The size of graphs with restricted rainbow 2-connection number

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Abstract

Let k be a positive integer, and G be a k-connected graph. An edge-coloured path is *rainbow* if all of its edges have distinct colours. The *rainbow k-connection number* of G, denoted by $rc_k(G)$, is the minimum number of colours in an edge-colouring of G such that, any two vertices are connected by k internally vertex-disjoint rainbow paths. The function $rc_k(G)$ was introduced by Chartrand, Johns, McKeon and Zhang in 2009, and has since attracted significant interest. Let $t_k(n,r)$ denote the minimum number of edges in a k-connected graph G on n vertices with $rc_k(G) \leq r$. Let $s_k(n,r)$ denote the maximum number of edges in a k-connected graph G on n vertices with $rc_k(G) \geq r$. The functions $t_1(n,r)$ and $s_1(n,r)$ have previously been studied by various authors. In this paper, we study

the functions $t_2(n, r)$ and $s_2(n, r)$. We determine bounds for $t_2(n, r)$ which imply that $t_2(n, 2) = (1 + o(1))n \log_2 n$, and $t_2(n, r)$ is linear in n for $r \geq 3$. We also provide some remarks about the function $s_2(n, r)$.

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1 Introduction

All graphs in this paper are finite, simple and undirected. For a graph G and vertices $x, y \in V(G)$, let $d_G(x, y)$ denote the distance (i.e., the length of a shortest path) from x to y in G, and let $\text{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}$ denote the diameter of G. Let $\deg_G(x)$ denote the degree of x in G. For $X \subset V(G)$, let $G[X]$ denote the subgraph of G induced by X. For disjoint subsets $X, Y \subset V(G)$, let $E_G(X, Y)$ denote the set of edges of G with one end-vertex in X and the other in Y. For a path P and $x, y \in V(P)$ (possibly $x = y$), we write xPy for the subpath of P with end-vertices x and y. A path $P = x_0 x_1 \cdots x_\ell$, for some $\ell \geq 1$, is called an ℓ-ear of G, or simply an ear, if $V(P) \cap V(G) = \{x_0, x_\ell\}$ and $E(P) \cap E(G) = \emptyset$. For $r \geq 1$, an r-edge-colouring of G, or simply an r-colouring, is a function $c : E(G) \to \{1, \ldots, r\}$. We think of $\{1, \ldots, r\}$ as a set of colours, and occasionally we use the terms *red, blue* and *green* if we are focussing on a small number of colours. We may use the terms *edge-colouring* or *colouring* if we do not wish to emphasize the number of colours. For any other undefined terms, the reader is referred to the books [4, 5, 10].

Let k be a positive integer. A set of internally vertex-disjoint paths connecting two vertices in a graph will simply be called *disjoint*. By Menger's theorem [20], a graph is k-connected if and only if every two vertices have k disjoint paths connecting them. An edge-coloured path is *rainbow* if all of its edges have distinct colours. Let G be a k-connected graph. An edge-colouring of G, not necessarily proper, is *rainbow k-connected* if every two vertices have k disjoint rainbow paths connecting them. The *rainbow k-connection number* of G, denoted by $rc_k(G)$, is the minimum possible number of colours in a rainbow k-connected colouring of G. Note that $rc_k(G)$ is well-defined if and only if G is k-connected. We write $rc(G)$ for $rc_1(G)$. The parameter $rc_k(G)$ was introduced by Chartrand, Johns, McKeon and Zhang ([8] for $k = 1$ (2008), and [9] for

general k (2009)), and has since attracted significant interest from many researchers. For an informative survey and book on the subject of rainbow connection of graphs, see [17, 18].

We consider the following two problems.

Problem 1. *For integers $r, k \geq 1$, let $t_k(n,r)$ denote the minimum number of edges in a k-connected graph G on n vertices with $rc_k(G) \leq r$. Determine $t_k(n,r)$.*

Problem 2. *For integers $r, k \geq 1$, let $s_k(n,r)$ denote the maximum number of edges in a k-connected graph G on n vertices with $rc_k(G) \geq r$. Determine $s_k(n,r)$.*

We note that Problems 1 and 2 have the following equivalent formulations.

Problem 1′. *Given integers $r, k \geq 1$, compute the maximum integer $g_k(n,r)$ such that, if G is a k-connected graph on n vertices and $|E(G)| \leq g_k(n,r)$, then $rc_k(G) \geq r$.*

Problem 2′. *Given integers $r, k \geq 1$, compute the minimum integer $f_k(n,r)$ such that, if G is a k-connected graph on n vertices and $|E(G)| \geq f_k(n,r)$, then $rc_k(G) \leq r$.*

It is easy to see that $t_k(n,r) = g_k(n, r+1) + 1$ and $s_k(n,r) = f_k(n, r-1) - 1$, whenever the terms are defined. Problems 1 and 2 are "Erdős-Gallai type problems". Loosely speaking, an Erdős-Gallai type problem involves the study of the bounds of a graph parameter (such as the number of edges), among all graphs that satisfy some property. For the case $k = 1$, note that if G is a connected graph on n vertices, then we have $|E(G)| \geq n - 1$, with equality if and only if G is a tree. Also, we have $1 \leq \text{diam}(G) \leq rc(G) \leq n - 1$, with $rc(G) = n - 1$ if and only if G is a tree. It is easy to see that

$$\binom{n}{2} = t_1(n, 1) \geq t_1(n, 2) \geq \cdots \geq t_1(n, n-1)$$
$$= t_1(n, n) = \cdots = n - 1, \tag{1}$$
$$\binom{n}{2} = s_1(n, 1) \geq s_1(n, 2) \geq \cdots \geq s_1(n, n-1) = n - 1, \tag{2}$$

and thus it suffices to consider $1 \leq r \leq n - 1$ in the above problems. Problem 1 for $k = 1$ was first considered by Schiermeyer [21], when he determined the values of $t_1(n, r)$ for $\frac{n}{2} \leq r \leq n - 1$, and the asymptotic answer for $r = 2$. Subsequently, Bode and Harborth [3], and Li et al. [14]

made some improvements, and Lo [19] determined the values of $t_1(n, r)$ for $3 \leq r < \frac{n}{2}$. Thus Problem 1 is essentially completely solved for $k = 1$.

Theorem 3. [3, 19, 21] *Let* $1 \leq r \leq n - 1$ *and* $n \geq 3$. *Then*

$$t_1(n, r) = \begin{cases} \binom{n}{2} & \text{if } r = 1, \\ (1 + o(1))n \log_2 n & \text{if } r = 2, \\ \left\lceil \frac{r(n-2)}{r-1} \right\rceil & \text{if } r \geq 3. \end{cases} \tag{3}$$

For $r \geq 2$, the constructions of a graph G on n vertices yielding the upper bounds in (3) are as follows. For $r = 2$, we take G to be the complete bipartite graph $K_{a,n-a}$ such that $2^{a-1} + a \leq n \leq 2^a + a$, so that $|E(G)| = (1 + o(1))n \log_2 n$. For $3 \leq r \leq n - 1$, we take $\lfloor \frac{n-3}{r-1} \rfloor$ cycles on r vertices having one vertex in common, and connect the remaining $n - 1 - \lfloor \frac{n-3}{r-1} \rfloor (r - 1) \leq r$ vertices to the common vertex of the cycles. Then $|E(G)| = \lceil \frac{r(n-2)}{r-1} \rceil$. For these graphs, it was shown in [8] and [3] that $rc(G) = r$.

Problem 2′ for $k = 1$ was first considered by Kemnitz and Schiermeyer [13], and their results were subsequently improved by Kemnitz et al. [12], and Li et al. [15]. When stated in terms of $s_1(n, r)$, we have the following result.

Theorem 4. [12, 13, 15] *Let* $2 \leq r \leq n-1$. *Then* $s_1(n, r) \geq \binom{n-r+2}{2} + r - 3$. *Equality holds for* $r \in \{2, 3, 4, 5, n - 5, n - 4, n - 3, n - 2, n - 1\}$.

The graph G which yields the lower bound in Theorem 4, as given in [13], is as follows. Take a complete graph K_{n-r+2} and delete one edge xy. Then take a path on $r - 1$ vertices and identify one end-vertex with x. Let G be the resulting graph. We have $|E(G)| = \binom{n-r+2}{2} + r - 3$, and $rc(G) \geq \text{diam}(G) = r$.

Finally, note that for $1 \leq k < \ell < n$, we have

$$t_k(n, r) \leq t_\ell(n, r), \tag{4}$$

whenever the terms are defined.

We remark that analogous Erdős-Gallai type problems have also been considered for other parameters similar to the rainbow connection number, such as the monochromatic connection number [6], and rainbow disconnection number [1, 2, 7], among others.

In this paper, we will focus on Problems 1 and 2 for the case $k = 2$. This paper is organised as follows. In Section 2, we gather some auxiliary

results about k-connected graphs and the rainbow k-connection number. In Section 3, we study the function $t_2(n, r)$. Our main result will be Theorem 11, from which we can conclude that $t_2(n, 2) = (1 + o(1)) n \log_2 n$, and $t_2(n, r)$ is linear in n for $r \geq 3$. In Section 4, we provide some remarks about the function $s_2(n, r)$.

2   Tools and some related results

In this section, we gather some results which will be useful for Sections 3 and 4. Throughout this section, let $n > k \geq 2$.

We first note that any k-connected graph G on n vertices has at least $\lceil \frac{kn}{2} \rceil$ edges, since G has minimum degree at least k. Harary [11] gave examples of k-connected graphs which show that the value $\lceil \frac{kn}{2} \rceil$ is best possible for the minimum number of edges.

Theorem 5. [11] *Let $n > k \geq 2$, and G be a k-connected graph on n vertices. Then $|E(G)| \geq \lceil \frac{kn}{2} \rceil$. Moreover, there exists a k-connected graph $H_{n,k}$ on n vertices with $|E(H_{n,k})| = \lceil \frac{kn}{2} \rceil$.*

Since we will mainly be studying 2-connected graphs, we state the well-known *ear decomposition theorem*, which gives a characterisation for the structure of 2-connected graphs. See for example [10] (Ch. 3, Proposition 3.1.1).

Theorem 6 (Ear decomposition). *A graph G is 2-connected if and only if G can be constructed from a cycle by successively adding ears to graphs that are already constructed.*

Remark. It is easy to see from Theorem 6 that, if G is a 2-connected graph on n vertices and $q \geq 0$, then G is constructed by successively adding q ears, starting with an initial cycle, if and only if $|E(G)| = n + q$.

We also have the following result from the book of Bollobás [4] (Ch. IV, Theorem 2.8).

Theorem 7. [4] *Let $m \geq 2$, and G be a 2-connected graph on n vertices with diameter at most $2m$. Then*

$$|E(G)| \geq \left(1 + \frac{1}{2m - 1}\right) n - 4(4m - 2)^{m-1}.$$

Now, we consider some results about the rainbow k-connection number. Note that for any k-connected graph G, we have $rc_k(G) \geq 2$. When n is

sufficiently large, Chartrand et al. [9] showed that equality holds for the complete graph K_n.

Theorem 8. [9]

(a) $rc_2(K_n) = 2$ for $n \geq 4$.

(b) $rc_k(K_n) = 2$ for $k \geq 2$ and $n \geq (k+1)^2$.

To see (a), we have a rainbow 2-connected 2-colouring of K_n as follows. For $n \geq 5$, we colour a Hamilton cycle of K_n with red, and all remaining edges with blue. For K_4, we take the 2-colouring where the colours red and blue both induce a path of length three.

Next, we have the following result of Li and Liu [16] about the rainbow 2-connection number of 2-connected graphs. This result is analogous to the fact that $rc(G) \leq n - 1$ if G is a connected graph on n vertices.

Theorem 9. [16] *Let G be a 2-connected graph on $n \geq 3$ vertices. Then $rc_2(G) \leq n$, with equality if and only if G is the cycle on n vertices.*

Finally, we prove the following useful lemma.

Lemma 10. *Let G be a 2-connected graph, and P be a path in G of length at least three such that, all internal vertices of P have degree two in G. Then for any rainbow 2-connected colouring of G, the edges of P must be rainbow coloured.*

Proof. Suppose that there exists a rainbow 2-connected colouring of G, and a path P in G as described in the lemma which is not rainbow coloured. We have $P = x_0 x_1 \cdots x_\ell$ for some $\ell \geq 3$, where $\deg_G(x_i) = 2$ for all $1 \leq i \leq \ell - 1$. There exist two edges $x_{i-1} x_i$ and $x_{j-1} x_j$ with the same colour, for some $1 \leq i < j \leq \ell$. If $i < j - 1$, let $u = x_i$ and $v = x_{j-1}$. If $i = j - 1$, let $u = x_{i-1}$ and $v = x_j$. Note that since $\ell \geq 3$, at least one of u and v is an internal vertex of P. We see that there do not exist two disjoint rainbow paths connecting u and v, a contradiction. $\square$

3 The function $t_2(n, r)$

Let $n > k \geq 2$, and G be a k-connected graph on n vertices. By Theorem 8(b), we see that for sufficiently large n ($n \geq (k+1)^2$ will do), $t_k(n, 2)$ is well-defined. Together with Theorem 5, we have the following analogue to

(1)
$$\binom{n}{2} \geq t_k(n,2) \geq t_k(n,3) \geq \cdots \geq t_k\left(n, \left\lceil \frac{kn}{2} \right\rceil\right)$$
$$= t_k\left(n, \left\lceil \frac{kn}{2} \right\rceil + 1\right) = \cdots = \left\lceil \frac{kn}{2} \right\rceil.$$

In this section, we study the function $t_2(n, r)$. For $n \geq 3$, the cycle C_n is the unique 2-connected graph on n vertices with the minimum number of edges, which is n. Note that $rc_2(C_n) = n$. Thus for $n \geq 4$, Theorem 8(a) implies

$$\binom{n}{2} \geq t_2(n,2) \geq t_2(n,3) \geq \cdots \geq t_2(n,n) = t_2(n, n+1) = \cdots = n. \quad (5)$$

It suffices to consider $t_2(n, r)$ for $2 \leq r \leq n - 1$. The following theorem is our main result.

Theorem 11.

(a) $t_2(n, 2) = (1 + o(1))n \log_2 n$.

(b)
$$t_2(n, r) \geq \begin{cases} \dfrac{4}{3}n - 24 & \text{if } r \in \{3, 4\} \text{ and } n \geq 18, \\ \dfrac{6}{5}n - \dfrac{1}{5}r(r-1) & \text{if } 5 \leq r \leq n-1 \text{ and } n > \dfrac{1}{6}r(r-1). \end{cases}$$

(c)
$$t_2(n, r) \leq \begin{cases} \dfrac{5}{2}n - 5 & \text{if } r \in \{3, 4\} \text{ and } n \geq r+1, \\ \dfrac{7n - 19}{3} & \text{if } r = 5 \text{ and } n \geq 7, \\ 2n - r + 2 & \text{if } 6 \leq r \leq n-3. \end{cases}$$

(d) $t_2(n, n-2) = n + 2$ for $n \geq 6$, and $t_2(n, n-1) = n + 1$ for $n \geq 4$.

Proof. (a) We have $t_2(n, 2) \geq t_1(n, 2) \geq n(\log_2 n - 4\log_2 \log_2 n - 2)$ for sufficiently large n, where the first inequality follows from (4), and the second inequality was proved by Li et al. [14]. Now we construct a graph G on n vertices, with a 2-colouring c, as follows. Take the complete bipartite graph with classes A and B, where $|A| = a \geq 4$ and $|B| = n - a$, such that $2^{a-2} + a - 1 \leq n \leq 2^{a-1} + a - 1$. Add a complete graph on A. It

is easy to obtain $|E(G)| = (1 + o(1))n \log_2 n$. Now, assign to the vertices of B distinct $(1,2)$-vectors of length a with a positive even number of 2s, such that the $a-1$ vectors $(2,2,1,\ldots,1),(2,1,2,1,\ldots,1),\ldots,(2,1,\ldots,1,2)$ are all present. Note that the assignment is possible, since $|B| = n - a \geq 2^{a-2} - 1 \geq a - 1$, so that the $a-1$ aforementioned vectors can all be assigned; and $|B| = n - a \leq 2^{a-1} - 1$, whence there are a total of $2^{a-1} - 1$ assignable vectors. Let $A = \{u_1, \ldots, u_a\}$. For $u_i \in A$ and $v \in B$, let $c(u_i v) = \vec{v}_i$, where $\vec{v}$ denotes the vector assigned to v, and $\vec{v}_i$ denotes the ith component of $\vec{v}$. Let $c(u_i u_j) = 1$ for all i, j. Then c is a rainbow 2-connected colouring of G. Indeed, for two vertices of G, we may find two disjoint rainbow paths connecting them as follows.

- If $u_i, u_j \in A$, then since the $a - 1$ aforementioned vectors are present, we may choose $v \in B$ such that $\vec{v}_i \neq \vec{v}_j$, and take the paths $u_i u_j$ and $u_i v u_j$.

- If $v, w \in B$, then there exist i, j such that $\vec{v}_i \neq \vec{w}_i$ and $\vec{v}_j \neq \vec{w}_j$, and we may take the paths $v u_i w$ and $v u_j w$.

- If $u_i \in A$ and $v \in B$, then we may choose $u_j \in A$ such that $\vec{v}_j = 2$, and take the paths $u_i v$ and $u_i u_j v$.

Thus $rc_2(G) = 2$, and $t_2(n, 2) \leq (1 + o(1))n \log_2 n$.

(b) Let $r \geq 3$, and G be a 2-connected graph on n vertices with $rc_2(G) \leq r$. Note that since $\operatorname{diam}(G) \leq rc_2(G) \leq 2\lceil \frac{r}{2} \rceil$, setting $m = \lceil \frac{r}{2} \rceil$ in Theorem 7 gives $|E(G)| \geq \left(1 + \frac{1}{2\lceil r/2 \rceil - 1}\right)n - c_r$ for some constant $c_r > 0$. This implies

$$t_2(n, r) \geq \left(1 + \frac{1}{2\lceil r/2 \rceil - 1}\right)n - c_r. \tag{6}$$

For $r \geq 5$, we will see that the constant $1 + \frac{1}{2\lceil r/2 \rceil - 1}$ in (6) can be improved to $\frac{6}{5}$. For $r \in \{3, 4\}$ and $n \geq 18$, setting $m = 2$ in Theorem 7 gives $|E(G)| \geq \frac{4}{3}n - 24$. Therefore, $t_2(n, r) \geq \frac{4}{3}n - 24$.

Now, suppose that $5 \leq r \leq n - 1$ and $n > \frac{1}{6}r(r-1)$. Note that $rc_2(G) \leq n - 1$, so that $G \neq C_n$. We prove that $|E(G)| \geq \frac{6}{5}n - \frac{1}{5}r(r-1)$, which implies $t_2(n, r) \geq \frac{6}{5}n - \frac{1}{5}r(r-1)$. Let $V_2 \subset V(G)$ be the subset defined by $V_2 = \{v \in V(G) : \deg_G(v) = 2\}$. Note that $G[V_2]$ is a linear forest, i.e., $G[V_2]$ is a union of vertex-disjoint paths. For an integer $i \geq 1$, let $\mathcal{P}_i$ be the family of path components in $G[V_2]$ such that $|V(P)| = i$ for every $P \in \mathcal{P}_i$. We have $V_2 = \bigcup_{i \geq 1} \bigcup_{P \in \mathcal{P}_i} V(P)$. By the definition of $\mathcal{P}_i$, together with the assumption that G is 2-connected, the following two properties hold:

(i) For any $P \in \mathcal{P}_i$ and $Q \in \mathcal{P}_j$ (possibly $i = j$), we have $E_G(V(P), V(Q)) = \emptyset$.

(ii) For any $P = x_1 \cdots x_i \in \mathcal{P}_i$, there exist two distinct vertices $x_0, x_{i+1} \in V(G) \setminus V_2$ such that $E_G(V(P), V(G) \setminus V(P)) = \{x_1 x_0, x_i x_{i+1}\}$, with $\deg_G(x_0) \geq 3$ and $\deg_G(x_{i+1}) \geq 3$.

Let $P = x_1 \cdots x_i \in \mathcal{P}_i$ be a path, for some i. In view of (ii), we will often look at the extended path of P containing x_0 and x_{i+1}. Thus, let P^+ be the path $P^+ = x_0 x_1 \cdots x_i x_{i+1}$. Note that the paths of $\{P^+ : P \in \bigcup_{i \geq 1} \mathcal{P}_i\}$ are pairwise edge-disjoint. Now, since $rc_2(G) \leq r$, we may fix a rainbow 2-connected r-colouring on G. By Lemma 10, we have P^+ is a rainbow path for any $P \in \bigcup_{i \geq 2} \mathcal{P}_i$. This implies that $\mathcal{P}_i = \emptyset$ for $i \geq r$.

Claim 12. *We have*

$$\left| V_2 \setminus \bigcup_{P \in \mathcal{P}_1} V(P) \right| = \sum_{i=2}^{r-1} i |\mathcal{P}_i| \leq \binom{r}{2}.$$

Proof. First, suppose that there exist two paths $P \in \mathcal{P}_i$ and $Q \in \mathcal{P}_j$, for some $2 \leq i, j \leq r - 1$, such that P^+ and Q^+ both contain two consecutive edges using the same pair of colours, say red and blue. Let $u \in V(P)$ and $v \in V(Q)$ be the two vertices between the red and blue edges, and note that $\deg_G(u) = \deg_G(v) = 2$. Since $i \geq 2$, we may choose a neighbour w of u such that $\deg_G(w) = 2$. Then w and v are not connected by two disjoint rainbow paths, a contradiction.

Now, $\bigcup_{i=2}^{r-1} \bigcup_{P \in \mathcal{P}_i} P^+$ contains $\sum_{i=2}^{r-1} i |\mathcal{P}_i|$ pairs of edges, with each pair being two consecutive edges of some path P^+. These pairs of edges use distinct pairs of colours, therefore

$$\left| V_2 \setminus \bigcup_{P \in \mathcal{P}_1} V(P) \right| = \sum_{i=2}^{r-1} i |\mathcal{P}_i| \leq \binom{r}{2}.$$

This proves Claim 12. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

If $|\mathcal{P}_1| \geq \frac{3}{5} n - \frac{1}{10} r(r-1) > 0$, then the bipartite subgraph of G induced by $V(G) \setminus V_2$ and $\{V(P) : P \in \mathcal{P}_1\}$ has $2|\mathcal{P}_1| \geq \frac{6}{5} n - \frac{1}{5} r(r-1)$ edges.

Otherwise, if $|\mathcal{P}_1| \leq \frac{3}{5}n - \frac{1}{10}r(r-1)$, then by Claim 12,

$$|E(G)| \geq \frac{1}{2}(2|V_2| + 3(n - |V_2|)) = \frac{3}{2}n - \frac{1}{2}|V_2|$$

$$= \frac{3}{2}n - \frac{1}{2}\sum_{i=2}^{r} i|\mathcal{P}_i| - \frac{1}{2}|\mathcal{P}_1| \geq \frac{3}{2}n - \frac{1}{2}\binom{r}{2} - \frac{3}{10}n + \frac{1}{20}r(r-1)$$

$$= \frac{6}{5}n - \frac{1}{5}r(r-1).$$

(c) We first consider the case $6 \leq r \leq n-3$. We construct a graph $G_{n,r}$ on n vertices with an r-colouring, as follows. First, let F_6 be the graph on six vertices x, y, a, b, c, d, with the 6-colouring as shown in Figure 1(a). Then, let H_m be the graph obtained by taking $m \geq 1$ identically 6-coloured copies of F_6, say $F_6^1, \ldots, F_6^m$, and identifying the vertices x and y. Let a_i, b_i, c_i, d_i be the vertices of F_6^i corresponding to a, b, c, d, for $1 \leq i \leq m$. See Figure 1(b) for the case of H_2. Next, let $H_{m,r}$ be obtained from H_m by adding an $(r-4)$-ear P at x and y. Colour the edges of P with colours $5, 6, \ldots, r$ such that colours 5 and 6 are incident with x and y. See Figure 1(c).
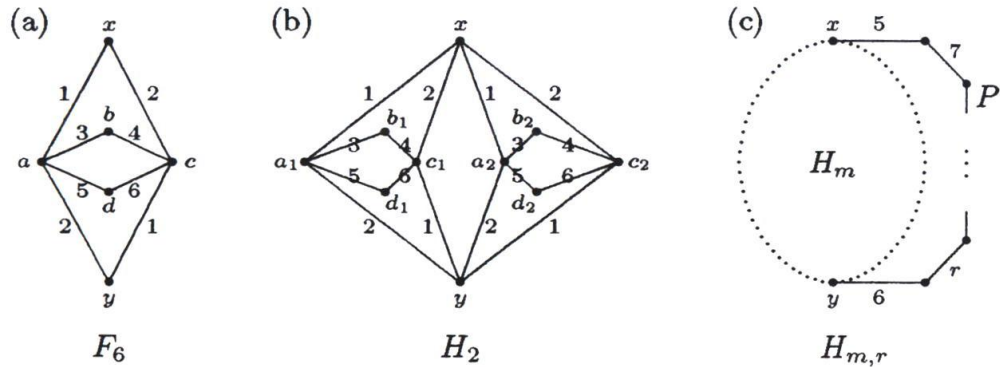

Figure 1. The graphs F_6, H_2 and $H_{m,r}$

We have $|V(H_{m,r})| = 4m + r - 3$. Let m and b be integers such that $n = 4m + r - 3 + b$, where $0 \leq b \leq 3$. Note that $m \geq 1$ since $n \geq r + 3$. Let $G_{n,r}$ be the graph obtained from $H_{m,r}$ by adding 2-ears $Q_1, \ldots, Q_b$ at x and y, with the colouring as shown in Figure 2. Note that $G_{n,r} = H_{m,r}$ if $b = 0$. If $b \geq 1$, let w_j be the middle vertex of Q_j for $1 \leq j \leq b$. We have $G_{n,r}$ is a 2-connected graph on n vertices, and $|E(G_{n,r})| = 8m + r - 4 + 2b = 2n - r + 2$.
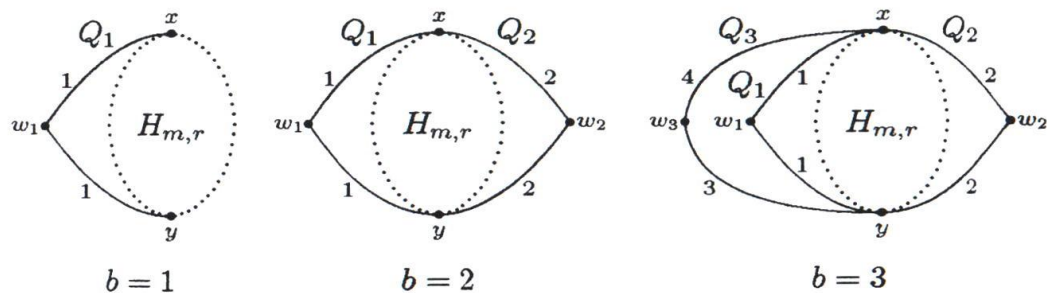
Figure 2. The graph $G_{n,r}$ for $b \in \{1, 2, 3\}$

To complete the third part of (c), it remains to prove the following claim.

Claim 13. *The r-colouring defined on $G_{n,r}$ is rainbow 2-connected.*

Proof. Let $u, v \in V(G_{n,r})$. We show that there are two disjoint rainbow paths connecting u and v. This is easy to check if at least one of u, v is x or y; or if u, v belong to the same copy of F_6; or $u, v \in V(P)$; or $u, v \in \{w_1, w_2, w_3\}$; or $u \in \{w_1, w_2, w_3\}$ and $v \in V(P)$. It remains to check the following cases.

- Let u and v belong to two different copies of F_6. It suffices to consider the cases $(u, v) = (a_1, a_2), (a_1, b_2), (a_1, c_2), (b_1, b_2), (b_1, d_2)$. By consulting Figure 1(b), it is easy to check that for these cases, u and v are connected by two disjoint rainbow paths.

- Let u belong to a copy of F_6, say F_6^1, and $v \in V(P) \setminus \{x, y\}$. It is easy to check that u and v are connected by two disjoint rainbow paths. In the case $u = d_1$, we take the paths ua_1yPv and uc_1xPv.

- Let u belong to a copy of F_6, say F_6^1, and $v \in \{w_1, w_2, w_3\}$. If $(u, v) = (a_1, w_1)$, we take ub_1c_1xv and uyv. The case $(u, v) = (a_1, w_2)$ is similar, and the case $(u, v) = (a_1, w_3)$ is easy. The case $u = c_1$ and $v \in \{w_1, w_2, w_3\}$ is similar to these previous three cases. If $u \in \{b_1, d_1\}$ and $v = w_1$, we take uc_1xv and ua_1yv. If $u \in \{b_1, d_1\}$ and $v \in \{w_2, w_3\}$, we take ua_1xv and uc_1yv.

This proves Claim 13. $\qquad\qquad\square$

Next, suppose that $r = 5$. Let F_5 be the 5-coloured graph as shown in Figure 3(a). As before, construct the graph H_m by taking $m \geq 1$ copies of F_5 and identifying the vertices x and y. Then, let $H_{m,5}$ be the graph

obtained from H_m by adding a 3-ear at x and y, with the colouring as shown in Figure 3(b). We have $|V(H_{m,5})| = 3m+4$. Let m and b be integers such that $n = 3m + 4 + b$, where $0 \leq b \leq 2$. Note that $m \geq 1$ since $n \geq 7$. Let $G_{n,5}$ be the graph obtained from $H_{m,5}$ by adding 2-ears $Q_1, \ldots, Q_b$ at x and y, with both edges of Q_i given colour i for $1 \leq i \leq b$. See Figure 3(c). Note that $G_{n,5} = H_{m,5}$ if $b = 0$. We have $G_{n,5}$ is a 2-connected graph on n vertices, and $|E(G_{n,5})| = 7m + 3 + 2b \leq \frac{7n-19}{3}$. By a similar argument as in Claim 13, the 5-colouring of $G_{n,5}$ is rainbow 2-connected. This proves the second part of (c).
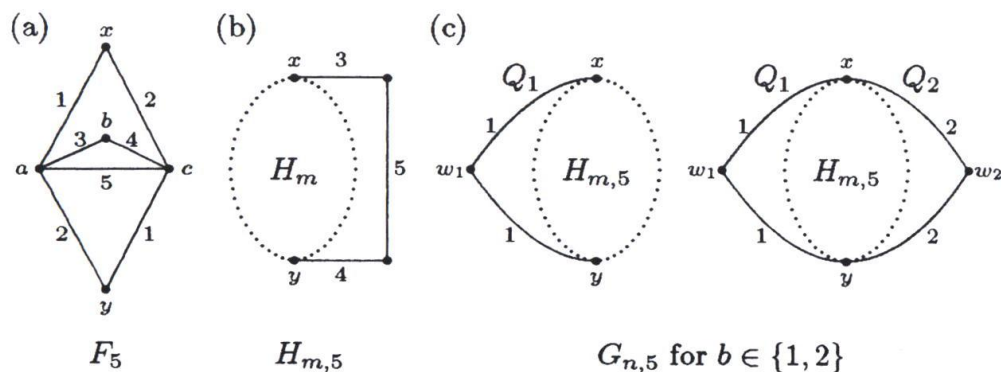

Figure 3. The graphs F_5, $H_{m,5}$ and $G_{n,5}$

Finally, let $r \in \{3, 4\}$. Let F_3 be the 3-coloured graph as shown in Figure 4(a). Let H_m be the graph obtained by taking $m \geq 1$ copies of F_3 and identifying the vertices x and y. We have $|V(H_m)| = 2m + 2$. Let $n = 2m + 2 + b$, where $0 \leq b \leq 1$, and note that $m \geq 1$ since $n \geq r + 1 \geq 4$. If n is even, set $G_{n,3} = H_m$. If n is odd, let $G_{n,3}$ be obtained by adding a 2-ear to H_m at x and y, with both edges of the ear given colour 3. See Figure 4(b). We have $G_{n,3}$ is a 2-connected graph on n vertices, and $|E(G_{n,3})| = 5m + 2b \leq \frac{5}{2}n - 5$. It is easy to check that the 3-colouring of $G_{n,3}$ is rainbow 2-connected. This proves the first part of (c).

(d) We first prove the two upper bounds. For $n \geq 4$, we construct a graph G_1 on n vertices, with $|E(G_1)| = n+1$ and $rc_2(G_1) \leq n-1$, as follows. We take the cycle $C_{n-1} = v_1 v_2 \cdots v_{n-1} v_1$, and connect another vertex x to v_1 and v_2. Now, define the $(n-1)$-colouring c_1 on G_1, where $c_1(v_i v_{i+1}) = i$ for $1 \leq i \leq n - 1$ (with $v_n = v_1$), $c_1(xv_1) = 2$, and $c_1(xv_2) = n - 1$. Then $|E(G_1)| = n + 1$, and it is easy to check that c_1 is a rainbow 2-connected colouring of G_1, so that $rc_2(G_1) \leq n-1$. It follows that $t_2(n, n-1) \leq n+1$.
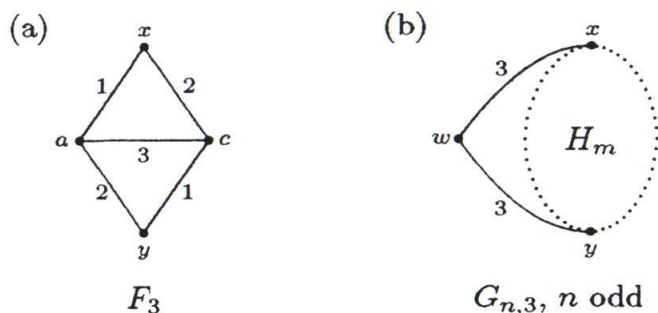
Figure 4. The graphs F_3 and $G_{n,3}$

Next, let $n \geq 6$. We construct a graph G_2 on n vertices, with $|E(G_2)| = n + 2$ and $rc_2(G_2) \leq n - 2$, as follows. We take the cycle $C_{n-2} = v_1 v_2 \cdots v_{n-2} v_1$, and connect a vertex x to v_1 and v_2, and a vertex y to v_2 and v_3. Now, define the $(n-2)$-colouring c_2 on G_2, where $c_2(v_i v_{i+1}) = i$ for $1 \leq i \leq n - 2$ (with $v_{n-1} = v_1$), $c_2(xv_1) = 2$, $c_2(xv_2) = n - 2$, $c_2(yv_2) = 3$, and $c_2(yv_3) = 1$. Then $|E(G_2)| = n + 2$, and it is easy to check that c_2 is a rainbow 2-connected colouring of G_2, so that $rc_2(G_2) \leq n - 2$. It follows that $t_2(n, n - 2) \leq n + 2$.

Now we prove the two lower bounds. First, let $n \geq 4$. Recall that the cycle C_n is the unique 2-connected graph on n vertices with the minimum of edges, which is n. However, $rc_2(C_n) = n$. It follows that $t_2(n, n-1) \geq n+1$.

Next, let $n \geq 6$. By (5) and the above, we have $t_2(n, n - 2) \geq t_2(n, n - 1) \geq n + 1$. Suppose that G is a 2-connected graph on n vertices, and $|E(G)| = n + 1$. We prove that $rc_2(G) \geq n - 1$, and this implies the lower bound $t_2(n, n - 2) \geq n + 2$. By the remark after Theorem 6, G must be a cycle with one ear attached. Thus G is a Θ-graph, i.e., G consists of three disjoint paths connecting two vertices, say x and y. Let Q_1, Q_2, Q_3 be the three paths, on $q_1, q_2 \geq 3$ and $q_3 \geq 2$ vertices. Let $x_i, y_i \in V(Q_i)$ be the neighbours of x and y for $i = 1, 2$, and for $i = 3$ if $q_3 \geq 3$. Assume that there exists a rainbow 2-connected colouring of G with at most $n - 2$ colours. In the two claims and the subsequent argument below, we obtain a contradiction by finding two vertices which are not connected by two disjoint rainbow paths. Call such a pair of vertices *bad*. For convenience, whenever we state an edge $wz \in E(Q_i)$ below, we have $d_{Q_i}(x, w) = d_{Q_i}(x, z) - 1$.

Claim 14. *If there exist two edges $ab \in E(Q_i)$ and $cd \in E(Q_j)$ with the same colour, for some $i \neq j$ with $q_i, q_j \geq 3$, then either $a = x$ and $d = y$, or $b = y$ and $c = x$.*

195

Proof. Suppose first that a, b are internal vertices of Q_i. Then $\{a, c\}$ (resp. $\{b, d\}$) is bad if c (resp. d) is an internal vertex of Q_j. Now suppose that $a = x$. If $d \neq y$, then $\{b, d\}$ is bad. Similarly, if $b = y$, then we have $c = x$. □

Claim 15. *There do not exist three edges with the same colour.*

Proof. Suppose that there exist three edges using the same colour, say red. If these red edges occur in all three paths, then we have a contradiction to Claim 14, unless (without loss of generality) $q_3 = 2$ and xx_1, y_2y, xy are all red. But then $\{x_1, y\}$ is bad. Thus, some Q_i contains two red edges. Lemma 10 implies that $q_i = 3$, so there is a red edge in Q_j for some $j \neq i$. Claim 14 then implies $q_j = 2$, so that $j = 3$, and we may assume that $i = 2$. Claim 14 again implies that there cannot exist a red edge in Q_1. Since $|E(G)| = n + 1$ and we have at most $n - 2$ colours, Q_1 must have two edges with the same colour other than red. But then $\{x, y\}$ is bad. □

Thus, since $|E(G)| = n + 1$ and we have at most $n - 2$ colours, Claim 15 implies that there exist three colours, where for each colour, there are exactly two edges using the colour. Let red, blue and green be these three colours.

Case 1. $q_3 = 2$.

By Claim 14, it follows that $Q_3 = xy$ must be one of the three colours, say green. If say Q_2 contains both blue edges, then Lemma 10 implies $q_2 = 3$. But then both red edges occur in Q_1, and $\{x, y\}$ is bad. Thus by Claim 14, we may assume that xx_1, y_2y are red, and xx_2, y_1y are blue. Now, the remaining green edge is in Q_i for some $i \in \{1, 2\}$, and $\{x, x_i\}$ is bad.

Case 2. $q_3 \geq 3$.

Suppose, say, that the two green edges are in Q_3. By Lemma 10, $q_3 = 3$, and xx_3, x_3y are green. If, say, the two blue edges are in Q_2, then Lemma 10 implies that $q_2 = 3$, and xx_2, x_2y are blue. But then $\{x, y\}$ is bad. Otherwise, Claim 14 implies that, we may assume xx_1, y_2y are red, and xx_2, y_1y are blue. But then $\{x, x_1\}$ is bad.

Hence, none of the three paths may contain two edges in red, blue, or green. Now by Claim 14, we may assume that xx_1, y_2y are red, xx_2, y_3y are blue, and xx_3, y_1y are green. But then, since $n \geq 6$, we have $q_i \geq 4$ for some i, and $\{x_i, y_i\}$ is bad.

Therefore, we have $rc_2(G) \geq n - 1$, as required.

This completes the proof of Theorem 11. □

4   The function $s_2(n,r)$

Let $n > k \geq 2$. If $n \geq (k+1)^2$, then by Theorem 8(b), we have $s_k(n,2) = \binom{n}{2}$. Now, define

$$M_{n,k} = \max\{r : rc_k(G) = r \text{ for some } k\text{-connected graph } G \text{ on } n \text{ vertices}\}.$$

Then $s_k(n,r)$ is well-defined if and only if $2 \leq r \leq M_{n,k}$. Together with Theorem 5, we obtain the following analogue to (2)

$$\binom{n}{2} = s_k(n,2) \geq s_k(n,3) \geq \cdots \geq s_k(n,M_{n,k}) \geq \left\lceil \frac{kn}{2} \right\rceil.$$

Now, we focus on the function $s_2(n,r)$. Let $n \geq 4$. Theorem 8(a) implies that $s_2(n,2) = \binom{n}{2}$. Theorem 9 then implies that $s_2(n,r)$ is well-defined if and only if $2 \leq r \leq n$. Moreover, if G is a 2-connected graph on n vertices with $|E(G)| \geq n+1$, then $G \neq C_n$, so Theorem 9 gives $rc_2(G) \leq n-1$. This implies that $s_2(n,n) = n$. Therefore, for $n \geq 4$,

$$\binom{n}{2} = s_2(n,2) \geq s_2(n,3) \geq \cdots \geq s_2(n,n) = n.$$

Thus, it remains to consider $3 \leq r \leq n-1$. We provide lower bounds for $s_2(n,r)$ in the following proposition.

Proposition 16.

(a) *Let* $3 \leq r \leq n-1$. *Then* $s_2(n,r) \geq \binom{n-r+2}{2} + r - 1$.

(b) *Let* $n \geq 6$, *and* $\frac{n}{2} + 2 \leq r \leq n-1$. *Then* $s_2(n,r) \geq \binom{n-r+3}{2} + r - 3$.

Proof. (a) Let G be the graph on n vertices, obtained by taking the complete graph K_{n-r+2} and attaching an $(r-1)$-ear P at two vertices v_0, v_{r-1}. Then $|E(G)| = \binom{n-r+2}{2} + r - 1$. We show that $rc_2(G) \geq r$, which implies that $s_2(n,r) \geq \binom{n-r+2}{2} + r - 1$.

Let $P = v_0 v_1 \cdots v_{r-1}$ and $u_1, \ldots, u_{n-r}$ be the remaining vertices of G. Suppose that there exists a rainbow 2-connected colouring c of G, using at most $r-1$ colours. The two disjoint rainbow paths connecting v_{r-1} and v_{r-2} must be $v_{r-1}v_{r-2}$ and $v_{r-1}v_0 v_1 \cdots v_{r-2}$, so we may assume that $c(v_{i-1}v_i) = i$ for $1 \leq i \leq r-2$ and $c(v_0 v_{r-1}) = r-1$. Similarly, by considering v_0 and v_1, the path $v_1 v_2 \cdots v_{r-1}v_0$ must be rainbow, and so $c(v_{r-2}v_{r-1}) = 1$. Now for any $1 \leq j \leq n-r$, to connect v_1 and u_j with two disjoint rainbow paths, one path must be $v_1 v_2 \cdots v_{r-1}u_j$. Thus $c(v_{r-1}u_j) =$

197

$r - 1$. Similarly, by considering v_{r-2} and u_j, we have $c(v_0 u_j) = r - 1$. But now, we see that there do not exist two disjoint rainbow paths connecting v_0 and v_{r-1}. Therefore, $rc_2(G) \geq r$.

(b) Note that we have $r \geq 5$. We construct the graph G on $n \geq 6$ vertices as follows. Take the complete graph K_{n-r+3} and delete one edge $v_0 v_{r-2}$. Then add an $(r-2)$-ear P at v_0 and v_{r-2}. We have $|E(G)| = \binom{n-r+3}{2} + r - 3$. We show that $rc_2(G) \geq r$, which implies that $s_2(n, r) \geq \binom{n-r+3}{2} + r - 3$.

Let $P = v_0 v_1 \cdots v_{r-2}$ and $u_1, \ldots, u_{n-r+1}$ be the remaining vertices of G. Suppose that there exists a rainbow 2-connected colouring c of G, using at most $r - 1$ colours. Since P has length $r - 2 \geq 3$, Lemma 10 implies that P must be rainbow coloured. We may assume that $c(v_{i-1} v_i) = i$ for $1 \leq i \leq r - 2$. Now for $1 \leq i \leq r - 2$, to connect v_{i-1} and v_i with two disjoint rainbow paths, one path must be $v_{i-1} P v_0 u_j v_{r-2} P v_i$, for some $1 \leq j \leq n - r + 1$. Note that $n - r + 2 \leq r - 2$, so that the vertices $v_0, v_1, \ldots, v_{n-r+2}$ all exist. By considering the vertices v_{i-1} and v_i for $1 \leq i \leq n - r + 1$, we may assume that for $1 \leq i \leq n - r + 1$, the path $v_0 u_i v_{r-2}$ uses the colours i and $r - 1$. But now, we see that there do not exist two disjoint rainbow paths connecting v_{n-r+1} and v_{n-r+2}. Therefore, $rc_2(G) \geq r$. □

For $3 \leq r \leq n - 1$, we believe that the lower bounds in Proposition 16 are close to the correct values of $s_2(n, r)$. We propose the following problem.

Problem 17. *For $3 \leq r \leq n-1$, determine the function $s_2(n, r)$. Do there exist n and r such that $s_2(n, r)$ is equal to either lower bound in Proposition 16?*

Acknowledgements

References

[1] X. Bai, R. Chang, Z. Huang, X. Li, More on the rainbow disconnection in graphs, Discuss. Math. Graph Theory, in press.

[2] X. Bai, X. Li, Erdős-Gallai-type results for the rainbow disconnection number of graphs, ArXiv preprint arXiv:1901.02740.

[3] J-P. Bode, H. Harborth, The minimum size of k-rainbow connected graphs of given order, Discrete Math. 313 (2013) 1924–1928.

[4] B. Bollobás, Extremal Graph Theory, Academic Press, London, 1978.

[5] B. Bollobás, Modern Graph Theory, Springer-Verlag, New York, 1998.

[6] Q. Cai, X. Li, D. Wu, Erdős-Gallai-type results for colorful monochromatic connectivity of a graph, J. Comb. Optim. 33 (2017) 123–131.

[7] G. Chartrand, S. Devereaux, T.W. Haynes, S.T. Hedetniemi, P. Zhang, Rainbow disconnection in graphs, Discuss. Math. Graph Theory 38 (2018) 1007–1021.

[8] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem. 133 (2008) 85–98.

[9] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, The rainbow connectivity of a graph, Networks 54 (2009) 75–81.

[10] R. Diestel, Graph Theory, 5th ed., Springer-Verlag, New York, 2018.

[11] F. Harary, The maximum connectivity of a graph, Proc. Nat. Acad. Sci. USA 48 (1962) 1142–1146.

[12] A. Kemnitz, J. Przybyło, I. Schiermeyer, M. Woźniak, Rainbow connection in sparse graphs, Discuss. Math. Graph Theory 33 (2013) 181–192.

[13] A. Kemnitz, I. Schiermeyer, Graphs with rainbow connection number two, Discuss. Math. Graph Theory 31 (2011) 313–320.

[14] H. Li, X. Li, Y. Sun, Y. Zhao, Note on minimally d-rainbow connected graphs, Graphs Combin. 30 (2014) 949–955.

[15] X. Li, M. Liu, I. Schiermeyer, Rainbow connection number of dense graphs, Discuss. Math. Graph Theory 33 (2013) 603–611.

[16] X. Li, S. Liu, A sharp upper bound for the rainbow 2-connection number of a 2-connected graph, Discrete Math. 313 (2013) 755–759.

[17] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: a survey, Graphs Combin. 29 (2013) 1–38.

[18] X. Li, Y. Sun, Rainbow Connections of Graphs, Springer-Verlag, New York, 2012.

[19] A. Lo, A note on the minimum size of k-rainbow-connected graphs, Discrete Math. 331 (2014) 20–21.

[20] K. Menger, Zur allgemeinen Kurventheorie, Fund. Math. 10 (1927) 96–115.

[21] I. Schiermeyer, On minimally rainbow k-connected graphs, Discrete Appl. Math. 161 (2013) 702–705.