G_k, let $\ell(v)$ denote its label. Thus, $\{\ell(v) : v \in V(G_k)\} = \mathcal{P}^*([k])$. Two vertices u and v of G_k are adjacent in G_k if and only if $\ell(u) \cap \ell(v) \neq \emptyset$. The vertex set $V(G_k)$ is partitioned into k subsets $V_1, V_2, \ldots, V_k$ where $V_i = \{v \in V(G_k) : |\ell(v)| = i\}$ for $1 \leq i \leq k$. Therefore, $G_k[V_k] = K_1$ and $G_k[V_1] = \overline{K}_k$ is empty. If $k = 2p+1$ is odd, then $G_k[V_{p+1} \cup V_{p+2} \cup \cdots \cup V_k] = K_{2^{k-1}}$. If $k = 2p$ is even, then let V'_p be the subset consisting of those elements S in V_p for which $1 \in S$. Then $|V'_p| = \frac{1}{2}\binom{k}{p}$ and $G_k\left[V'_p \cup V_{p+1} \cup V_{p+2} \cup \cdots \cup V_k\right] = K_{2^{k-1}}$. Let m_k be the size of G_k. The graph G_3 of order $7 = 2^3 - 1$ has size $m_3 = 15$ and is shown in Figure 10.

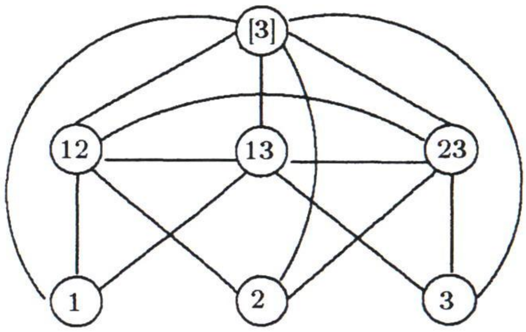

Figure 10: The graph G_3 of order $7 = 2^3 - 1$ and size $m_3 = 15$

There is an immediate condition under which a connected graph cannot be a royal-zero graph. As we mentioned earlier, it was shown in [4] that G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$, then $\text{sroy}(G) \leq k + 2$.

Observation 3.1 *Let G be a connected graph of order $n \geq 4$ and size m where $2^{k-1} \leq n \leq 2^k - 1$ for an integer k. If G is not a subgraph of the graph G_k, then either* $\mathrm{sroy}(G) = k+1$ *or* $\mathrm{sroy}(G) = k+2$, *and so G is not a royal-zero graph. Consequently, if $m \geq m_k + 1$, then G is not a royal-zero graph.*

Since $\mathrm{sroy}(T) = 3$ for each tree T of order n where $4 \leq n \leq 7$, it follows by Observation 1.2 that if G is a connected graph of order n where $4 \leq n \leq 7$, then $\mathrm{sroy}(G)$ is either 3 or 4. If G is a connected graph of order 7 that is not isomorphic to a subgraph of G_3 of Figure 10, then $\mathrm{sroy}(G) \neq 3$ and so $\mathrm{sroy}(G) = 4$. Since the size of G_3 is 15, it follows that if G is a connected graph of order 7 with size at least 16, then $\mathrm{sroy}(G) = 4$. Figure 11 shows the graphs H_4, H_5, and H_6 of order 4, 5, and 6, respectively, of greatest size that are subgraphs of G_3. For each graph H_i where $i = 4, 5, 6$, if every edge uv of H_i is assigned the color $c(uv) = \ell(u) \cap \ell(v)$, then $c'(v) = \bigcup_{e \in E_{H_i}(v)} c(e) = \ell(v)$, resulting in a strong royal 3-edge coloring of H_i. Hence, $\mathrm{sroy}(H_i) = 3$ for $i = 4, 5, 6$. The graph $H_4 = K_4$, while H_5 has size 9 and H_6 has size 12. So, if G is a connected graph of order 5 whose size is at least 10 (that is, $G = K_5$) or if G is a connected graph of order 6 whose size is at least 13, then $\mathrm{sroy}(G) = 4$.

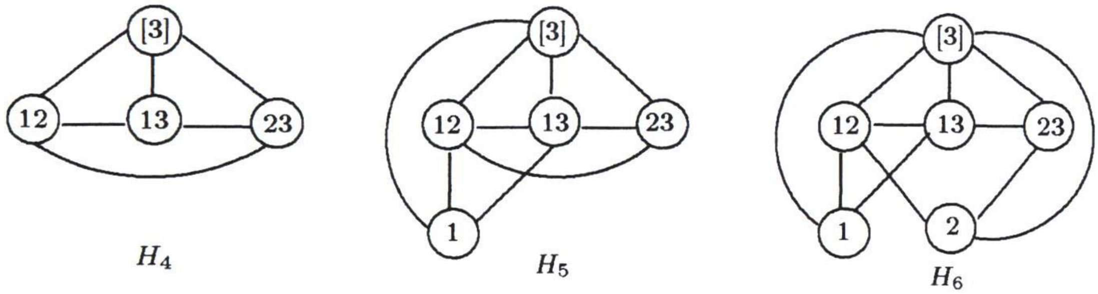

Figure 11: Subgraphs of G_3

By Observation 3.1, if G is a connected graph of order $n \geq 4$ and size m where $2^{k-1} \leq n \leq 2^k - 1$ such that $m > m_k$, which implies that $G \nsubseteq G_k$, then $\mathrm{sroy}(G) \geq k+1$. In fact, if G possesses any property that implies that $G \nsubseteq G_k$, then $\mathrm{sroy}(G) \geq k+1$. For example, if the order of G is $n = 2^k - 1$ and $\delta(G) \geq \delta(G_k) + 1$ or G has more than one vertex of degree $n-1$, then $\mathrm{sroy}(G) \geq k+1$. On the other hand, even though $C_7 \subseteq G_3$ (where $n = 2^3 - 1$ and $k = 3$), $|E(C_7)| = 7 < m_3$, and $\delta(C_7) < \delta(G_3)$, we saw that $\mathrm{sroy}(C_7) = 4 = k+1$. Furthermore, for every chord e of C_7, $\mathrm{sroy}(C_7 + e) = 3$ (see Figure 12). Consequently, even though one might suspect that $\mathrm{sroy}(G + uv) \geq \mathrm{sroy}(G)$ for every connected graph G and every pair u, v of nonadjacent vertices of G, such is not the case.

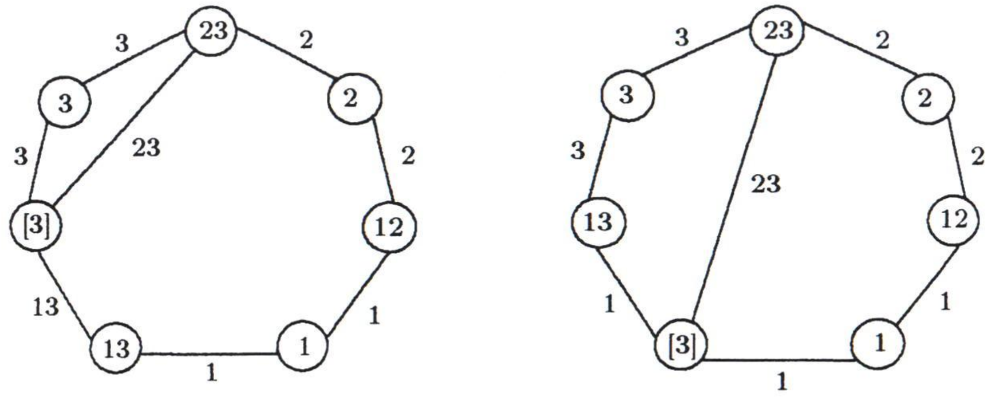

Figure 12: Showing that $\text{sroy}(C_7 + e) = 3$ for each $e \notin E(C_7)$

What we have seen is that if G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ having a sufficiently large size, then $\text{sroy}(G) \neq k$. However, if G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ having a small size, then we are not guaranteed that $\text{sroy}(G) = k$. Indeed, even the strong royal index of trees is in doubt.

If Conjecture 1.3 is true, then for every connected graph G of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$, either $\text{sroy}(G) = k$ or $\text{sroy}(G) = k + 1$. In order to present a sufficient condition for $\text{sroy}(G) \neq k$ in terms of the size and minimum degree of G, we describe an expression for the size m_k of the graph G_k (as it is easier in general to compare two numbers than to determine whether a graph contains a subgraph isomorphic to a given graph).

Recall that we label the $2^k - 1$ vertices of G_k with the distinct elements of $\mathcal{P}^*([k])$. The label of each vertex v of G_k is denoted by $\ell(v)$ and so $\{\ell(v) : v \in V(G_k)\} = \mathcal{P}^*([k])$. Let $\{V_1, V_2, \ldots, V_k\}$ be the partition of of $V(G_k)$ described earlier, where then $V_i = \{v \in V(G_k) : |\ell(v)| = i\}$ for $1 \leq i \leq k$. Let $v \in V_i$ for some integer i with $1 \leq i \leq k$. Then $\ell(v) = S$ is some i-element subset of $[k]$. There are $2^i - 1$ nonempty subsets of S and 2^{k-i} subsets of $[k] - S$. For each nonempty subset S' of S and each subset T of $[k] - S$, the vertex v is adjacent to that vertex w of G_k for which $\ell(w) = S' \cup T$. Since v is not adjacent to itself, however, it follows that $\deg_{G_k} v = (2^i - 1)2^{k-i} - 1$. Furthermore, there are $\binom{k}{i}$ vertices in V_i

for $1 \le i \le k$. Therefore,

$$\begin{aligned}
m_k &= \frac{1}{2}\sum_{i=1}^{k}\binom{k}{i}\left[(2^i-1)2^{k-i}-1\right] = \frac{1}{2}\sum_{i=1}^{k}\binom{k}{i}(2^k-2^{k-i}-1) \\
&= \frac{1}{2}\left[\sum_{i=1}^{k}\binom{k}{i}2^k - \sum_{i=1}^{k}\binom{k}{i}2^{k-i} - \sum_{i=1}^{k}\binom{k}{i}\right] \\
&= \frac{1}{2}\left[2^k\sum_{i=1}^{k}\binom{k}{i} - 2^k\sum_{i=1}^{k}\binom{k}{i}\left(\frac{1}{2}\right)^i - \sum_{i=1}^{k}\binom{k}{i}\right] \\
&= \frac{1}{2}\left\{2^k(2^k-1) - 2^k\left[\left(1+\frac{1}{2}\right)^k - 1\right] - (2^k-1)\right\} \\
&= \frac{1}{2}(4^k-3^k-2^k+1).
\end{aligned}$$

In particular, if $k = 3$, then the size of G_3 is $m_3 = 15$, as we saw in Figure 10.

Proposition 3.2 *Let G be a graph of order $n \ge 4$ and size m where $2^{k-1} \le n \le 2^k - 1$ for some integer $k \ge 3$. If $m > \frac{1}{2}(4^k - 3^k - 2^k + 1)$, then either* $\text{sroy}(G) = k+1$ *or* $\text{sroy}(G) = k+2$, *and so G is not a royal-zero graph.*

For each integer $k \ge 3$, the minimum degree $\delta(G_k)$ of the graph G_k is $2^{k-1} - 1$. Consequently, if G is a graph of order $n \ge 4$ and size m where $2^{k-1} \le n \le 2^k - 1$ for which $\delta(G) \ge 2^{k-1}$, then it may occur that $m < m_k$ but yet G is not a subgraph of G_k, and so (by Observation 3.1) $\text{sroy}(G) \ge k+1$. However, in this case, more can be said. It is useful to recall that every path P_n for $n \ge 4$ is royal-zero (see [4, 8]).

Proposition 3.3 *Let G be a connected graph of order $n \ge 4$ where $2^{k-1} \le n \le 2^k - 1$ for some integer $k \ge 2$. If $\delta(G) \ge 2^{k-1}$, then* $\text{sroy}(G) = k+1$ *and so G is a royal-one graph.*

Proof. We have already observed that $\text{sroy}(G) \ge k+1$ for such a graph. Since $\delta(G) \ge 2^{k-1}$ and $n \le 2^k - 1$, it follows that $\delta(G) \ge (n+1)/2$ and therefore G has a Hamiltonian path (in fact, a Hamiltonian cycle). Since $\text{sroy}(P_n) = k$ for every path P_n of order n, it follows by Observation 1.2 that $\text{sroy}(G) \le k+1$ and so $\text{sroy}(G) = k+1$. ■

We have seen that both K_7 and C_7 (a spanning subgraph, or factor, of K_7) are royal-one graphs. The complement $\overline{C_7}$ of C_7 is a 4-regular graph of order 7 and so it is not a subgraph of the graph G_3 shown in Figure 10. Hence, $\overline{C}_7$ is also a royal-one graph. The size of $\overline{C}_7$ is 14 which is less than the size 15 of G_3 (the graph of order 7 having the maximum size that is royal-zero). This brings up the problem of determining for each integer $n \geq 3$, the minimum size of a graph of order n that is royal-one. Of course, the minimum size is 7 when $n = 7$.

The graph $\overline{C}_7$ can itself be factored into two copies of C_7. Therefore, the royal-one graph K_7 can be factored into three royal-one graphs. However, K_7 can also be factored into three graphs satisfying any of the following: (1) all three factors are royal-zero, (2) exactly two factors are royal-zero, (3) exactly one factor is royal-zero. Consequently, there is a host of additional problems that arise with strong royal colorings of graphs.

Acknowledgment We thank the anonymous referees whose valuable suggestions resulted in an improved paper.

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