The homomorphism order of signed graphs

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Abstract

A signed graph (G, σ) is a graph G together with a mapping σ which assigns to each edge of G a sign, either positive or negative. The sign of a closed walk in (G, σ) is the product of the signs of its edges (considering multiplicity). Considering signs of closed walks as one of the key structures of signed graphs, a homomorphism of a signed graph (G, σ) to a signed graph (H, π) is defined to be a mapping that maps vertices to vertices, edges to edges, and that preserves incidences, adjacencies and signs of closed walks. This is a recently defined notion, closely related to sign-preserving homomorphisms of signed graphs (or, equivalently, to homomorphisms of 2-edge-colored graphs), that helps, in particular, to establish a stronger connection between the theories of coloring and homomorphisms of graphs and the minor theory of graphs.

When there exists a homomorphism of (G, σ) to (H, π), one may write $(G, \sigma) \to (H, \pi)$, thus extending the graph homomorphism order to a partial order on the classes of homomorphically equivalent signed graphs. In this work, studying this order, we prove that this order forms a lattice. That is to say, for each pair (G_1, σ_1) and (G_2, σ_2) of signed graphs, representing their respective classes, both their join and meet exist. While proving this result, we also show the existence of categorical products for signed graphs.

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1 Introduction

A signed graph (G, σ) is a graph G together with a mapping σ which assigns to each edge either a positive or a negative sign. The mapping σ can be given by the set E^- of negative edges, in which case we may write (G, E^-) instead. Given a signed graph (G, σ), the *signature* of G normally denotes the set of its negative edges, but it may also refer to the function σ in an equivalent way.

For the purpose of this work, graphs are finite and are allowed to have loops and multi-edges. However, multi-edges of a same sign will be irrelevant to our work and will thus not be considered. A pair of parallel edges of different signs (together with its endpoints) is referred to as a *digon*.

Two key notions in the study of signed graphs are the *sign of a substructure*, which is defined to be the product of the signs of the edges in the substructure, and *switching*, which is to change the sign of each edge in a given subset of edges. In most studies, including this one, these two notions are restricted to two types of substructures which are closely related: sign will be defined for cycles, or more generally for closed walks, and switching can be done on an edge cut, noting that these two structures are rather dual. They are formally defined as follows.

Definition 1 (Sign of cycles and closed walks, switching, switching equivalence)**.** Given a signed graph (G, σ) and a closed walk W of G, the *sign of* W, denoted as $\sigma(W)$, is the product of the signs of the edges of W, considering multiplicities. To *switch an edge cut* $[X, Y]$ of G in (G, σ) is to change the signs of all edges in $[X, Y]$ and to *switch a vertex subset* $X \subseteq V(G)$ is to switch the edge cut $[X, V(G) \setminus X]$. Furthermore, (G, σ') is said to be *switching equivalent* to (G, σ), if (G, σ') is a switching of (G, σ). In that case, we may also say that σ and σ' are switching equivalent.

The following observations directly follow from these definitions.

Observation 2. *Given a signed graph (G, σ), the sign of every closed walk of G is invariant under the switching operation.*

Observation 3. *Given two switching equivalent signed graphs (G, σ) and (G, σ'), the symmetric difference of the two signatures σ and σ' is an edge cut of G.*

Switching equivalence partitions the set of all signatures on a graph G into equivalence classes, called *switch-equivalence classes*. Each member of such a class partitions the set of cycles (or closed walks) into two parts, the negative ones and the positive ones. The following theorem of Zaslavsky shows that this partition uniquely determines the switch-equivalence class to which a signature belongs.

Theorem 4 (Zaslavsky [7]). *Two signed graphs (G, σ_1) and (G, σ_2) are switching equivalent if and only if they have the same set of negative cycles.*

It should be noted here that not every partition of cycles or closed walks of a graph G into two parts would correspond to a set of positive and negative cycles of a signature on G. A classification of such partitions, using the notion of *exclusive 3-walk property*, can be found in [5] and [6].

One should also note that a signed graph (G, σ) could be simply viewed as a 2-edge-colored graph, usually associating positive edges with color blue (or with solid lines) and negative edges with color red (or with dashed lines). With such a view, all results on 2-edge-colored graphs indeed hold for signed graphs. However, the notion of switching, which is based on the basic mathematical relation between + and − (as opposed to colors red and blue which are not related), adds an intriguing level of complexity which gives birth to a number of beautiful theories.

A particular example is the notion of homomorphisms of signed graphs. While it remains strongly related to homomorphisms of 2-edge-colored graphs, it becomes of higher interest in the study of a number of conjectures, specially those in relation with the theory of graph minors, such as the four-color theorem and its possible extensions (see [4] or [5] for more details on this connection).

In this paper, we study the homomorphism order of signed graphs. Because of the strong relation with homomorphisms of 2-edge-colored graphs, we will also speak about the homomorphism order of 2-edge-colored graphs, but, to emphasize the connection, and towards a uniform presentation, we will consider homomorphisms of 2-edge-colored graphs under the umbrella of edge-sign-preserving homomorphisms of signed graphs.

We continue with proper definitions of the two homomorphism relations and their connections in the next section.

2 Homomorphisms of signed graphs

Considering signs of cycles and closed walks as one of the key structures of a signed graph, the following is the natural definition of homomorphisms of signed graphs.

Definition 5 (Homomorphism of signed graphs [5]). Given two signed graphs (G, σ) and (H, π), a homomorphism of (G, σ) to (H, π) is a mapping under which a vertex maps to a vertex, an edge maps to an edge and adjacencies, incidences and signs of closed walks are preserved. When such a homomorphism exists, we write $(G,\sigma) \to (H,\pi)$.

To present a homomorphism of (G,σ) to (H,π) as a function which not only verifies that adjacencies and incidences are preserved but also verifies that signs of closed walks are preserved, we use Theorem 4. More precisely, let ϕ be such a mapping. Let σ' be the signature on G which is induced from (H,π) by ϕ, i.e., $\sigma'(e) = \pi(\phi(e))$. The condition that signs of closed walks are preserved by ϕ together with Theorem 4 implies that σ' is switching equivalent to σ, and thus σ' is obtained by switching a set X (or $V - X$) of vertices of G. Therefore, the mapping ϕ can be viewed as a function composed of three components, $\phi = (\phi_1, \phi_2, \phi_3)$, where $\phi_1 : V(G) \to \{+,-\}$, $\phi_2 : V(G) \to V(H)$, and $\phi_3 : E(G) \to E(H)$ in such a way that it preserves adjacencies, incidences and for any edge uv of (G,σ), we have $\sigma(uv) = \phi_1(u)\phi_1(v)\pi(\phi_3(uv))$. We note that, since switching a subset X of vertices of G is the same as switching $V-X$, we have $\phi_1 = -\phi_1$.

This view of homomorphisms of signed graphs leads to a strong connection with the notion of edge-sign-preserving homomorphisms of signed graphs, formally defined as follows.

Definition 6 (Edge-sign-preserving homomorphism of signed graphs). Given two signed graphs (G,σ) and (H,π), an *edge-sign-preserving homomorphism* of (G,σ) to (H,π) is a mapping ϕ which maps $V(G)$ to $V(H)$ and $E(G)$ to $E(H)$, in such a way that adjacencies, incidences and signs of edges are preserved. When such a homomorphism exists, we write $(G,\sigma) \xrightarrow{sp} (H,\pi)$.

As in this work we only consider signs for edges, we may simply write *sign-preserving homomorphism*, or in short *sp-homomorphism*, in place of edge-sign-preserving homomorphism.

Observe that, in a mapping of (G,σ) to (H,π), when vertices x and y of an edge $e = xy$ of G are mapped to the two ends of a digon in (H,π), or to the same vertex which has both a negative and a positive loop, then one must indicate to which of the two possible edges e is mapped to. In all other cases, specially when H has no multi-edge, the edge mapping is induced by the vertex mapping and one may simply refer to the vertex mapping.

A key difference between the definitions of homomorphisms of signed graphs and of sp-homomorphisms of signed graphs is that in the former an edge might map to an edge of a different sign, as shown in Figure 2. However, there are also strong connections between them, that will be discussed later. For more on that point, we refer to [5].

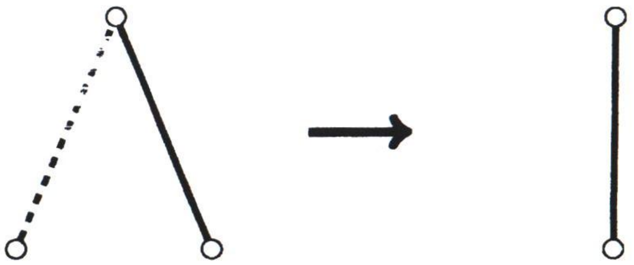

Figure 1: An example of a homomorphism of signed graphs

It is easily observed that both relations $\rightarrow$ and $\xrightarrow{sp}$ are reflexive and associative. However, considered on the class of all signed graphs, none of them is antisymmetric. The natural way to form a partial order from either of them is to put all homomorphically equivalent signed graphs into a same class, represented by any of its members (two signed graphs (G, σ) and (H, π) are homomorphically equivalent if and only if $(G, \sigma) \rightarrow (H, \pi)$ and $(H, \pi) \rightarrow (G, \sigma)$). Such a class will be called a *hom-equivalence class*. This natural partial order obtained from homomorphisms or sp-homomorphisms of signed graphs is called the *homomorphism order* or *sp-homomorphism order* of signed graphs, respectively. The main focus of this paper is to study the homomorphism order of signed graphs.

It is a well known fact that the homomorphism order of graphs is a lattice, that is to say that any pair of graphs admits a join and a meet. The *join* of two given graphs is simply their disjoint union, and their *meet* is given by the graph product defined below. This product is known under various names such as categorical product, direct product, tensor product, Hedetniemi product, etc. We refer to [2] for more details.

Definition 7 (Direct product of graphs). Given two graphs G and H, the *direct product of* G *and* H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ where a vertex (x, u) is adjacent to a vertex (y, v) if and only if x is adjacent to y in G and u is adjacent to v in H.

This product has been naturally extended to edge-colored graphs and to directed graphs, implying the fact that the sp-homomorphism order is a lattice.

Definition 8 (Sp-direct product of signed graphs). Given two signed graphs (G, σ) and (H, π), the *sp-direct product of* (G, σ) *and* (H, π), denoted by $(G, \sigma) \overset{sp}{\times} (H, \pi)$, is the signed graph with vertex set $V(G) \times V(H)$, where a vertex (x, u) is adjacent to a vertex (y, v) with a positive edge (respectively, with a negative edge) if and only if x is adjacent to y in (G, σ) with a positive edge (respectively, with a negative edge), and u is adjacent to v

in (H, π) with a positive edge (respectively, with a negative edge).

We will show in the next section that the homomorphism order of signed graphs is also a lattice. The direct product of two signed graphs will be defined in two equivalent ways. One is constructive and easy to visualize while the other provides more intuition on why such a construction works. The later has a strong connection with sign-preserving homomorphisms, which is described below. We refer to [5] and references therein for more on this connection.

One such important connection we would like to mention is based on the notion of double switching graph, which was introduced in [1] within a larger context, defined as follows.

Definition 9 (Double switching graph of a signed graph). Given a signed graph (G, σ), the *double switching graph of* (G, σ), denoted by $DSG(G, \sigma)$, is the signed graph whose set of vertices is made of two disjoint copies V^+ and V^- of the vertices of G, where each copy induces a copy of (G, σ) and, furthermore, where two vertices $x^+ \in V^+$ and $y^- \in V^-$ are adjacent with a positive edge (respectively, with a negative edge) if and only if the vertices x and y are adjacent in (G, σ) with a negative edge (respectively, with a positive edge).

Observe that if (G, σ') is obtained from (G, σ) by switching a set X of vertices, then the corresponding sets $X^- \subset V^-$ and $(V \setminus X)^+ \subset V^+$ induce a copy of (G, σ'), with their complement inducing another copy of the same signed graph. This explains why this construction is called the double switching graph.

The following result is a special case of a general theorem given in [1].

Theorem 10. *Given two signed graphs (G, σ) and (H, π), $(G, \sigma) \to (H, \pi)$ if and only if $(G, \sigma) \xrightarrow{sp} DSG(H, \pi)$.*

One more important notion that we will use in the next section is the notion of the *category of signed graphs.* This is the category where objects are switch-equivalence classes of signed graphs and where homomorphisms play the role of morphisms from one object to another. But instead of dealing with switch-equivalence classes of signed graphs, we will consider any of its elements, taken as a representative of the whole class.

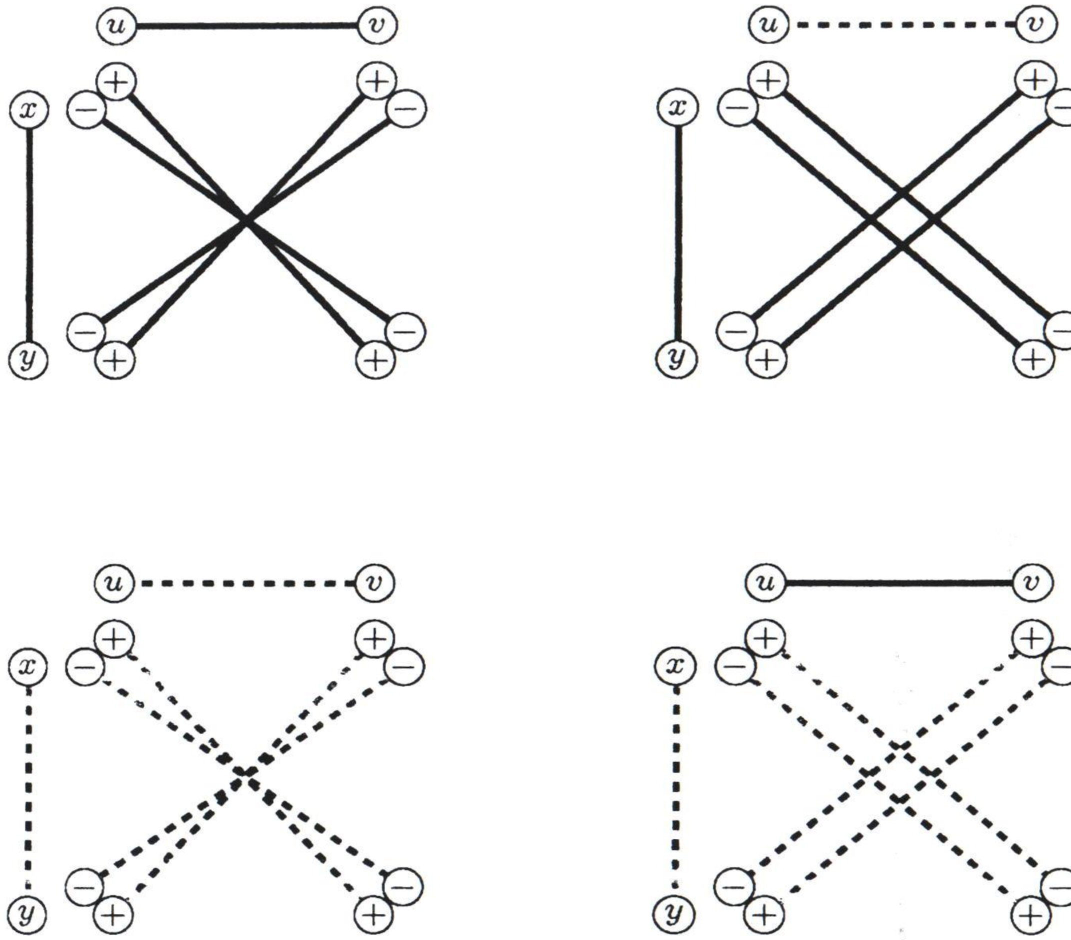

Figure 2: Possible products of two edges

3 The homomorphism order of signed graphs

Our goal in this section is to prove that the homomorphism order of signed graphs is a lattice. Observe that it is enough to show that for any pair of signed graphs (G, σ) and (H, π) (representing their respective hom-equivalence classes), both their join and meet exists. The join and meet are also given by two signed graphs, which are representative of their respective hom-equivalence classes. We will now define the direct sum and direct product for signed graphs and later on justify their names.

Definition 11 (Direct sum of two signed graphs)**.** Given two signed graphs (G, σ) and (H, π), the *direct sum of* (G, σ) *and* (H, π), denoted by $(G, \sigma) + (H, \pi)$, is the disjoint union of (G, σ) and (H, π).

Definition 12 (Direct product of two signed graphs)**.** Given two signed graphs (G, σ) and (H, π), the *direct product of* (G, σ) *and* (H, π), denoted by $(G,\sigma) \times (H,\pi)$, is the signed graph (F,ς) defined as follows.

1. $V(F) = \{+,-\} \times V(G) \times V(H)$.
2. To have an edge between (i,x,u) and (j,y,v), first of all we must have an edge xy in G and an edge uv in H. In such a case, the corresponding eight vertices (respectively, four vertices if either $x = y$ and $u = v$, or two vertices if both $x = y$ and $u = v$) will induce a matching of order four (respectively, two, one), all of its edges having the same sign as $\sigma(xy)$. The choice of these edges then depends on whether uv has the same sign as xy or not, and is described as follows (see Figure 2).
   (a) If $\sigma(xy) = \pi(uv)$, then (i,x,u) is adjacent to (i,y,v) and (i,x,v) is adjacent to (i,y,u) for every $i \in \{+,-\}$ (left side of Figure 2).
   (b) If $\sigma(xy) = -\pi(uv)$, then (i,x,u) is adjacent to $(-i,y,v)$ and (i,x,v) is adjacent to $(-i,y,u)$ for every $i \in \{+,-\}$ (right side of Figure 2).

We will now show that $(G,\sigma) + (H,\pi)$ and $(G,\sigma) \times (H,\pi)$ are the join and meet, respectively, of (G,σ) and (H,π) in the homomorphism order. Before going to that, let us review the definition of join and meet for the homomorphism order.

Definition 13 (Join and meet of two signed graphs). Let (G,σ) and (H,π) be two signed graphs.

The *join* of (G,σ) and (H,π) is a signed graph (J,μ) satisfying $(G,\sigma) \to (J,\mu)$ and $(H,\pi) \to (J,\mu)$ such that $(G,\sigma),(H,\pi) \to (C,\eta)$ implies $(J,\mu) \to (C,\eta)$.

The *meet* of (G,σ) and (H,π) is a signed graph (F,ς) satisfying $(F,\varsigma) \to (G,\sigma)$ and $(F,\varsigma) \to (H,\pi)$ such that $(C,\eta) \to (G,\sigma)$ and $(C,\eta) \to (H,\pi)$ implies $(C,\eta) \to (F,\varsigma)$.

Usually, for (undirected, directed) graphs, the standard way of showing that join and meet exist is to prove that the direct sum and the direct product are the categorical co-products and products, respectively. For signed graphs, we will do something similar. Let us first provide the definitions of co-product and product in the category of signed graphs.

Definition 14 (Categorical co-product of two signed graphs). Given two signed graphs (G,σ) and (H,π), their *categorical co-product* is a signed graph (J,μ) together with two (inclusion) homomorphisms $i_G : (G,\sigma) \to (J,\mu)$ and $i_H : (H,\pi) \to (J,\mu)$ such that, for any signed graph (C,η), if there exist homomorphisms $f_G : (G, \sigma) \to (C, \eta)$ and $f_H : (H, \pi) \to (C, \eta)$, then there exists a unique homomorphism $f : (J, \mu) \to (C, \eta)$ satisfying $f_G = i_g \circ f$ and $f_H = i_H \circ f$.

Definition 15 (Categorical product of two signed graphs)**.** Given two signed graphs (G, σ) and (H, π), their *categorical product* is a signed graph (F, η) together with two (projection) homomorphisms $p_G : (F, \varsigma) \to (G, \sigma)$ and $i_H : (F, \varsigma) \to (H, \pi)$ such that, for any signed graph (C, η), if there exist homomorphisms $f_G : (C, \eta) \to (G, \sigma)$ and $f_H : (C, \eta) \to (H, \pi)$, then there exists a unique homomorphism $f : (C, \eta) \to (F, \varsigma)$ satisfying $p_G = f \circ f_G$ and $p_H = f \circ f_H$.

Thus from the definitions of join and categorical co-product it is clear that, if the categorical co-product of two signed graphs exists, then the join of their corresponding hom-equivalence classes also exists and is the same as the hom-equivalence class of their categorical co-product. Similarly, if the categorical product of two signed graph exists, then the meet of their corresponding hom-equivalence classes also exists and is the same as the hom-equivalence class of their categorical product.

Thus let us first show that $(G, \sigma) + (H, \pi)$, together with the two natural inclusion homomorphisms, is actually the co-product of (G, σ) and (H, π).

Theorem 16. *The signed graph $(G, \sigma) + (H, \pi)$, along with the two natural inclusion homomorphisms, is the co-product of (G, σ) and (H, π).*

Proof. First let us explicitly describe the natural inclusion maps. The homomorphisms $i_G : (G, \sigma) \to (G, \sigma) + (H, \pi)$ and $i_H : (H, \varsigma) \to (G, \sigma) + (H, \pi)$ do not switch any vertex and map a vertex or an edge to itself. Let now (C, μ) be a signed graph admitting homomorphisms $f_G : (G, \sigma) \to (C, \eta)$ and $f_H : (H, \pi) \to (C, \eta)$. Consider then the homomorphism $f : (G, \sigma) + (H, \pi) \to (C, \eta)$, given by $f(x) = f_G(x)$ for $x \in V(G) \cup E(G)$ and $f(y) = f_H(y)$ for $y \in V(H) \cup E(H)$. From the definition of f it follows that $f_G = i_g \circ f$ and $f_H = i_H \circ f$. Moreover, the uniqueness of f is obvious. □

Next we are going to show that $(G, \sigma) \times (H, \pi)$, along with the two natural projections, is the categorical product of (G, σ) and (H, π). We will explicitly describe the projections inside the proof.

Theorem 17. *The signed graph $(G, \sigma) \times (H, \pi)$, along with the two natural projections, is the categorical product of (G, σ) and (H, π).*

Proof. We must first present the natural projections p_G and p_H of $(G, \sigma) \times (H, \pi)$ onto (G, σ) and onto (H, π), respectively. The projection p_G maps each vertex (i, x, v) to the vertex x of (G, σ) without switching. It is straightforward to check that p_G is a homomorphism of $(G, \sigma) \times (H, \pi)$ to (G, σ). The projection p_H first switches all vertices of the form $(-, x, v)$, and then maps each vertex (i, x, v) to the vertex v of (H, π). That means, due to the switching, that an edge between two vertices of the form (i, x, u) and (j, y, v) will change sign if and only if $i \neq j$. Thus, from the definition of $(G, \sigma) \times (H, \pi)$, it follows that p_H is a homomorphism of $(G, \sigma) \times (H, \pi)$ to (H, π).

We may now consider a signed graph (C, η) and assume that f_G and f_H are homomorphisms of (C, η) to (G, σ) and of (C, η) to (H, π), respectively. We would like to define a natural mapping f of (C, η) to the product $(G, \sigma) \times (H, \pi)$, using f_G and f_H. Recall that a homomorphism f of signed graphs consists of three components, f_1, f_2 and f_3, the first one deciding for each vertex whether it must be switched or not, the second one being a vertex mapping and the third one an edge mapping.

To define the mapping f, we choose f_1 to be identical to the first component of f_G, that is to say, f_1 switches a vertex z of (C, η) if and only if f_G switches z. For the component f_2, if f_H switches a vertex z of (C, η), then f_2 maps z to $(-, f_G(z), f_H(z))$, otherwise f_2 maps z to $(+, f_G(z), f_H(z))$.

Finally, we must determine the component f_3, that is the edge mapping. Let e be an edge of (C, η) with endpoints z and z'. Let i be the sign of e after applying the possible switches given by f_G at vertices z and z'. Observe that there must be an edge with sign i between $f(z)$ and $f(z')$. We then set $f_3(e)$ to be this edge. It is then straightforward to check that f is a homomorphism of (C, η) to $(G, \sigma) \times (H, \pi)$ and, furthermore, that $p_G \circ f = f_G$ and $p_H \circ f = f_H$.

The uniqueness of f is obvious. □

Note that the direct product of two signed graphs (G, σ) and (H, π) has order $2 \times |V(G)| \times |V(H)|$. We will understand the reason behind this unusual form of the product at the end of this section.

For now, we can say that both the categorical co-product and the product of two signed graphs exist. Therefore, the homomorphism order of signed graphs is a lattice.

Theorem 18. *The homomorphism order of signed graphs is a lattice.*

Proof. Follows directly from Theorems 16 and 17. □

In the following we are going to describe yet another way of presenting the categorical product. This alternative construction has relation with the sp-categorical product of graphs. To describe this construction, we must introduce a few more concepts.

Given a signed graph (G, σ) and a vertex v of G, we denote by $N^+(v)$ (respectively, by $N^-(v)$) the set of vertices of G that are adjacent to v with a positive edge (respectively, with a negative edge), allowing $N^+(v)$ (respectively, $N^-(v)$) to contain v only if there is a positive loop at v (respectively, a negative loop at v). Two vertices u and v of a signed graph are said to be *twins* if and only if $N^+(v) = N^+(u)$ and $N^-(v) = N^-(u)$. On the other hand, two vertices u and v are *antitwins* if $N^+(v) = N^-(u)$ and $N^-(v) = N^+(u)$, or, equivalently, if after having switched (only) u, v and u are twins in the so-obtained switching equivalent signed graph (G, σ'). Observe that our definition of twins deals with loops as well. Indeed, if u is an antitwin of v in (G, σ) and u has, say, a positive loop, then u must be adjacent to v by a negative edge, and v must also have a positive loop.

Observe also that in $DSG(G, \sigma)$, the two copies in V^+ and V^- of any vertex x are antitwins. The converse is also true in the following sense.

Theorem 19. *A signed graph (H, π) is a double switching graph if and only if its set of vertices can be partitioned into pairs of anti-twins.*

Proof. One direction directly follows from the definition of a double switching graph. For the other direction, simply delete one vertex from each pair of antitwin vertices in (H, π) and let (G, σ) be the resulting signed graph. It is clear that $(H, \pi) = DSG(G, \sigma)$. Note also that different choices of vertices will lead to different switching equivalent copies of (G, σ). □

Let us now recall Definition 8 and use it along with Theorem 19 to redefine the categorical product of two signed graphs (G, σ) and (H, π) with respect to homomorphisms of signed graphs (see Definition 12), as the signed graph (F, η) whose double switching graph is the product $DSG(G, \sigma) \overset{sp}{\times} DSG(H, \pi)$. This is well defined since we have the following result.

Theorem 20. *Given two signed graphs (G, σ) and (H, π), the set of vertices of the direct product $DSG(G, \sigma) \overset{sp}{\times} DSG(H, \pi)$ can be partitioned into pairs of antitwins.*

Proof. Recall that, for each vertex x of G, there are two vertices x^+ and x^- in $DSG(G, \sigma)$. Thus, for each pair of vertices (x, u) with $x \in V(G)$ and $u \in V(H)$, we have four vertices (x^+, u^+), (x^+, u^-), (x^-, u^+) and (x^-, u^-) in $DSG(G, \sigma) \overset{sp}{\times} DSG(H, \pi)$. Furthermore, if any vertex, say (y^+, v^+), is adjacent to one of these four vertices, say (x^+, u^+), with a positive edge, that would mean that xy is a positive edge of (G, σ) and uv is a positive edge of (H, π). Thus, x^-y^+ is a negative edge of $DSG(G, \sigma)$ and u^-v^+ is a negative edge of $DSG(H, \pi)$, which gives that (x^-, u^-) is the antitwin of (x^+, u^+) in $DSG(G, \sigma) \overset{sp}{\times} DSG(H, \pi)$. Similarly, (x^-, u^+) and (x^+, u^-) form a pair of antitwin vertices in $DSG(G, \sigma) \overset{sp}{\times} DSG(H, \pi)$. All other cases are similar. □

In other words, the product (F, ς) of (G, σ) and (H, π) satisfies

$$DSG(F, \varsigma) = DSG(G, \sigma) \overset{sp}{\times} DSG(H, \pi),$$

where (x^+, u^+) is taken as the vertex $(+, x, u)$ of (F, ς), which represents the pair $((x^+, u^+), (x^-, u^-))$ of antitwin vertices in $DSG(G, \sigma) \overset{sp}{\times} DSG(H, \pi)$, and (x^+, u^-) is taken as the vertex $(-, x, u)$, which represents the pair $((x^-, u^+), (x^+, u^-))$. Hence, the given signature ς is a function of σ, π, and their order in the product. However, the class of switching equivalent signatures to which ς belongs is only a function of σ and π, and is independent of the order in which we multiply the two signed graphs. That is, $(G, \sigma) \times (H, \pi)$ and $(H, \pi) \times (G, \sigma)$ are switching equivalent. Furthermore, if (G, σ') is switching equivalent to (G, σ) and (H, π') is switching equivalent to (H, π), then $(G, \sigma') \times (H, \pi')$ is switching equivalent to (F, ς).

4 Remarks

In this work, given two signed graphs (G, σ) and (H, π), we have defined their direct sum $(G, \sigma) + (H, \pi)$ and their direct product $(G, \sigma) \times (H, \pi)$. We then defined the homomorphism order of signed graphs, whose elements are the hom-equivalence classes of signed graphs, each class containing all the signed graphs which are pairwise homomorphically equivalent.

Furthermore, we proved that the direct sum and the direct product of two signed graphs are, respectively, their categorical co-product and product. We also provided two ways of constructing the categorical product, having order $2 \times |V(G)| \times |V(H)|$, one of them easier to visualize and the other intuitively clearer. After that we showed that the homomorphism order of signed graphs is a lattice, by showing that the categorical co-product and the categorical product of (G, σ) and (H, π) represent the classes of their join and of their meet, respectively, in the order.

The *core* of a signed graph (G, σ) is its smallest subgraph which is also a homomorphic image of (G, σ). The fact that this notion is well defined follows from the fact that any two such minimal subgraphs are necessarily switching equivalent, as shown in [4]. For example, the signed graph on the left side of Figure 2 is switching equivalent to the signed graph obtained by switching its negative edge to a positive one, and its core is $(K_2,+)$, the graph consisting of a positive edge (right side of the same figure). This signed graph is also switching equivalent to the signed graph obtained by switching its positive edge to a negative one, and its core is $(K_2,-)$, the graph consisting of a negative edge. Clearly, $(K_2,+)$ and $(K_2,-)$ are switching equivalent signed graphs.

A signed graph (G,σ) whose core is (G,σ) itself is called a *core*. Note that, if we restrict ourselves only to cores, then homomorphisms of signed graphs induce a partial order, equivalent to the homomorphism order. By what we have proved, given any two cores (G,σ) and (H,π), in the above mentioned order, the core of their direct sum and direct product will correspond to their join and meet, respectively. Thus, the partial order induced by homomorphisms of signed graphs on the set of cores is a lattice.

The two examples below will show that (1) the core of the meet of a given pair of signed graphs could be of smaller order than the order of the categorical product of the two given signed graphs, and (2) as a general construction of meet that works for all pairs of signed graphs, $2 \times |V(G)| \times |V(H)|$ is the smallest order one may work with.

Given an integer $k \neq 0$, let C_k be a signed graph on the cycle of length $|k|$ (i.e., $C_{|k|}$) with an assignment of signs to the edges in such a way that the sign of the resulting cycle is the same as the sign of k. That is to say, C_k is a negative cycle if $k < 0$ and a positive cycle, otherwise. In particular, C_{-1} is the negative loop and C_{+1} is the positive loop. The meet of the two cycles C_p and C_q can then be determined as follows.

1. If p and q have the same parity and the same sign, then, assuming $|p| \leq |q|$, $C_p \times C_q$ is homomorphically equivalent to C_q. That is simply because $C_q \to C_p$, and thus signed graphs admitting a homomorphism to both C_p and C_q are those admitting a homomorphism to C_q.

2. If either the parity or the sign of p and q are not the same, then the meet of C_p and C_q is simply $(K_2,+)$, which is homomorphically equivalent to $(K_2,-)$. To see that, let us denote by $g_{ij}(G,\sigma)$, $ij \in \mathbb{Z}_2^2$, the smallest length of a non-trivial closed walk of type ij in (G,σ), that is to say, a non-trivial closed walk whose sign is positive if $i=0$ and negative if $i=1$, and whose length is even if $j=0$ and odd if $j=1$ (see [5] for more details on this notion). If no such closed walk exists in (G,σ), we let $g_{ij}(G,\sigma)=\infty$. Observe now that a signed graph (G,σ) which maps to C_p has $g_{ij}(G,\sigma)=\infty$ for at least two choices of $ij \in \mathbb{Z}_2^2$, where these two choices depend on the parity and

sign of p. As the same holds for q, and since p and q cannot lead to the same choices, a signed graph (G, σ) which maps to both of them must have $g_{ij}(G, \sigma) = \infty$ for at least three choices of $ij \in \mathbb{Z}_2^2$. Therefore, the only possible closed walks in (G, σ) are closed walks of type 00.

Thus, on the one hand $C_{-1} \times C_{+1}$ has $2 \times |V(C_{-1})| \times |V(C_{+1})|$ vertices, and on the other hand, for large values of $|p|$ and $|q|$ where either the parity or the sign of p and q are not the same, the core of the categorical product $C_p \times C_q$, that is their meet, is of order 2.

Finally, it is worth noting here that similar constructions can be given, leading to similar results, for oriented graphs with respect to pushable homomorphisms, as introduced by Klostermeyer and MacGillivray in [3].

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