The complexity of monitoring a network with both watchers and listeners

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Abstract

We consider the problem of detecting an intruder in a network where there are two types of detecting devices available. One device can determine the distance from itself to the intruder and the other can see the vertex it occupies as well as all adjacent vertices. The problem is to determine how many devices are required and where they should be placed in order to determine a single intruder's location. We show that on the one hand, finding the minimum number of devices required to do this is easy when the network is a tree with at most one leaf adjacent to any vertex, while on the other hand finding this number is an NP-complete problem even for trees with at most two leaves adjacent to any vertex.

Key words: Domination, Metric Basis, Locating Set, Detection Pair, NP-Complete

This article is dedicated to Gary MacGillivray *in honour and recognition of his extremely active and prolific career in the delightful art of problem solving.*

1 Introduction

Our objective is to be able to locate a single intruder (a fault) if one appears (occurs) at any vertex of a given network. We introduce the problem of using two different detection devices: *watchers* and *listeners*. A watcher can determine the location of the vertex that the intruder occupies provided

JCMCC 116 (2021), pp.233-244

that the watcher is located at a vertex in the closed neighborhood of the vertex that the intruder occupies. A listener can determine the distance from the vertex it occupies to the vertex that the intruder occupies. The goal is to place the minimum number of devices so that the intruder can be found regardless of its location in the network.

Consider a finite simple loopless graph $G = (V, E)$. For a vertex x in G, the *open neighborhood of* x is the set $N_G(x) = \{v \in V(G) \,|vx \in E(G)\}$. The *closed neighborhood of* x is the set $N_G[x] = N_G(x) \cup \{x\}$, and if $S \subseteq V(G)$ then $N_G[S] = \bigcup\{N_G[x]|x \in S\}$. If x and y are in $V(G)$, then the *distance* $d(x, y)$ between x and y is the number of edges in a shortest path from x to y.

Definition 1.1 *Let v be a vertex in the graph G, and let $S = \{s_1, s_2, \ldots, s_k\}$ be an ordered subset of $V(G)$. The S-*location vector *for v is the vector $[d(v, s_1), d(v, s_2), \ldots, d(v, s_k)]$. If two distinct vertices u and v of G have the same S-location vector, we say that u and v are* confused *by S.*

Definition 1.2 *A* detection pair *P for the graph G is a pair of subsets $P = (W, L)$ of $V(G)$ such that if $V(G) \setminus N[W] \neq \varnothing$ then $L \neq \varnothing$ and with the property that no two vertices in $V(G) \setminus N[W]$ are confused by L. We say that* the size of *P is $|W| + |L|$ and denote it by $||P||$.*

Thus a detection pair represents a pair of subsets (W, L) of $V(G)$ where watchers are stationed at each vertex in W and listeners are stationed at each vertex in L. Note that W and L need not be disjoint and that either W or L could be empty. Then we say a *minimum detection pair D* is a detection pair of smallest size and define the *detection number $L\gamma(G)$* of G to be the size of a minimum detection pair in G.

Suppose that (W, L) is a detection pair (minimum detection pair) for G. If on the one hand, $L = \varnothing$, then $V(G) \subseteq N[W]$ and thus W is a *dominating set* (*minimum dominating set*) for G. The cardinality of a minimum dominating set is called the *dominating number* $\gamma(G)$ of G. A comprehensive introduction to the work done in the area of domination can be found in [6].

If on the other hand $W = \varnothing$, then no pair of vertices in G can be confused by L, and thus L is a *metric basis* (*minimum metric basis*) for G. The cardinality of a minimum metric basis is called the *metric dimension* $L(G)$ of G. There is a considerable body of literature on metric bases (also known as *reference sets* and *locating sets*) see for example [1, 5, 6, 7, 11] and more recently [3].

Note that these notions are distinct from *locating-dominating sets*, [9, 10], where the sets need to be simultaneously locating and dominating.

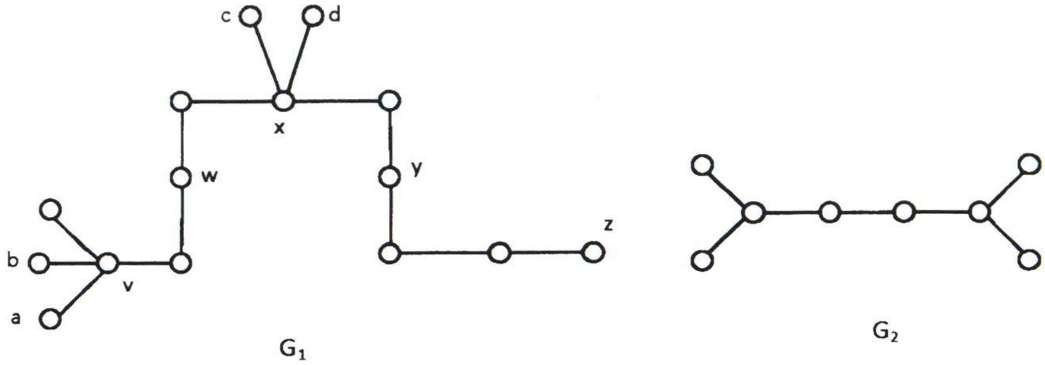

Figure 1 The Graphs G_1 and G_2

Example 1.1 *In Figure 1,* $\gamma(G_1) = 5$ *with a sample minimum dominating set* $\{v, w, x, y, z\}$, $L(G_1) = 4$ *with a sample minimum metric basis* $\{a, b, c, z\}$, *and* $L\gamma(G_1) = 3$ *with a sample minimum protection pair* $(\{x, v\}, \{z\})$. *On the other hand,* $\gamma(G_2) = L(G_2) = L\gamma(G_2) = 2$, *but there is no minimum protection pair* (W, L) *such that both* W *and* L *are nonempty.*

We define a *leaf* to be a vertex of degree 1 and a *stem* to be a vertex which is adjacent to at least one leaf. For a stem s, we will let $\mathfrak{L}(s)$ be the set of leaves that are adjacent to s and let $\mathfrak{L}[s] = \{s\} \cup \mathfrak{L}(s)$.

Remark 1.3 *Let* G *be a graph with cut vertex* $v \in V(G)$ *and suppose that* $G[V(G) \setminus \{v\}]$ *consists of components* $C_1, C_2, \ldots, C_k$ *where* $k \geq 2$. *Let* n *be an integer with* $1 \leq n < k$, *let* $\{a, b\} \subseteq \bigcup_{i=1}^{n} V(C_i)$ *and let* x *and* y *be distinct vertices in* $\bigcup_{i=n+1}^{k} V(C_i)$. *Then* $\{a\}$ *confuses* x *and* y *if and only if* $\{b\}$ *confuses* x *and* y *since either condition follows precisely when* x *and* y *are the same distance from* v.

In the mid 1970's, algorithms to compute the metric dimension of a tree were independently found by Slater in [11] and Harary and Melter [5]. In a tree T, any vertex of degree at least 3 is called a *branch point*. Each leaf x is assigned to the branch point $B(x)$ that is closest to it. We call vertices in $\{B(x) | x \text{ is a leaf of } T\}$ *special branch points*. For each special branch point y, we set $B^{-1}(y) = \{x | x \text{ is a leaf of } T \text{ and } B(x) = y\}$. Note that the special branch points were denoted *stems* by Slater in [11] where he proved the following result and produced a linear time algorithm for finding a minimum metric basis of a tree.

Proposition 1.4 *Let T be a tree with $k \geq 3$ leaves and let B be the set of special branch points in T. Then the metric dimension of T is $k - |B|$.*

The key observation in proving 1.4 is that all but one of the paths from a special branch point to one of its assigned leaves must contain an element of the metric basis (else as noted in 1.3, confusion would result).

2 Leaves and minimum detection pairs

We begin this section by observing that if one uses only listening devices and if $r+1$ leaves are adjacent to some stem v, then it may not be possible to determine the location of an intruder in $N(v)$ unless devices are located on at least r of the leaves. Then we have the following:

Lemma 2.1 *If a graph G has a vertex x, with four or more leaves attached to it, and if $P = (W, L)$ is a minimum detection pair, then $x \in W$. If x has exactly three leaves attached to it, then there is a minimum detection pair $D = (U, S)$ with $x \in U$.*

Proof. Suppose that vertex x is adjacent to a set K of $r \geq 3$ leaves and let $P = (W, L)$ be a minimum detection pair for G. Suppose that $x \notin W$.

Then since P is a detection pair, a minimum of $r-1$ devices must be located in K. Choose any leaf $k_0 \in K$ and define the new pair $D = ((W \setminus K) \cup \{x\}, (L \setminus K) \cup \{k_0\}) = (U, S)$. We claim that D is a detection pair with $x \in U$.

Indeed suppose that u and v are distinct vertices in $V(G) \setminus N[U]$. Then since $N[W] \subset N[U]$, u and v are both in $V(G) \setminus N[W]$ and are hence distinguished by L. Now, if on the one hand they are distinguished by $L \setminus K$, then they are distinguished by S. If on the other hand they are not distinguished by $L \setminus K$, then there is a vertex $k \in K$ such that $d(k, u) \neq d(k, v)$. But $d(k, u) = d(k_0, u)$ and $d(k, v) = d(k_0, v)$ and thus u and v are distinguished by S. Hence D is a detection pair.

Now $||D|| = ||P|| - (r-1) + 2$ and hence on the one hand, if $r = 3$, we have $||D|| = ||P||$ and thus D is a minimum detection pair; but on the other hand, if $r \geq 4$ then $||D|| = ||P|| - (r-1) + 2 \leq ||P|| - 3 + 2 < ||P||$, contradicting the hypothesis that P is a minimum detection set. This contradiction shows that $x \in W$ when $r \geq 4$. ■

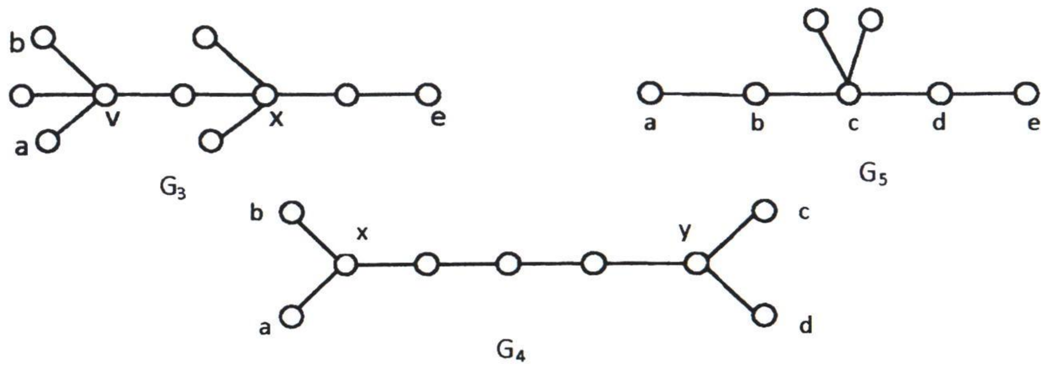

Figure 2 The Graphs G_3, G_4 and G_5

Example 2.1 *In Figure 2, $L\gamma(G_3) = 3$ and the stem v is adjacent to 3 leaves, but $(\{x\}, \{a, b\})$ is a minimum protection pair with no watcher on v. On the other hand, $(\{v, x\}, \{a\})$ is also a minimum protection pair with a watcher on v as promised in Lemma 2.1.*

Turning our attention to stems with 2 leaves attached, we note the following:

Remark 2.2 *Let G be a graph with a detection pair $P = (W, L)$ and suppose that G contains a stem s, with at least 2 adjacent leaves. Then there must be at least one detection device in $\mathfrak{L}[s]$, for otherwise $\mathfrak{L}(s) \cap N[W] = \varnothing$ and L will confuse each pair of vertices in $\mathfrak{L}(s)$.*

Example 2.2 *In Figure 2, $L\gamma(G_4) = 2$ with a sample minimum protection pair $(\varnothing, \{a, c\})$ while any protection pair (W, L) such that $W \neq \varnothing$ is not minimum and hence neither stem can be occupied by a watcher in a minimum protection pair. On the other hand, $L\gamma(G_5) = 2$, and c is a stem adjacent to 2 leaves in G_5, but there is no minimum protection pair (W, L) such that $c \notin W$.*

Lemma 2.3 *Suppose that a graph G has a minimum detection pair $P = (W, L)$ and that $x \in V(G)$ is adjacent to exactly two leaves. If either $W \cap \mathfrak{L}[x] \neq \varnothing$ or $|L \cap \mathfrak{L}[x]| \geq 2$ then there is a minimum detection pair $D = (U, S)$ with $x \in U$. Moreover P can be converted to D in polynomial time.*

Proof. Suppose first that $W \cap \mathfrak{L}[x] \neq \varnothing$. If $x \in W$, the proof is complete. Otherwise set $U = (W \setminus \mathfrak{L}(x)) \cup \{x\}$ and observe both that $|U| \leq |W|$ and that $N[W] \subseteq N[U]$, and hence $D = (U, S)$ is the required minimum detection pair.

Now suppose that $|L \cap \mathfrak{L}[x]| \geq 2$. One of the listeners must be positioned at a vertex $y \in \mathfrak{L}(x)$. Let $A = L \cap \mathfrak{L}[x] \setminus \{y\}$ and set $U = W \cup \{x\}$ and $S = L \setminus A$. Observe both that $|U| + |S| \leq |W| + |L|$ and that $N[W] \subseteq N[U]$. Hence in view of Remark 1.3, $D = (U, S)$ is the required minimum detection pair. ■

Proposition 2.4 *If a tree T has no vertex with more than one leaf as a neighbor, then a minimum detection pair can be constructed using only listeners in polynomial time.*

Proof. If T consists of one vertex, place a listener on that vertex. If T is a path, let x be a leaf of T and place a listener on x. Henceforth we assume that T has at least 3 leaves.

Let $P = (W, L)$ be a detection pair for T and let A be the set of special branch points of T . For each $a \in A$, let $k_a = |B^{-1}(a)|$ and let $\mathfrak{P}_a$ be the set of paths from a to the leaves in $B^{-1}(a)$. By the definition of the special branch points, if $\pi \in \mathfrak{P}_a$ and $x \in V(\pi) \setminus \{a\}$ then $deg_G(x) \leq 2$.

Claim 2.5 *If π and ρ are distinct paths in $\mathfrak{P}_a$, each containing at least 3 vertices, then there must be at least one detection device located in $(V(\pi) \cup V(\rho)) \setminus \{a\}$.*

Proof of Claim. Suppose that π and ρ are distinct paths in $\mathfrak{P}_a$, each containing at least 3 vertices and let $x \in V(\pi)$ and $y \in V(\rho)$ be at distance 2 from a. Then x and y will be confused by any listener in $\{a\} \cup (G \setminus (\pi \cup \rho))$. Hence there must be at least one detection device located in $(V(\pi) \cup V(\rho)) \setminus \{a\}$ thus proving the claim.

It follows that if all the paths in $\mathfrak{P}_a$ contain at least 3 vertices then at least $k_a - 1$ of these paths must have at least one detection device at a vertex other than a. On the other hand, if a is adjacent to a leaf, then since, by hypothesis, a has at most one leaf, it follows that $k_a - 1$ of the paths in $\mathfrak{P}_a$ must contain at least 3 vertices. If each of them has a detection device at a vertex other than a, then the total number of detection devices on the paths in $\mathfrak{P}_a$ must be at least $k_a - 1$. If, on the other hand there is a path π_0 in $\mathfrak{P}_a$, containing at least 3 vertices such that $V(\pi_0) \setminus \{a\}$ contains no detection device, let $x \in V(\pi_0)$ be adjacent to a and let y be the leaf adjacent to a. Then x and y will be confused by any listener in $G \setminus (\pi_0 \cup \{y\})$ and hence there must be a detection device located on either vertex a or y. Also, by Claim 2.5, each path in $\mathfrak{P}_a \setminus \{\pi_0\}$ which contains at least 3 vertices must host a detection device, none of which is located on the vertex a. Hence in this case we also conclude that there are at least $k_a - 1$ detection devices located on the paths in $\mathfrak{P}_a$.

Thus the total number of devices required for the tree is at least the number of leaves minus the number of special branch points. By Proposition 1.4, this can be achieved by using only listening devices. In particular, for each special branch point a, place a listener on all but one of the leaves in $B^{-1}(a)$. ■

3 The complexity of finding a minimum detection pair

When G is a tree, both finding a minimum dominating set and finding a smallest metric basis can be done in linear time (see [2] for finding a minimum dominating set and [11] for a metric basis). It would seem reasonable to expect, especially in view of Proposition 2.4, that it would be equally easy to find a minimum detection pair in a given tree. Somewhat surprisingly, this is not the case. First we give a formal statement of the problem.

DETECTION PAIR
Input: A finite simple loopless graph G and a positive integer j.
Question: Is there a detection pair D for G such that $||D|| = j$?

Note that given a pair of subsets of $V(G)$ it can be determined in polynomial time both whether it is a detection pair and whether its size is j. Hence **DETECTION PAIR** is in NP.

We will show that **DETECTION PAIR** is in NP-complete even when the input graph is restricted to trees whose stems are adjacent to two or fewer leaves.

Consider the NP-complete problem known as *the set packing problem*, one of Karp's 21 problems [8]. It can be stated as follows (see [4], page 221):

SP3
Input: A finite collection S of subsets of $\{1, 2, \ldots, m\}$ and a positive integer k.
Question: Is there a subcollection R of S of mutually disjoint sets such that $|R| = k$?

In order to reduce **SP3** to **DETECTION PAIR** it will be useful to employ graphs from the following family: we say that a tree is an *n-vine with base c_0 and end c_{3n-1}* provided that it consists of the path $c_0c_1 \ldots c_{3n-1}$

together with two leaves adjacent to each vertex c_{3j} where $0 \le j \le n-1$ (see Figure 3). Observe that an n-vine has $5n$ vertices.

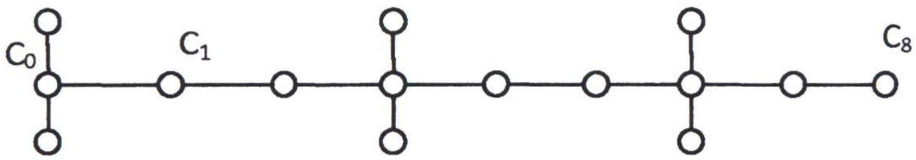

Figure 3 A 3-Vine

Lemma 3.1 *Let C be an n-vine consisting of a path $c_0c_1 \ldots c_{3n-1}$ together with two leaves adjacent to each vertex c_{3j} where $0 \le j \le n-1$. Let H be the graph generated by the disjoint union of some graph G and C together with the edge xc_0 where $x \in V(G)$. If $P = (W, L)$ is a minimum detection pair for H, then there is a minimum detection pair $D = (U, S)$ for H with $\{c_{3j} | 0 \le j \le n-1\} \subseteq U$. Moreover P can be converted to D in polynomial time.*

Proof. Suppose that $P = (W, L)$ is a minimum detection pair for H. By Remark 2.2, there is a detection device in each $\mathfrak{L}[c_{3j}]$ for $0 \le j \le n-1$. Furthermore by Lemma 2.3, we may assume for each j, $0 \le j \le n-1$, there is either a watcher located at c_{3j} or the only device located in $\mathfrak{L}[c_{3j}]$ is a single listener. If the second option fails to be true for any $0 \le j \le n-1$, then we are done.

Otherwise, let k be the largest integer such that $\mathfrak{L}[c_{3k}]$ contains exactly one listener and no watcher. This listener cannot occupy the vertex c_{3k}, or else the vertices in $\mathfrak{L}(c_{3k})$ would be confused. Set $\mathfrak{L}(c_{3k}) = \{v, r\}$ and, without loss of generality, assume that the listener occupies v. Now observe that the listener confuses r and c_{3k+1} and hence there is either a watcher in $\{c_{3k+1}, c_{3k+2}\}$, or a listener in the set $N[\{c_j | 3k+2 \le j \le 3n-1\}]$.

Thus we observe that in C, the number of devices $|V(C) \cap W| + |V(C) \cap L| \ge n+1$.

Hence, in view of Remark 1.3, we conclude that if $U' = (W \setminus V(C)) \cup$

$\{c_{3j} | k \leq j \leq n-1\}$ and $S' = (L \setminus V(C)) \cup \{c_{3n-1}\}$ then $D' = (U', S')$ is a minimum detection pair. ■

Theorem 3.2 *The problem* **DETECTION PAIR***for trees is NP-complete.*

Proof. We proceed by polynomially reducing **SP3** to **DETECTION PAIR**. Let $S = \{S_1, S_2, \ldots, S_t\}$ be a finite collection of nonempty subsets of $M = \{1, 2, \ldots, m\}$ and $k > 1$ be an integer. For $0 \leq i \leq t$ let P_i be the path $v_{i,0}v_{i,1} \ldots v_{i,m+1}$. Now form a tree T as follows.

First, set $J = \{(i,p) | 1 \leq i \leq t$ and $p \in M \cup \{m+1\} \setminus S_i\}$. Now fix i, $1 \leq i \leq t$ and for each $(i,p) \in J$ let $C_{i,p}$ be a $(1 + i\lceil \frac{m}{3} \rceil)$-vine with base labeled $u_{i,p}$ and end labeled $e_{i,p}$. Next, join each such $C_{i,p}$ to P_i via the new edge $v_{i,p}u_{i,p}$ to obtain T_i. Finally, let T be the graph induced by a $(1 + \lceil \frac{m}{3} \rceil)$-vine with base x, together with both $(\bigcup\{T_1, T_2, \ldots, T_t\}) \cup \{P_0\}$ and with the edges $xv_{i,0}$ for $0 \leq i \leq t$. (See Example 3.1)

Note that since an n-vine has $5n$ vertices we have

$$\begin{aligned} |V(T)| &\leq 5(1 + \lceil \frac{m}{3} \rceil) + (t+1)(m+2) \qquad (1) \\ &+ \sum_{i=1}^{t}((m+1)5(1 + i\lceil \frac{m}{3} \rceil)) \\ &= 5\lceil \frac{m}{3} \rceil (1 + \frac{(m+1)t(t+1)}{2}) + 6mt + 7t + m + 7. \end{aligned}$$

Set $j = 1 + \lceil \frac{m}{3} \rceil + \sum_{(i,p)\in J}(1 + i\lceil \frac{m}{3} \rceil) + 1 + t - k$. We complete the proof by establishing the following Claim.

Claim 3.3 *The tree T has a detection pair D with $||D|| = j$ if and only if there is a set $K \subseteq M$, $|K| = k$ such that the sets in $\{S_i | i \in K\}$ are pairwise disjoint.*

Suppose that $D = (W, L)$ is a detection pair of T with $||D|| = j$. By Lemma 3.1, without loss of generality, we may assume that the set W' of the stems in T that are adjacent to exactly 2 leaves is a subset of W. Note that $|W'| = 1 + \lceil \frac{m}{3} \rceil + \sum_{(i,p)\in J}(1 + i\lceil \frac{m}{3} \rceil)$ and thus that at most $t + 1 - k$ devices can occupy vertices in $T \setminus W'$. Since $k > 1$, there is at least one $i_0, 1 \leq i \leq t$ such that T_{i_0} contains no device that is not in W'. We observe that, since S_{i_0} is nonempty, it must contain an element, say a. But then, by construction, the vertex $v_{i_0,a}$ is not adjacent to a vertex in W' and is distance $a + 1$ from x. But the vertex $v_{0,a}$ (on the path P_0) is also distance

$a+1$ from x. It follows that there is at least one device on a vertex in P_0. This implies that of the $t+1-k$ devices that can occupy vertices in $T \setminus W'$, at least one is in P_0, leaving at most $t-k$ devices that can occupy vertices in $\bigcup\{T_1, T_2, \ldots, T_t\}$.

Hence there is a set $K \subseteq M$, $|K| = k$ such that no device occupies a vertex in $T_i \setminus W'$ for $i \in K$. Observe that by construction $\{d(x, v_{i,q})-1 | q \geq 1$ and $(i,q) \notin J\} = S_i$ for each $1 \leq i \leq t$. But since D is a detection pair and since for each $k \in K$, no vertex in $\{v_{i,p} | (i,p) \notin J\}$ can be in $N[W]$, the sets in $\{S_i | i \in K\}$ must be pairwise disjoint.

Conversely, if there is a set $K' \subseteq M$, $|K'| = k$ such that the sets in $\{S_i | i \in K'\}$ are pairwise disjoint, then set $D' = (W', \{v_{0,m+1}\} \cup \{e_{i,m+1} | i \notin K'\})$. Since no two ends e and f of the vines in $\{C_{i,p} | (i,p) \in J\}$ are confused by $v_{0,m+1}$, and since $d(v_{0,m+1}, v_{i,p}) < d(v_{0,m+1}, e)$ for $1 \leq i \leq t$ and $1 \leq p \leq m$, no end is confused with a node on some P_i.

Now, on one hand, if $i \notin K'$, then all the vertices in P_i are distinguished by $e_{i,m+1}$. On the other hand, if $i \in K'$, then all the vertices in $\{v_{i,p} | i \in K'$ and $p \in S_i\} \cup P_0$ are distinguished by $v_{0,m+1}$. The remaining vertices in T are in $N[W']$ and hence D' is a detection pair for T with $||D'|| = j$. ■

Example 3.1 *An illustration of the tree T generated in Theorem 3.2.*

Consider $M = \{1,2,3,4,5,6,7\}$ with subsets $S_1 = \{1,2,4,6,7\}$, $S_2 = \{2,3,4,6\}$, $S_3 = \{1,2,4,5,6,7\}$ and $S_4 = \{1,5,7\}$. In this case, the overall structure of T is pictured in Figure 4, and the structure of T_1 (associated with the set S_1) is pictured in Figure 5.

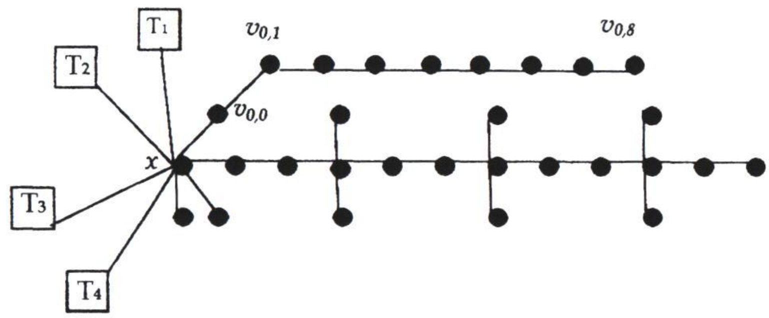

Figure 4 The Graph T

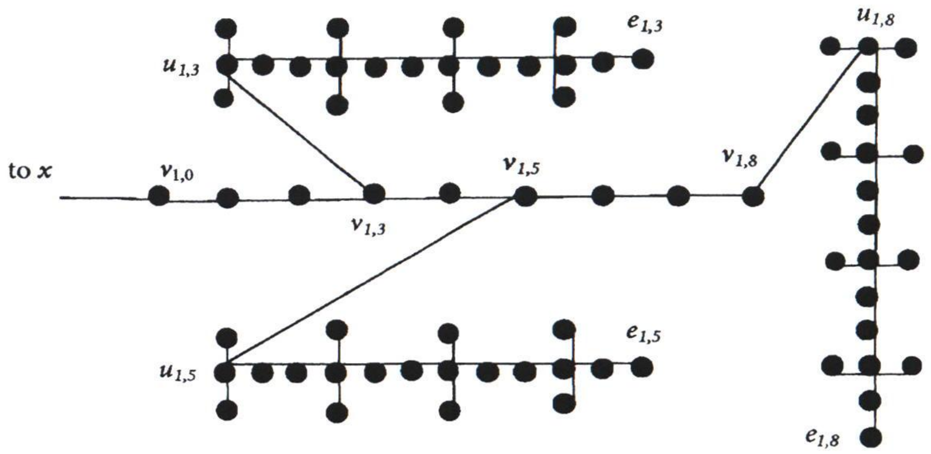

Figure 5 The Graph T_1

4 Conclusion and future directions

To summarize, suppose we are given a tree T. Even though it is easy to determine the minimum number of watchers to guard T and similarly straight forward to find the smallest number of listeners to guard T; if one has both types of devices available it can, in the case of some T, become a difficult problem to minimize the total number of devices needed to do the job. Given this somewhat surprising result it would seem to be of interest to characterize, if possible, a significant collection of non-trivial graphs for which the optimal number can actually be found in polynomial time.

From a different point of view, if each watcher costs $C_W > 0$ and each listener costs $C_L > 0$, then we can look for the minimum cost to guard a graph. We have just seen that if $C_W = C_L$, the problem of minimizing the cost of guarding a tree is in general hard.

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