

On the cone of weighted graphs generated by triangles

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Abstract

Motivated by problems involving triangle-decompositions of graphs, we examine the facet structure of the cone τ_n of weighted graphs on n vertices generated by triangles. Our results include enumeration of facets for small n , a construction producing facets of τ_{n+1} from facets of τ_n , and an arithmetic condition on entries of the normal vectors. We also point out that a copy of τ_n essentially appears via the perimeter inequalities at one vertex of the metric polytope.

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1 Introduction

1.1 Overview

The problem of partitioning the edges of a graph G into triangles has been studied extensively. When G is a complete graph, the problem is equivalent to the existence of a Steiner triple system, an object dating back at least to the mid-19th century, [15, 22]. For general graphs, the decision problem for existence of a triangle decomposition is known [9] to be NP-complete. In view of this, attention has turned to sufficient conditions for special graph families. Planar graphs are considered in [17]. The problem for certain 3-partite graphs G has been studied [2, 3] for its connection to partial latin square completion. Otherwise, most research on triangle decompositions has focused on host graphs G having high minimum degree. A sequence of papers including [14], then [1, 10] and more recently [4] have lowered the sufficient minimum degree threshold toward the conjectured limit of Nash-Williams, [18].

Of course, for a graph (or even a multigraph) G to admit a partition of its edges into triangles, it is necessary for the number of edges of G to be a multiple of three, and for each degree of G to be even. These are ‘arithmetic’ necessary conditions, since they arise from the integrality of triangle weights used. Recent work on the problem often considers the fractional relaxation, in which triangle weights can be nonnegative reals. Indeed, concerning the minimum degree threshold, the breakthrough result [1] essentially reduces the problem to its fractional relaxation.

In the fractional relaxation, arithmetic conditions disappear, but there remain other necessary conditions. Suppose we map the vertices of G into the unit interval. The perimeter of every triangle of G in this embedding has length at most 2. It follows that for G to have a triangle decomposition, the average length of an edge cannot exceed $2/3$. But there exist graphs on n vertices with minimum degree approaching $3n/4$ from below which fail this condition; for instance, a blow-up $C_4 \cdot K_m$ of the four-cycle by equal-sized cliques has, when vertices are mapped to $\{0, 1\}$ according to a 2-colouring of the underlying C_4 , average edge-length

$$\frac{(2m)^2}{(2m)^2 + 4\binom{m}{2}} > \frac{2}{3}.$$

This example was given (though analyzed somewhat differently) by Ron Graham at the end of Nash-Williams’ note [18]. Combining this with the arithmetic conditions, the conjecture attributed to Nash-Williams normally reads as follows.

Conjecture 1.1 ([18]). *For sufficiently large n , every graph G on n vertices with even degrees, number of edges a multiple of three, and minimum degree at least $3n/4$ has an edge-decomposition into triangles.*

We remark that an equivalent formulation could replace ‘even degrees’ with an Eulerian assumption. It is natural to call the condition on average edge length a ‘geometric’ necessary condition since it is witnessed by a Euclidean embedding. But there is an even more general geometric viewpoint, namely that a graph built as a nonnegative combination of triangles inherits ‘by convexity’ any linear constraint on its triangles.

The purpose of this paper is to take some preliminary steps toward organizing geometric necessary conditions for triangle decompositions. We introduce and study a polyhedral cone relevant for the problem (although the cone has essentially appeared before in some other contexts). The description of this cone by facets seems very complicated, but offers an encoding of all constraints for (the fractional relaxation of) the triangle decomposition problem. We believe an understanding of this cone is possibly useful for making further steps toward Conjecture 1.1.

1.2 Set-up and notation

For our purposes, a *weighted graph* on a vertex set V is a function $f : \binom{V}{2} \rightarrow \mathbb{R}$ assigning a real number to each edge of the complete graph on V . For $e \in \binom{V}{2}$, we say that $f(e)$ is the *weight* of e . A weighted graph is *nonnegative* if every edge has nonnegative weight. Alternatively, a nonnegative weighted graph on V is a triple (V, E, f) , where $G = (V, E)$ is a (simple) graph and $f : E \rightarrow \mathbb{R}_+$ is an assignment of positive reals to the edges; here, it is understood that pairs in $\binom{V}{2} \setminus E$ get weight 0. A similar notion may be used in the presence of negative edges.

Here we assume a finite vertex set, typically $V = [n] := \{1, 2, \dots, n\}$. The set of weighted graphs forms a vector space of dimension $\binom{n}{2}$ over the reals. Thus we may identify weighted graphs with vectors in $\mathbb{R}^{\binom{n}{2}}$. We shall adopt the colexicographic order $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \dots$ on $\binom{[n]}{2}$, in which edge e precedes edge f if and only if $\max(e \oplus f) \in f$, and index vectors accordingly. In this way, for $1 \leq m \leq n$, the prefix of first $\binom{m}{2}$ coordinates in a vector corresponds to the subgraph induced by $\{1, 2, \dots, m\}$.

Recall that a *cone* in \mathbb{R}^d is a set κ which is closed under both addition and scalar multiplication by nonnegative reals. The cone *generated* by v_1, \dots, v_k is $\{\sum_{i=1}^k a_i v_i : a_i \geq 0\}$. For instance, the set of nonnegative weighted graphs forms a cone corresponding to the nonnegative orthant of $\mathbb{R}^{\binom{n}{2}}$, and hence it is generated by the standard basis.

We are interested here in the cone $\tau_n \subset \mathbb{R}^{\binom{n}{2}}$ of weighted graphs on n vertices generated by triangles. A triangle is understood to mean a copy of K_3 on V , that is, a weighted graph assuming the value 1 on edges in a 3-subset $\{x, y, z\} \subseteq V$ and 0 otherwise. The cone τ_3 is simply the ray $(1, 1, 1)\mathbb{R}$ in \mathbb{R}^3 . In the case $n = 4$, it is easy to see that a weighted graph is a linear combination of triangles if and only if the sum of weights on any two disjoint edges is a constant. For $n \geq 5$, the set of all triangles spans $\mathbb{R}^{\binom{n}{2}}$ by linear combinations; see Proposition 2.2 to follow. It follows that τ_n has full dimension $\binom{n}{2}$ if $n \geq 5$.

1.3 Graph decompositions

An F -decomposition of a graph $G = (V, E)$ is a partition of its edge set E into (edge sets of) subgraphs, each isomorphic to F . We consider the case $F = K_3$, and also refer to a K_3 -decomposition as a triangle decomposition. A triangle decomposition of K_n is equivalent to a Steiner triple system of order n , which exists if and only if $n \equiv 1$ or $3 \pmod{6}$.

A fractional K_3 -decomposition of G is an assignment of nonnegative real weights to the triangles in G such that, for every edge e of G , the sum of weights assigned to triangles containing e equals 1. From our formulation above, G has a fractional K_3 -decomposition if and only if it belongs to the cone τ_n . For dense graphs G , the recent paper [4] establishes the current record minimum degree threshold for existence of a fractional triangle decomposition. Here and in what follows, $\delta(G)$ denotes the minimum degree of G .

Theorem 1.2 ([4]). *For sufficiently large n , every graph G on n vertices with $\delta(G) > 0.827n$ belongs to τ_n .*

Barber, Kühn, Lo and Osthus showed in [1] showed that a minimum degree threshold sufficient for fractional K_3 -decomposition is also roughly sufficient for the exact triangle decomposition problem. This gives considerable motivation to studying degree thresholds for fractional triangle decompositions, and the cone τ_n in general. Indeed, reducing 0.827 to $\frac{3}{4}$ in Theorem 1.2 would nearly establish Conjecture 1.1 except for some cases very close to the boundary.

1.4 Organization

We initiate a detailed study of τ_n , especially its facets, and the connection with triangle decompositions of (weighted) graphs. Section 2 contains

some additional background relevant for our problem, including language for polyhedral cones and some reformulations of τ_n . One such related object is the metric polytope, which we briefly discuss in Section 2.6. In Section 3, we identify some simple arithmetic constraints on entries of the normal vectors. Then, in Section 4, we report on a computer-aided classification of facets of τ_n for $n \leq 8$ (this being essentially contained in earlier computations on the metric polytope) and in addition push the computation to a probably complete classification for $n = 9$. Section 5 contains a ‘vertex splitting’ operation which generates many infinite families of facets. In spite of this partial inductive structure, there is a surprising level of complexity to τ_n . In Section 6, we examine a class of facets having a hybrid combinatorial-geometric structure. From these, we produce a new class of graphs having no fractional triangle decomposition but minimum degree approaching $3n/4$ from below.

2 Background

2.1 Cones

First we review some background on cones. For our purposes, all cones are assumed to be ‘polyhedral’ (finitely generated). Unless otherwise specified, cones are ‘pointed’ ($u, -u \in \kappa$ implies $u = 0$) and of full dimension in their vector space.

Let κ be a cone. A *face* of κ is a cone $\eta \subseteq \kappa$ such that for all $u \in \eta$, if $u = u_1 + u_2$ with $u_1, u_2 \in \kappa$, then $u_1, u_2 \in \eta$. A face of dimension 1 is called an *extremal ray* of κ , while a face of codimension 1 is called a *facet* of κ . Two extremal rays are *adjacent* if they span a face of dimension 2. Likewise, two facets are adjacent if they intersect in a face of codimension 2.

The discussion from now on focuses on cones in real Euclidean space \mathbb{R}^m . The usual inner product $\langle \cdot, \cdot \rangle$ is used. When matrices are involved, we adopt the convention that in $\langle a, b \rangle$, a is a (dual) row vector and b is an (ordinary) column vector.

A *supporting vector* for a cone κ in \mathbb{R}^m is a nonzero vector $y \in \mathbb{R}^m$ such that $\langle y, u \rangle \geq 0$ for all $u \in \kappa$. From this inequality, y defines a half-space containing κ . A result of fundamental importance is that a cone κ is the intersection of all half-spaces defined by supporting vectors of κ . Theorem 2.1 below states this in the concrete setting which shall be used herein.

Given an $m \times n$ matrix A , the set $\text{cone}(A) = \{Ax : x \in \mathbb{R}^n, x \geq 0\}$ is a closed and polyhedral cone in \mathbb{R}^m . The dimension of $\text{cone}(A)$ is equal to the rank of A . The following well known result provides necessary and sufficient conditions for a point to belong to $\text{cone}(A)$.

Theorem 2.1 (Farkas Lemma). *Let A be an $m \times n$ matrix over \mathbb{R} , and let $b \in \mathbb{R}^m$. Then $Ax = b$ has a solution $x \geq 0$ if and only if $\langle y, b \rangle \geq 0$ for all $y \in \mathbb{R}^m$ such that $yA \geq 0$.*

REMARKS. One direction of this result is immediate. Suppose $Ax = b$ has a nonnegative solution $x \in \mathbb{R}^n$, and let y be such that $yA \geq 0$. Then $\langle y, b \rangle = \langle y, Ax \rangle = \langle yA, x \rangle \geq 0$. The converse asserts the existence of a ‘separating hyperplane’ between $\text{cone}(A)$ and a point $b \notin \text{cone}(A)$.

It is enough to check the condition in Theorem 2.1 for y corresponding to facets of $\text{cone}(A)$. Adapting the simplex algorithm or Fourier-Motzkin elimination gives a procedure to enumerate the facets of $\text{cone}(A)$. Indeed, testing for membership in $\text{cone}(A)$ is a linear programming problem whose dual is described by Theorem 2.1.

2.2 An inclusion matrix

We return to the setting of (edge-weighted) graphs. For a simple graph G on vertex set $[n]$, let $\mathbf{1}_G$ denote the characteristic vector of $E(G)$ in $\mathbb{R}^{\binom{n}{2}}$, where again coordinates are indexed by $\binom{[n]}{2}$. That is,

$$\mathbf{1}_G(e) = \begin{cases} 1 & \text{if } e \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

We define W_n as the inclusion matrix of 2-subsets versus 3-subsets of $[n]$. That is, for $e \in \binom{[n]}{2}$ and $f \in \binom{[n]}{3}$,

$$W_n(e, f) = \begin{cases} 1 & \text{if } e \subseteq f, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, W_n is the matrix whose columns are the characteristic vectors of all triangles in K_n . It follows that $\tau_n = \text{cone}(W_n)$. That is, the existence of a fractional triangle decomposition of G is equivalent to the existence of a nonnegative solution x to $W_n x = \mathbf{1}_G$.

The following simple fact is well-known but is important for our analysis to follow.

Proposition 2.2. *The 10×10 inclusion matrix matrix W_5 is invertible with*

$$W_5^{-1}(f, e) = \begin{cases} 1/3 & \text{if } |e \cap f| \in \{0, 2\}, \\ -1/6 & \text{otherwise.} \end{cases} \quad (2.1)$$

PROOF. Let $f, f' \subseteq \{1, \dots, 5\}$ with $|f| = |f'| = 3$. The inner product of row f of the claimed inverse (2.1) and column f' of W_5 is computed in cases. If $f' = f$, the product is $3 \cdot 1/3$. If $|f' \cap f| = 1$ or 2 , exactly one $e \subset f'$ satisfies $|e| = 2$ and $|f \cap e| \in \{0, 2\}$ (this being $e = f^c = f' \setminus f$ or $e = f \cap f'$, respectively). In either of these cases, the inner product is $1/3 - 2 \cdot 1/6 = 0$. \square

Proposition 2.2 is useful for checking whether a set \mathcal{K} of triangles on vertex set $[n]$ spans $\mathbb{R}^{\binom{n}{2}}$ for $n > 5$. If there exists a set S of five points, all of whose triangles can be spanned by \mathcal{K} , then $\mathbb{1}_e$ can be spanned for each $e \in \binom{S}{2}$. Applying this with different choices of S can produce ‘new’ triangles in the span. Note we have identified triangles with their characteristic vectors, a slight abuse of language which is convenient in what follows.

2.3 Some example facets

For results on fractional triangle decompositions, the Farkas lemma motivates a study of the facets of τ_n . A vector $y \in \mathbb{R}^{\binom{n}{2}}$, or alternatively an edge-weighted graph on n vertices, is normal to a facet of τ_n if: (1) $\langle y, \mathbb{1}_K \rangle \geq 0$ for all triangles K , and (2) the span of triangles K for which $\langle y, \mathbb{1}_K \rangle = 0$ has codimension 1. We call such vectors ‘facet normals’ in what follows.

Example 2.3. Let $n \geq 5$ and suppose (A, B) is a partition of $[n]$ with $|A|, |B| \geq 2$. Consider the vector y defined by

$$y(e) = \begin{cases} 2 & \text{if } e \subseteq \binom{A}{2} \cup \binom{B}{2}, \\ -1 & \text{otherwise.} \end{cases} \quad (2.2)$$

It is easy to check that y as defined is nonnegative on triangles. We show that y is a facet normal of τ_n . Let \mathcal{K}_0 be the set of triangles crossing the partition (A, B) . We have y orthogonal to each triangle in \mathcal{K}_0 . Put $\mathcal{K} = \mathcal{K}_0 \cup \{K\}$, where K is a triangle inside (say) A . By Proposition 2.2 and the discussion following it, any $\mathbb{1}_e$, $e \in \binom{B}{2}$ is spanned by \mathcal{K} ; this is seen by considering the five-point-set set $e \cup V(K)$. It follows, then, that every triangle inside B is spanned by \mathcal{K} . Finally, by choosing three points in B and two in A , one has every $\mathbb{1}_e$, $e \in \binom{A}{2}$, spanned by \mathcal{K} , and this is enough to span the entire space. It follows that y is a facet normal of τ_n .

We call y as in (2.2) an $(|A|, |B|)$ -cut. Proposition 5.1 provides an alternate verification that such vectors are facet normals of τ_n . We note that there are exponentially many (in n) facets of this type.

Suppose $4 \mid n$ and recall the graph $G = C_4 \cdot K_{n/4}$ mentioned in Section 1. There exists an equipartition of the vertices (A, B) so that the number of edges of G within A or B equals $4\binom{n/4}{2} = n^2/8 - n/2$, while the number of edges crossing the partition equals $n^2/4$. Taking y to be the $(n/2, n/2)$ -cut facet defined by (A, B) , we see that $\langle y, \mathbb{1}_G \rangle < 0$. In fact, this same cut witnesses many other graphs with minimum degree near $3n/4$ also failing to have a (fractional) triangle decomposition.

Example 2.4. Let $n \geq 6$. For any $e \in \binom{[n]}{2}$, the vector $y = \mathbb{1}_e$ is a facet normal of τ_n , since $\dim y^\perp = \binom{n}{2} - 1$. We call (positive multiples of) such y and their corresponding facets *trivial*.

In practice, it is simple to check whether a vector y supports our cone. For it to be a facet normal, it must be ‘indecomposable’: if $y = y_1 + y_2$ with y_1 and y_2 both supporting vectors, then $y_1 = cy$ and $y_2 = (1 - c)y$ for some $c \in [0, 1]$. This, together with Example 2.4 leads to an easy observation.

Lemma 2.5. Let y be a nontrivial facet normal of τ_n . Then every edge in $\binom{[n]}{2}$ is contained in a triangle K such that $\langle y, \mathbb{1}_K \rangle = 0$.

PROOF. This can be shown by direct verification for $n = 5$ using that the only facets of τ_5 are $(2, 3)$ -cuts; see Proposition 2.2. Suppose $n \geq 6$ and y is a facet normal of τ_n . If e is an edge in no such triangle, then it is possible to decrease the weight of e in y such that the resulting vector still supports τ_n . Therefore, y is a positive multiple of $\mathbb{1}_e$. \square

Example 2.6. Let $n \geq 6$. The edge-weighted graph y on vertex set $[n]$ with

$$y(e) = \begin{cases} -1 & \text{if } e = \{1, n\}, \\ 1 & \text{if } e = \{i, n\} \text{ for } i \in \{2, \dots, n-1\}, \\ 0 & \text{otherwise} \end{cases}$$

is a facet normal of τ_n . This is easy to verify directly. Any triangle avoiding vertex n is orthogonal to y , as is any triangle containing edge $\{1, n\}$. If we include with these triangles one of positive weight, say on $\{2, 3, n\}$, it is possible to span the whole space by a similar argument as in Example 2.3. For more details, including why $n \geq 6$ is needed, we refer the reader to [12].

It is natural to call the facet y described by Example 2.6 a *star* facet of τ_n . It has a combinatorial interpretation for decomposition into triangles, albeit of very mild significance: the inequality $\langle y, \mathbb{1}_K \rangle \geq 0$ is asserting that,

should G have a fractional triangle decomposition, there cannot exist any vertex of degree 1 in G . The star y centered at such a vertex and aligned so that the negative edge is the pendant edge would have $\langle y, \mathbf{1}_G \rangle = -1$.

2.4 Symmetry

Consider the natural action of the symmetric group \mathcal{S}_n on $\mathbb{R}^{\binom{n}{2}}$. For an edge-weighted graph y and permutation $\alpha \in \mathcal{S}_n$, we have $y^\alpha(e) = y(\alpha^{-1}e)$; that is, the action is induced on edges (edge weights) by permutations of the vertices.

Let us define the *stabilizer* of y to be $\text{stab}(y) = \{\alpha \in \mathcal{S}_n : y^\alpha = y\}$. By the orbit-stabilizer theorem, the number of distinct edge-weighted graphs isomorphic to y on n vertices is $n!/|\text{stab}(y)|$.

Suppose G has automorphism group Γ . Testing whether G is in the cone τ_n amounts to checking $\langle \bar{y}, \mathbf{1}_G \rangle \geq 0$ on all \bar{y} of the form

$$\bar{y} = \frac{1}{|\Gamma|} \sum_{\alpha \in \Gamma} y^\alpha$$

for some facet normal y of τ_n . The set of weighted graphs invariant under Γ is a subspace of $\mathbb{R}^{\binom{n}{2}}$. Therefore, its intersection with τ_n is a sub-cone. For example, if $\Gamma = \mathcal{S}_a \times \mathcal{S}_b$ for $n = a + b$, then the (a, b) -cuts (and the nonnegative orthant) give a complete description; see [11]. It would be interesting to study invariant sub-cones for other specific groups $\Gamma \leq \mathcal{S}_n$.

2.5 A matrix formulation

As an alternative to placing edge-weighted graphs in correspondence with $\mathbb{R}^{\binom{n}{2}}$, we can use the $n \times n$ symmetric matrices with zero diagonal. Under this slight change in notation, τ_n is equivalent to the cone generated by the $\binom{n}{3}$ matrices

$$P^\top \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} P \in \mathbb{R}^{n \times n}, \quad (2.3)$$

where P is comprised of three rows of an $n \times n$ permutation matrix. This alternate presentation has some advantages. For example, the characteristic polynomial $\chi_y(t)$ of the matrix corresponding to an edge-weighted graph y is preserved under vertex permutation, making it a useful invariant. Moreover, the factorization of $\chi_y(t)$ in $\mathbb{Q}[t]$ carries algebraic information, such as the stabilizer of y .

In this context, a graph G has a fractional triangle decomposition if and only if its adjacency matrix A_G is a conical combination of matrices of the form (2.3). A referee suggested a nice variant on this: G admits a fractional triangle-decomposition if and only if its Laplacian matrix $L_G := D_G - A_G$, where D_G is the diagonal matrix of degrees in G , is a conical combination of matrices of the form

$$P^\top \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} P \in \mathbb{R}^{n \times n}, \quad (2.4)$$

where P is as before. A feature of this reformulation is that both L_G and the matrices in (2.4) are positive semidefinite.

2.6 The metric polytope

The cone τ_n appears ‘locally’ inside a well-studied polytope. Recall that a *metric* d on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ such that, for all $x, y, z \in X$,

a. $d(x, y) \geq 0$,

(a') $d(x, y) = 0$ if and only if $x = y$,

b. $d(x, y) = d(y, x)$,

c. $d(x, z) \leq d(x, y) + d(y, z)$.

A *semi-metric* is a function that satisfies only conditions (a), (b) and (c). The *metric cone* Met_n consists of all semi-metrics on an n -set. If we bound Met_n by considering only those semi-metrics which satisfy the ‘perimeter inequalities’

$$d(x, y) + d(x, z) + d(y, z) \leq 2 \quad (2.5)$$

for all 3-subsets $\{x, y, z\} \subseteq [n]$, then we obtain the *metric polytope* met_n . Each of the triangle inequalities and perimeter inequalities defines a half-space bounding met_n . Taking $X = [n]$ and writing d_{ij} for $d(i, j)$, we can consider the metric d as a vector $(d_{12}, d_{13}, d_{23}, \dots, d_{n-1, n})$ in $\mathbb{R}^{\binom{n}{2}}$, and thus embed Met_n and met_n in $\mathbb{R}^{\binom{n}{2}}$.

The point $(2/3, \dots, 2/3)$ is a vertex of met_n for $n > 3$ since the vector obtained by incrementing any coordinate by $\epsilon > 0$ and simultaneously decrementing a vertex-disjoint coordinate by ϵ fails to satisfy (2.5). Near this vertex (within $2/9$ in the box norm) the triangle inequalities are automatically satisfied. It follows that $(2/3, \dots, 2/3) - \text{met}_n$, near the origin, is described by the inequalities $a_{ij} + a_{ik} + a_{jk} \geq 0$ for 3-subsets $\{i, j, k\}$. The following result is an immediate consequence.

Proposition 2.7. *The halfspaces bounding τ_n admit a natural bijection with the edges of met_n incident with $(2/3, \dots, 2/3)$.*

3 Arithmetic and combinatorial structure

In this section, we sample some structure forced on facets of τ_n , specifically on the entries of their normal vectors. We begin with an easy observation on the relative sizes of extreme entries in facet normals.

Proposition 3.1. *Let y be a nontrivial facet normal of τ_n . Let a and b denote, respectively, the maximum and minimum entry in y . Then we have $-b/2 \leq a \leq -2b$.*

PROOF. First, suppose $a \geq -b$. Using Lemma 2.5, choose a triangle K in y^\perp containing an edge of weight a . The other two edges of K must have negative weight, since a is largest among the entries in magnitude. It follows that one of these other edges in K has weight at most $-a/2$. This establishes $a \leq -2b$. The case $a < -b$ is similar. \square

REMARKS. It is easy to see that $a = 1, b = -2$ is not possible for facets (although it is possible for supporting vectors). Indeed, it may be the case that the first inequality can be strengthened to $a > -b/2$ or even $a \geq -b$.

Presumably, Proposition 3.1 only scratches the surface of constraints on the signs and relative magnitudes in facet normals. We do not explore this further here, although it seems reasonable to guess that entries have some central tendency and are roughly symmetric about their mean.

Proposition 3.2. *Let y be a facet normal of τ_n and let G be the simple graph carrying the nonpositive entries of y . If G is bipartite, then it is complete bipartite.*

PROOF. Suppose (A, B) is a bipartition of the vertices of G such that all edges in $\binom{A}{2}$ and $\binom{B}{2}$ are positive. Let z be the cut facet corresponding to (A, B) ; see Example 2.3. Since all entries of y on edges within A and B are positive, it follows that for some $\epsilon > 0$, $y - \epsilon z$ also supports τ_n . This is a contradiction to the indecomposability of y . \square

Next, we offer a purely arithmetic constraint. Let us say that a vector is in *standard form* if its entries are integers with greatest common divisor equal to 1.

Proposition 3.3. *Let $n \geq 5$ and suppose y is a facet normal of τ_n in standard form. The following are equivalent:*

- (a) $\langle y, \mathbf{1}_K \rangle \equiv 0 \pmod{3}$ for all triangles K ;
- (b) no entry of y is $0 \pmod{3}$; and
- (c) all entries of y are $1 \pmod{3}$ or all entries of y are $2 \pmod{3}$.

PROOF. The implication (c) \Rightarrow (a) is obvious, so it is enough to prove (a) \Rightarrow (b) and (b) \Rightarrow (c).

Suppose y satisfies (a) but that $y_e = 0$ for some edge $e = \{u, v\}$. For every triangle involving e , the entries of y on the other two edges must be different and nonzero $\pmod{3}$, using (a). Since $n \geq 5$, there are at least three such triangles. It follows that there is a 4-cycle, say w_1, u, w_2, v in which the entries of y on its edges are $1, 1, 2, 2 \pmod{3}$, in this order. There is now no possible value for y on $w_1 w_2$: it must be $1 \pmod{3}$ from the triangle with u and $2 \pmod{3}$ from the triangle with v . This contradiction shows (a) \Rightarrow (b).

For the next implication, suppose y satisfies (b). For $i = 1, 2$, let E_i be the set of edges e such that $y_e \equiv i \pmod{3}$. Let \mathcal{K} be the set of all triangles K such that $\langle y, \mathbf{1}_K \rangle = 0$. Any such triangle must have all three edges in E_1 or all three edges in E_2 , by (b). Let \mathcal{K}_i be the set of triangles K such that $\langle y, \mathbf{1}_K \rangle = 0$ and all three edges belong to E_i . Now, take any edge e , and assume without loss of generality that $e \in E_1$. Since y is a facet-normal and $y_e \neq 0$, we have that $\{\mathbf{1}_K : K \in \mathcal{K}\} \cup \{\mathbf{1}_e\}$ spans $\mathbb{R}^{E_1 \cup E_2}$. Examining the coordinates indexed by E_2 , it follows that $\{\mathbf{1}_K : K \in \mathcal{K}_2\}$ spans $\{0\} \oplus \mathbb{R}^{E_2}$. Letting y_2 denote the projection of y onto \mathbb{R}^{E_2} , we have $\dim\{y_2\}^\perp = |E_2|$. It follows that $y_2 = 0$, and this implies $E_2 = \emptyset$. This proves y is constant $\pmod{3}$ and we have shown (c). \square

As a result of Proposition 3.3, we can partition all facets of τ_n into three categories according to their normal vectors in standard form. It is natural to label these 0, 1, 2 according to whether some zero is present, all entries are $1 \pmod{3}$, or all entries are $2 \pmod{3}$, respectively. The trivial and star facets belong to category 0. The cut facets belong to category 2. Example 4.2 to follow gives the first example of a facet in category 1, which occurs for $n = 7$.

The following example shows that Proposition 3.3 is somewhat tight.

Example 3.4. The vector

$$\begin{pmatrix}
 10, \\
 8, \quad 6, \\
 2, \quad 4, \quad 2, \\
 2, \quad 4, \quad 2, \quad -4, \\
 2, \quad 2, \quad -4, \quad 2, \quad 2, \\
 -1, \quad -1, \quad 5, \quad 5, \quad -1, \quad -1, \\
 -1, \quad -1, \quad 5, \quad 5, \quad -1, \quad -1, \quad 2, \\
 -4, \quad -2, \quad -4, \quad 2, \quad 2, \quad 8, \quad 5, \quad 5, \\
 -4, \quad -6, \quad 0, \quad 2, \quad 2, \quad 4, \quad 7, \quad 7, \quad 8)
 \end{pmatrix}$$

is in standard form and normal to a facet of τ_{10} . There are only three entries congruent to 0 (mod 3), namely 6, 0, -6; these are supported on edges joined in a path.

4 Classification for small n

The case $n = 5$ is the first in which τ_n has full dimension. As a direct consequence of Proposition 2.2, the only facets of τ_5 are the (2,3)-cuts, giving a total of 10 facets. Moreover, their structure is ‘simplicial’ (any two are adjacent) since the ten facet normals are linearly independent in \mathbb{R}^{10} . In this section we enumerate facets (and the adjacent facets for each) for the next few values of n . Small cases can be easily replicated using, e.g., the built-in function `Cone()` in Sage [20]. We remark that much of the enumerative work to follow has been done in the context of the metric cone, [5].

A classification of facets of τ_6 is shown in Table 1. We note that the examples in Section 2.3, namely trivial, star, (3,3)-cut, and (2,4)-cut, give a complete description of τ_6 . In the table, each row gives a vector, with coordinates corresponding to the colex order on $\binom{[6]}{2}$ for an isomorphism class. The column with heading # gives the number of distinct copies induced under the action of S_6 . The rightmost column gives the degree, or number of facets to which the corresponding facet is adjacent. The three nontrivial isomorphism types are displayed as weighted graphs in Figure 1. A {blue,red}-edge-coloring illustrates the positive and negative weights, with magnitudes as labeled.

The following result is a straightforward consequence of the classification.

Proposition 4.1. *A multigraph on six vertices admits a fractional triangle decomposition if and only if it satisfies all inequalities corresponding*

	representative	#	deg
1.	(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	15	32
2.	(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0, 0)	30	14
3.	(2, 2, 2, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, 2, 2)	10	57
4.	(2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, 2)	15	32
	total	70	

Table 1: Isomorphism classes of facet normals of τ_6 , in standard form

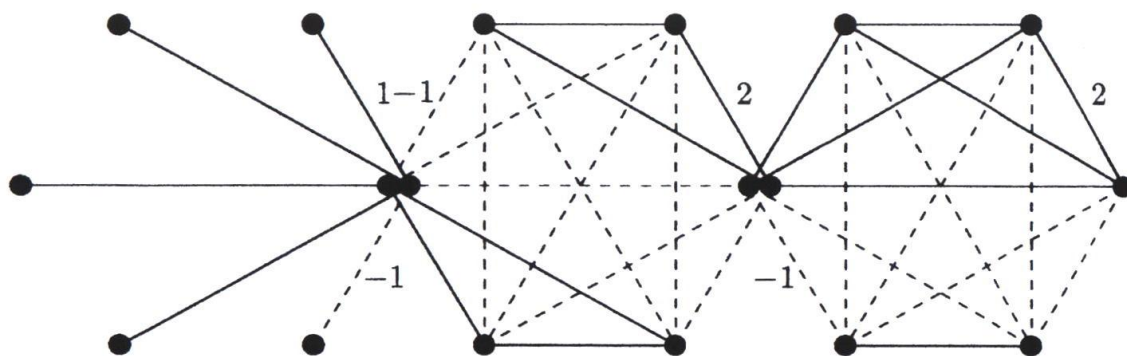


Figure 1: Weighted graphs for the nontrivial facet normals of τ_6

to cuts and its underlying simple graph has no pendant vertices.

For $n = 7$, the results appear in Table 2 with headings as before and weighted graphs in Figure 2.

We note here the emergence of a facet normal (number 7) with all entries 1 (mod 3). In terms of triangle decompositions, this constraint has interesting implications which we consider in Section 6.

Example 4.2. The facet normal in row 7 of Table 2 has all entries 1 (mod 3); it follows that both alternatives in Proposition 3.3(c) are possible.

The classification of facets of τ_8 is displayed in Table 3. The total count of 52367 facets also appears in [5] as the degree of anti-cuts in the metric polytope met_8 .

Here, for the first time, we encounter facets with all entries 2 (mod 3) other than cuts. We also notice that the trivial facet and balanced cut facets have by far the largest degrees; this is likely due to the relatively large number of triangles orthogonal to the corresponding vectors.

The number of facets of τ_n for $n = 5, 6, 7, 8$ is now sequence A246427 in the OEIS database; see [21].

	representative	#	deg
1.	(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	21	340
2.	(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, -1, -1, 0, 0, 0, 0, 1)	420	20
3.	(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0)	42	75
4.	(2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 1, 1, 1, 1)	105	75
5.	(2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, -1, -1, 2, 2)	35	340
6.	(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 2)	21	75
7.	(4, 4, -2, 1, 1, 1, 1, 1, 1, -2, -2, 4, -2, 1, 1, -2, -2, 4, 1, 1, 4)	252	20
	total	896	

Table 2: Isomorphism classes of facet normals of τ_7 , in standard form

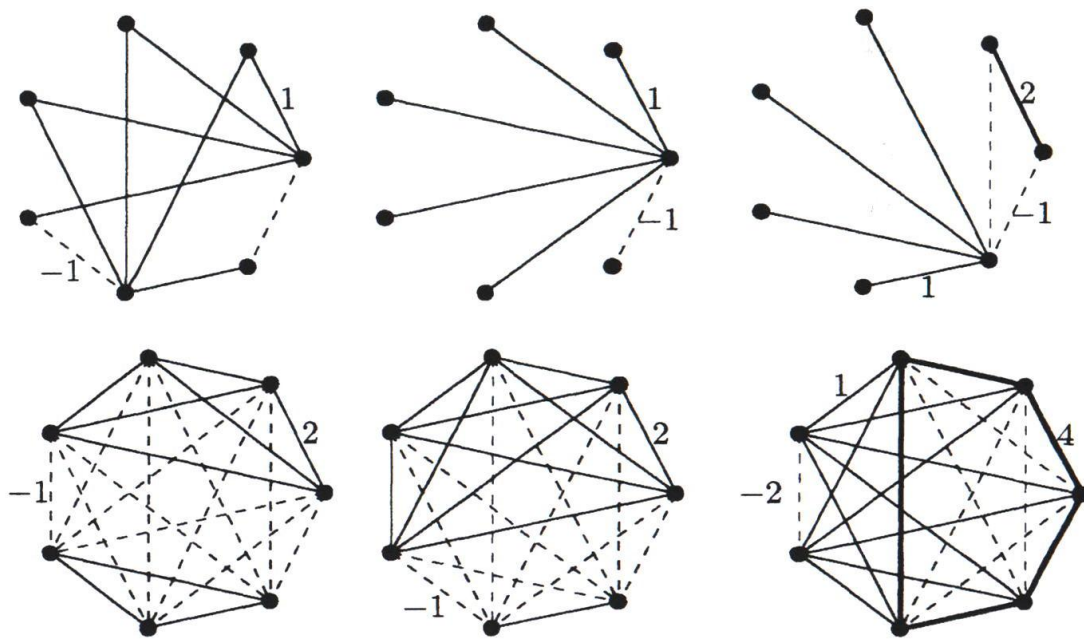


Figure 2: Weighted graphs for the nontrivial facet normals of τ_7

5 Lifting facets

Our aim here concerns lifting facets of τ_n to facets of τ_{n+1} via a ‘vertex splitting’ operation.

Proposition 5.1. *Let $n \geq 5$ and suppose y is a facet normal of τ_n . Suppose there exists a triangle K on $[n-1]$ with $\langle y, \mathbf{1}_K \rangle > 0$. Define the vector y^{spl} on $\binom{[n+1]}{2}$ by*

$$y^{spl}(e) = \begin{cases} y(e) & \text{if } e \subset [n], \\ y(\{i, n\}) & \text{if } e = \{i, n+1\} \text{ for } i \in [n-1], \\ -2 \min\{y(\{i, n\}) : i \in [n-1]\} & \text{if } e = \{n, n+1\}. \end{cases}$$

Then y^{spl} is a facet normal of τ_{n+1} .

PROOF. It is straightforward to check that y^{spl} is nonnegative on all triangles. Let \mathcal{K} consist of K , together with all triangles L on $[n+1]$ such that $\langle y^{spl}, \mathbf{1}_L \rangle = 0$. Using that y is a facet normal of τ_n and vertices $n, n+1$ are clones with respect to y^{spl} , every edge in $\binom{[n+1]}{2}$, except possibly $\{n, n+1\}$, is a linear combination of triangles in \mathcal{K} . By the choice of weight on $\{n, n+1\}$, there exists j such that the triangle on $\{j, n, n+1\}$ belongs to \mathcal{K} . Then, since $\{j, n\}$ and $\{j, n+1\}$ are spanned by \mathcal{K} , so is $\{n, n+1\}$. \square

The third author’s dissertation presents a similar construction which allows copies of a facet of τ_n to be glued together on a common positive triangle to produce facets of τ_m for $m > n \geq 5$. This allows for somewhat more general lifts of facets. See [12, Proposition 3.8] for details.

Figure 3 shows the effect of vertex splitting small facets; refer to Tables 1, 2, 3 for facet labels, which are indicated in the figure as subscripts.

Example 5.2. The weighted graph shown in Figure 4 defines a facet of τ_8 . Repeatedly applying Proposition 5.1 to pendant vertices gives an infinite family of facets of τ_n , $n \geq 8$, as follows. Given any partition (A, B) of $\{3, \dots, n\}$ with $|A|, |B| \geq 3$, a facet arises from the normal vector y defined by

$$y(e) = \begin{cases} -1 & \text{if } e = \{1, 2\}, \\ 1 & \text{if } e = \{1, a\} \text{ for } a \in A, \text{ or } e = \{2, b\} \text{ for } b \in B, \\ 0 & \text{otherwise.} \end{cases} \quad (5.1)$$

The terminology ‘binary star’ was used for such facets in [12]. These have some significance for triangle decompositions. Consider the question

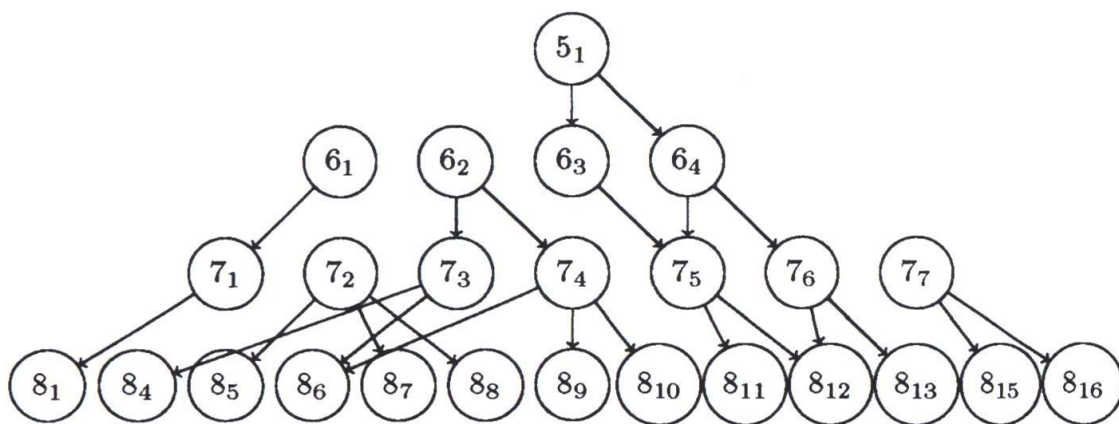


Figure 3: Vertex splitting facets of τ_n for $n = 5, 6, 7$

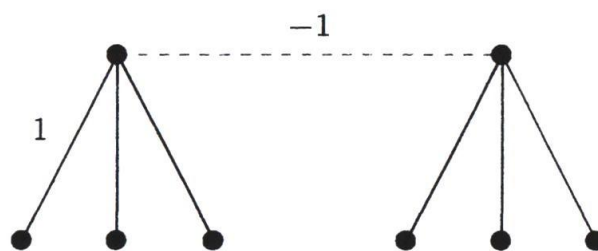


Figure 4: The weighted graph corresponding to the 'binary star' facet of τ_8

of whether G has a fractional triangle decomposition under the assumption that it has at least $\frac{3}{4}\binom{n}{2}$ edges and has minimum degree $\delta(G) \geq cn$. The binary star y reveals that we cannot take $c < \frac{1}{2}$. For instance, suppose (A, B) is a roughly balanced partition of $\{3, \dots, n\}$. Build the graph G on vertex set $[n]$ from a clique on $A \cup B$, and join every vertex in A to 2, every vertex in B to 1, and finally include the edge $\{1, 2\}$. We have $\langle y, \mathbb{1}_G \rangle = -1$ where y is as in (5.1); indeed, the edge $\{1, 2\}$ belongs to no triangle in G .

6 A class of nearly-Euclidean facets

Recall the interpretation of cut facets as maps $V(G) \rightarrow \{0, 1\}$, where the implied constraint on triangle decomposability of G amounts to the perimeter bound. In a sample of several hundred facets of τ_n in the range $8 \leq n \leq 12$, we notice many that roughly resemble a similar Euclidean embedding to the interval $[0, 1]$. Here, we examine in detail one such class of facets.

Let us start by considering an analog of cuts with three ‘bins’: we let $\omega : [n] \rightarrow \{0, \frac{1}{2}, 1\}$, and define $y_\omega \in \mathbb{R}^{\binom{n}{2}}$ to have entries

$$y_\omega(\{i, j\}) = 4 - 6|\omega(i) - \omega(j)| \in \{-2, 1, 4\}.$$

With $A = \omega^{-1}(0)$, $B = \omega^{-1}(\frac{1}{2})$ and $C = \omega^{-1}(1)$, the corresponding weighted graph is depicted in Figure 5. Note that when $B = \emptyset$, y_ω reduces to a cut with partition (A, C) .

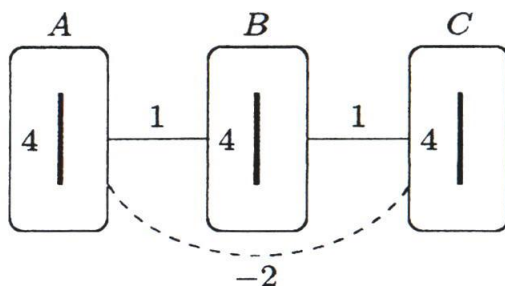


Figure 5: Weighted graph for the supporting vector y_ω

It is easy to see that y_ω supports τ_n . However, y_ω is not a facet normal unless $B = \emptyset$, since otherwise it is the sum of two cuts with partitions $(A \cup B, C)$ and $(A, B \cup C)$. However, if we slightly modify y_ω by reducing some values in the middle bin $\binom{B}{2}$ from 4 to -2 , then in some cases the resulting vector is a facet normal. For example, the facet of τ_7 mentioned in Example 4.2 has precisely this structure, where A and C are singletons

and all edges in a cycle on the five vertices of B have been reduced from 4 to -2 . More generally, one can subtract $6\mathbf{1}_H$ from y_ω , where H is a maximal triangle-free graph with $V(H) = B$ and still have the resulting vector support τ_n . We note that, although maximal triangle-free, the complete bipartite graphs do not induce facets in this way.

Proposition 6.1. *The vector $y_\omega - 6\mathbf{1}_H$ is not a facet normal if H is complete bipartite.*

PROOF. Let $B_1 \cup B_2$ be the bipartition of H . Then $y_\omega - 6\mathbf{1}_H = y_1 + y_2$, where y_1 is the cut corresponding to $(A \cup B_1, C \cup B_2)$ and y_2 is the cut corresponding to $(A \cup B_2, C \cup B_1)$. It follows that the given vector is not a facet normal. \square

From the limited experimentation we have done, it appears the complete bipartite graphs may in fact be the only maximal triangle-free graphs which do not induce facets. We now describe a class of graphs H which do induce facets.

A C_5 -blow-up is a graph obtained by replacing every vertex of a 5-cycle by an independent set (of possibly different sizes). Such graphs are clearly maximal triangle-free; moreover, they can be constructed recursively by starting with a ‘seed’ C_5 , and adding vertices one at a time, joining them to maximal independent sets.

Proposition 6.2. *Let $\omega : [n] \rightarrow \{0, \frac{1}{2}, 1\}$ with nonempty level sets A, B, C . Let H be a C_5 -blow-up spanning the vertex set $B = \omega^{-1}(\frac{1}{2})$. Then $y_\omega - 6\mathbf{1}_H$ is a facet normal of τ_n .*

PROOF. By Proposition 5.1, it suffices to consider the case $|A| = |C| = 1$, so that $k := |B| = n - 2$. Assume $k \geq 5$. When $k = 5$, the only candidate graph for H is C_5 , resulting in the facet normal of Example 4.2. Suppose for induction hypothesis that the result holds for n vertices, where $n \geq 7$. Consider ω with level sets A, B, C , where $|A| = |C| = 1$ and with a C_5 -blow-up H on the $k + 1$ vertices of B . This H can be obtained by cloning a vertex, wlog k , in a C_5 -blow-up H' of order k on vertex set B' . Using the induction hypothesis, let z be the normal to the $(1, k, 1)$ -facet based on H' . Choose $e \in \binom{B' \setminus \{k\}}{2}$ which is not an edge of H' . Then $e \cup A$ defines a triangle K such that $\langle z, \mathbf{1}_K \rangle = 1 + 1 + 4 > 0$. By Proposition 5.1, the vector $z^{\text{spl}} = y_\omega - 6\mathbf{1}_H$ obtained by splitting at k produces a facet normal of τ_{n+1} . \square

Let us call facets of the type described in Proposition 6.2 C_5 -facets. The non-isomorphic C_5 -blow-ups on k vertices admit a natural bijection with the bracelets made from 5 black beads and $k - 5$ white beads. The count-

ing problem for such bracelets appears in the OEIS database as sequence A032279, see [21]. Let a_k denote the k th term of this sequence. By a formula of Robert Israel, $a_k = k^4/240 + o(k^3)$. It follows that there are

$$\sum_{k=5}^{n-2} \left\lfloor \frac{n-k}{2} \right\rfloor a_k = \Omega(n^6)$$

non-isomorphic C_5 -facets of τ_n .

To display the utility of these facets, we construct graphs of minimum degree near $3n/4$ which are excluded from τ_n by a C_5 -facet but no cut facets. The construction is a minor perturbation of the canonical example $C_4 \cdot K_{n/4}$ rejected by cut facets.

Construction 6.3. Fix $\epsilon > 0$, and let J_p denote a graph on p vertices of degree $(1-\epsilon)p$. Let $n = 4p+5q$ and construct G as the join $C_4 \cdot J_p + C_5 \cdot \overline{K}_q$ on vertex set $[n]$.

We build a C_5 -facet based on $H = C_5 \cdot \overline{K}_q$ which witnesses that G is not in τ_n . Assign the copies of J_p ‘alternately’ into A and C , and let H be the subgraph $C_5 \cdot \overline{K}_p$ on vertex set B . Then, we compute

$$\langle y_\omega - 6\mathbb{1}_H, \mathbb{1}_G \rangle = 4 \times 2(1-\epsilon)p^2 + 1 \times (4p)(5q) - 2 \times (4p^2 + 5q^2) = -8\epsilon p^2 + 20pq - 10q^2. \quad (6.1)$$

If, instead, we attempt to place vertices of H into $A \cup C$, we get at least p^2 edges of H internal to A or C . It follows that such a cut z satisfies

$$\langle z, \mathbb{1}_G \rangle \geq 8(1-\epsilon)p^2 + 20pq - 4q^2. \quad (6.2)$$

Let p be large and take q to be an integer near $(\frac{2}{5}\epsilon + \frac{8}{125}\epsilon^2)p$. With this choice, the right side of (6.1) is negative while the right side of (6.2) is positive.

The minimum degree of G is $\delta(G) = (3-\epsilon)p+5q$, or roughly $\frac{75+25\epsilon+8\epsilon^2}{100+50\epsilon+8\epsilon^2}n$, which approaches (from below) the threshold in Conjecture 1.1.

7 Concluding Remarks

We have seen many properties of the cone τ_n , and connected its halfspace description with the triangle decomposition problem for graphs. We have pointed out several places where additional work could lead to a better understanding of facets. As another next step, we feel it would be useful to attempt an approximation of τ_n for large n by a cone with simpler structure.

The interested reader may wish to explore some related directions we have neglected to mention so far. First, the cone τ_n even long ago attracted some interest in quantum physics for its connection with the ‘ N -representability problem’; see for instance [16]. Next, it is worth mentioning the ‘cut cone’ in $\mathbb{R}^{\binom{n}{2}}$, generated by all vertex cuts in the complete graph K_n . More information can be found in [7, 8]. In particular, the ‘uniform cut cone’ studied by Neto [19] is closely related to τ_n .

Finally, our cone τ_n is a special case of the family of cones introduced in [11]. This more general setting considers the cone generated by the inclusion matrix of t -subsets $\binom{[n]}{t}$ versus k -subsets $\binom{[n]}{k}$, where t, k, n are positive integers satisfying $k \geq t$ and $n \geq k + t$. Alternatively, this is the cone of weighted t -uniform hypergraphs on n vertices generated by k -vertex cliques $K_k^{(t)}$. It was shown in [11] that various inequalities for t -designs arise from certain supporting vectors described by orthogonal polynomials.

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