Transversals in 4-Uniform Linear Hypergraphs

Michael A. Henning[1,*] and Anders Yeo[1,2]

[1]Department of Mathematics and Applied Mathematics
University of Johannesburg
Auckland Park, 2006 South Africa
Email: mahenning@uj.ac.za

[2]Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55, 5230 Odense M, Denmark
Email: andersyeo@gmail.com

Abstract

Let H be a hypergraph of order $n_H = |V(H)|$ and size $m_H = |E(H)|$. The transversal number $\tau(H)$ of a hypergraph H is the minimum number of vertices that intersect every edge of H. A linear hypergraph is one in which every two distinct edges intersect in at most one vertex. A k-uniform hypergraph has all edges of size k. For $k \geq 2$, let $\mathcal{L}_k$ denote the class of k-uniform linear hypergraphs. We consider the problem of determining the best possible constants q_k (which depends only on k) such that $\tau(H) \leq q_k(n_H + m_H)$ for all $H \in \mathcal{L}_k$. It is known that $q_2 = \frac{1}{3}$ and $q_3 = \frac{1}{4}$. In this paper we show that $q_4 = \frac{1}{5}$, which is better than for non-linear hypergraphs. Using the affine plane $AG(2,4)$ of order 4, we show there are a large number of densities of hypergraphs $H \in \mathcal{L}_4$ such that $\tau(H) = \frac{1}{5}(n_H + m_H)$.

Keywords: Transversal; Hypergraph; Linear hypergraph; Affine plane
AMS subject classification: 05C65, 51E15

Dedicated to Gary MacGillivray
on the special occasion of his 60th birthday
to honour his many contributions to the graph theory community

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1 Introduction

In this paper we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*. The *order* of H is $n(H) = |V|$ and the *size* of H is $m(H) = |E|$. For simplicity, we sometimes we denote $n(H)$ and $m(H)$ by n_H and m_H, respectively. A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are a special instance of hypergraphs. For $i \geq 2$, we denote the number of edges in H of size i by $e_i(H)$. The *degree* of a vertex v in H, denoted by $d_H(v)$, is the number of edges of H which contain v. The minimum and maximum degrees among the vertices of H is denoted by $\delta(H)$ and $\Delta(H)$, respectively.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. The *neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from the context, is the set of all vertices different from v that are adjacent to v. A vertex in $N(v)$ is a *neighbor* of v. The *neighborhood* of a set S of vertices of H is the set $N_H(S) = \cup_{v \in S} N_H(v)$, and the *boundary* of S is the set $\partial_H(S) = N_H(S) \setminus S$. Thus, $\partial_H(S)$ consists of all vertices of H not in S that have a neighbor in S. If H is clear from context, we simply write $N(S)$ and $\partial(S)$ rather than $N_H(S)$ and $\partial_H(S)$. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of H is a *component* of H. Thus, no edge in H contains vertices from different components. A component of H isomorphic to a hypergraph F we call an *F-component* of H.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A transversal of size $\tau(H)$ is called a $\tau(H)$-transversal. Transversals in hypergraphs are well studied in the literature (see, for example, [1, 6, 9, 10, 11, 18, 19, 20, 21, 22, 23, 32, 33, 37, 39]).

A hypergraph H is called an *intersecting hypergraph* if every two distinct edges of H have a non-empty intersection, while H is called a *linear hypergraph* if every two distinct edges of H intersect in at most one vertex. We say that two edges in H *overlap* if they intersect in at least two vertices. A linear hypergraph therefore has no overlapping edges. Linear hypergraphs are well studied in the literature (see, for example, [2, 5, 8, 13, 14, 29, 31, 35, 36, 38]), as are uniform hypergraphs (see, for example, [6, 7, 8, 14, 15, 23, 25, 33, 34, 35, 36, 38]). A set S of vertices in a hypergraph H is *independent* (also called strongly independent in the literature) if no two vertices in S belong to a common edge. Independence in hypergraphs is well studied in the literature (see, for example, [3, 26, 32], for recent papers on this topic).

Given a hypergraph H and subsets $X, Y \subseteq V(H)$ of vertices, we let $H(X,Y)$ denote the hypergraph obtained by deleting all vertices in $X \cup Y$ from H and removing all (hyper)edges containing vertices from X and removing the vertices in Y from any remaining edges. If $Y = \emptyset$, we simply denote $H(X,Y)$ by $H - X$; that is, $H - X$ denotes that hypergraph obtained from H by removing the vertices X from H, removing all edges that intersect X and removing all resulting isolated vertices, if any. Further, if $X = \{x\}$, we simply write $H - x$ rather than $H - X$. When we use the definition $H(X,Y)$ we furthermore assume that no edges of size zero are created. That is, there is no edge $e \in E(H)$ such that $V(e) \subseteq Y \setminus X$. In this case we note that if we add X to any $\tau(H(X,Y))$-set, then we get a transversal of H, implying that $\tau(H) \leq |X| + \tau(H(X,Y))$. We will often use this fact throughout the paper.

In geometry, a *finite affine plane* is a system of points and lines that satisfy the following rules: (R1) Any two distinct points lie on a unique line. (R2) Each line has at least two points. (R3) Given a point and a line, there is a unique line which contains the point and is parallel to the line, where two lines are called parallel if they are equal or disjoint. (R4) There exist three non-collinear points (points not on a single line). A finite affine plane $AG(2,q)$ of *order* $q \geq 2$ is a collection of q^2 points and $q^2 + q$ lines, such that each line contains q points and each point is contained in $q+1$ lines. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

2 Motivation and Known Results

Let H be a hypergraph of order $n_H = n(H)$ and size $m_H = m(H)$. For $k \geq 2$, let $\mathcal{H}_k$ denote the class of all k-uniform hypergraphs. Tuza [39] proposed the problem of determining or estimating the best possible constants c_k (which depends only on k) such that $\tau(H) \leq c_k(n_H + m_H)$ for all $H \in \mathcal{H}_k$. These constants are given by

$$c_k = \sup_{H \in \mathcal{H}_k} \frac{\tau(H)}{n_H + m_H}.$$

It is a simple exercise to show ([12, p. 1180], or see [37]) that $c_2 = \frac{1}{3}$. Chvátal and McDiarmid [9] and Tuza [39] independently established that $c_3 = \frac{1}{4}$, while Lai and Chang [33] showed that $c_4 = \frac{2}{9}$. Applying probabilistic arguments, Alon [1] determined the asymptotic behaviour of c_k as k grows.

Theorem 1 (Alon [1]) $c_k = (1 + o(1))\left(\frac{\ln(k)}{k}\right)$ *as* $k \to \infty$.

As remarked by Alon [1], "it would be extremely interesting to determine precisely the value of c_k for every k. The considerable effort made in [33] to show that $c_4 = \frac{2}{9}$ suggests that this may be difficult." Indeed, the precise value of c_k has yet to be determined for any values of k with $k \geq 5$.

Very few papers give bounds on the transversal number for linear hypergraphs, even though these appear in many applications, as it seems difficult to utilise the linearity in the known techniques. In this paper, we nevertheless consider the class of k-uniform *linear* hypergraphs, which we denote by $\mathcal{L}_k$. Motivated by Tuza [39], we propose an analogous problem of determining or estimating the best possible constants q_k (which depends only on k) such that $\tau(H) \leq q_k(n_H + m_H)$ for all $H \in \mathcal{L}_k$. These constants are given by

$$q_k = \sup_{H \in \mathcal{L}_k} \frac{\tau(H)}{n_H + m_H}.$$

For $k \geq 2$, the family $\mathcal{L}_k$ is a (proper) subfamily of $\mathcal{H}_k$, implying that $q_k \leq c_k$. For $k \geq 2$, let E_k denote the k-uniform hypergraph on k vertices with exactly one edge. If $H = E_k$, then $H \in \mathcal{L}_k$ and $\tau(H)/(n_H + m_H) = 1/(k+1)$, implying that $q_k \geq 1/(k+1)$. This yields the following observation.

Observation 1 *For* $k \geq 2$, $c_k \geq q_k \geq \frac{1}{k+1}$.

If $q_k = \frac{1}{k+1}$ and the affine plane $AG(2,k)$ of order k exists for some $k \geq 2$, then the authors [27] show that the bound $\tau(H) \leq (n_H + m_H)/(k+1)$ where $H \in \mathcal{L}_k$ is tight for average degree 1 and average degree k and for a number of average degrees in the interval from 1 to k.

We note that the family $\mathcal{L}_2$ is precisely the family $\mathcal{H}_2$, and so $q_2 = c_2 = \frac{1}{3}$. For $k \geq 3$, the family $\mathcal{L}_k$ is a proper subfamily of $\mathcal{H}_k$, implying that $q_k \leq c_k$ and that strict inequality may be possible. As a consequence of results due to Tuza [39], Chvátal and McDiarmid [9], Henning and Yeo [20, 22], and Dorfling and Henning [11], $q_3 = c_3 = \frac{1}{4}$. Further, for $k \in \{2,3\}$, if

$H \in \mathcal{L}_k$, then $\tau(H) = q_k(n_H + m_H)$ if and only if H consists of a single edge or H is obtained from the affine plane $AG(2, k)$ of order k by deleting one or two vertices. We state these known results formally as follows.

Theorem 2 ([27]) $q_2 = \frac{1}{3}$ *and* $q_3 = \frac{1}{4}$.

Applying probabilistic arguments, the authors [27] showed that the asymptotic behaviour of q_k as k grows is the same as that of c_k, namely of the order $\ln(k)/k$.

3 Main Result

As observed earlier, $q_k \leq c_k$ for all $k \geq 2$. Further, $q_2 = c_2 = \frac{1}{3}$ and $q_3 = c_3 = \frac{1}{4}$. In this paper, we determine the precise value of q_4.

Theorem 3 $q_4 = \frac{1}{5}$.

Recall that $c_4 = \frac{2}{9}$, and so, by Theorem 3, $q_4 < c_4$. Therefore, the best possible upper bound on the transversal number for 4-uniform linear hypergraphs is better than that for 4-uniform non-linear hypergraphs. We remark that the result of Theorem 3 was conjectured by the authors in [25].

As shown by the authors [27] if the affine plane $AG(2, k)$ of order k exists for some $k \geq 2$, then there are a large number of densities of hypergraphs $H \in \mathcal{L}_k$ such that $\tau(H) = \frac{1}{k+1}(n_H + m_H)$. In particular, when $k = 4$ we know that $AG(2, 4)$ exists. Let F_{16} be the linear, 4-uniform, 5-regular hypergraph of order 16 which is equivalent to the affine plane $AG(2, 4)$ of order 4. Let e be an arbitrary edge in F_{16} and let X be an arbitrary non-empty subset of vertices belonging to the edge e. As shown in [27], if $H = F_{16} - X$, then $H \in \mathcal{L}_4$ and $\tau(H) = \frac{1}{5}(n_H + m_H)$. Hence, the bound $\tau(H) \leq \frac{1}{5}(n_H + m_H)$ is tight for average degree 1 and average degree 4 and for a number of average degrees in the interval from 1 to 4. We summarize this result in Table 1.

$H = F_{16}(X)$					
	$\tau(H)$	n_H	m_H	$\frac{1}{5}(n_H + m_H)$	Average degree
$\lvert X\rvert = 1$	6	15	15	6	$60/15 = 4$
$\lvert X\rvert = 2$	5	14	11	5	$44/14 \approx 3.14\ldots$
$\lvert X\rvert = 3$	4	13	7	4	$28/13 \approx 2.15\ldots$
$\lvert X\rvert = 4$	3	12	3	3	$12/12 = 1$

Table 1. Hypergraphs $H \in \mathcal{L}_4$ achieving equality in the bound $\tau(H) \le \frac{1}{5}(n_H + m_H)$.

By Theorem 3, $\tau(H) \le \frac{1}{5}(n_H + m_H)$ for every hypergraph $H \in \mathcal{L}_4$. We remark that this bound is not necessary true for the non-linear hypergraph case as may be seen, for example, by taking $k = 4$ and letting $H = \overline{F}$ be the complement of the Fano plane F, where the Fano plane is shown in Figure 1 and where its complement $\overline{F}$ is the hypergraph on the same vertex set $V(F)$ and where e is a hyperedge in the complement if and only if $V(F) \setminus e$ is a hyperedge in F. In this case, $H = \overline{F} \in \mathcal{H}_4 \setminus \mathcal{L}_4$ and $\tau(H) = 3 = \frac{3}{14}(n_H + m_H) > \frac{1}{5}(n_H + m_H)$.

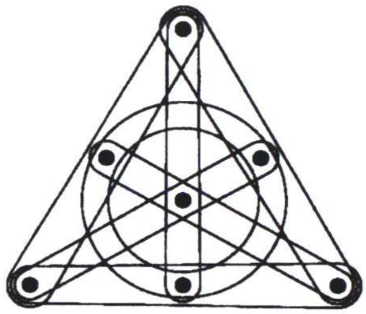


Figure 1: The Fano plane F

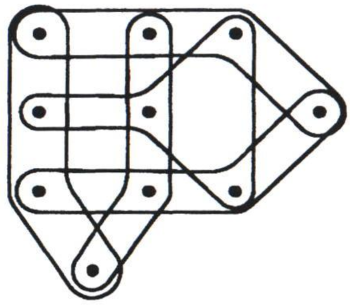
We proceed as follows. In Section 4, we present a proof of Theorem 3. We begin by defining fifteen special hypergraphs in Section 4.1 and introducing the concept of the deficiency of a hypergraph in Section 4.2. Thereafter in Section 4.4 we prove a key result, namely Theorem 6, about the deficiency of a hypergraph that will enable us to prove Theorem 3. Finally in Section 4.5, we present a proof of Theorem 3 and in Section 5 we present several applications of Theorem 3. In Section 6 we present some open problems, questions and conjectures that we have yet to settle.

4 Proof of Theorem 3

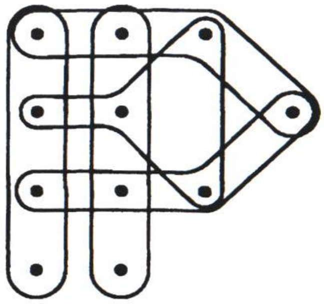
Throughout this section, we let $\mathcal{L}_{4,3}$ be the class of all 4-uniform, linear hypergraphs with maximum degree at most 3. We first define fifteen special hypergraphs, which are shown in Figure 2.

4.1 Special Hypergraphs

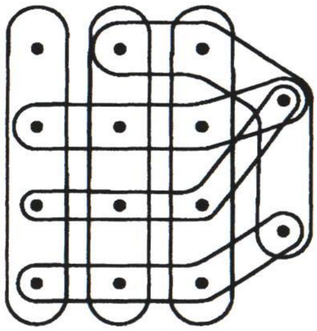
(a) H_4



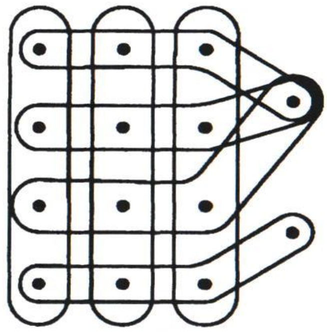
(b) H_{10}



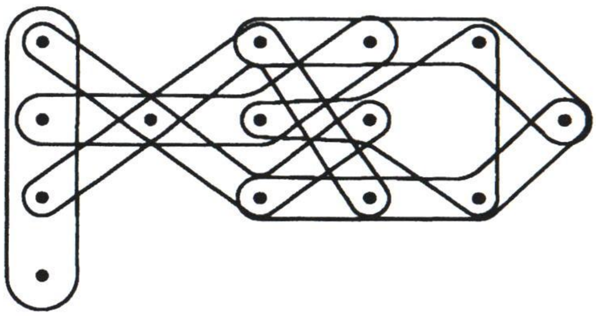
(c) H_{11}



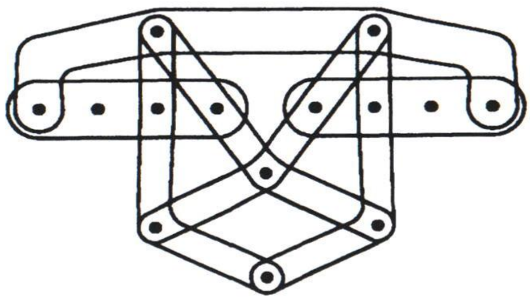
(d) $H_{14,1}$



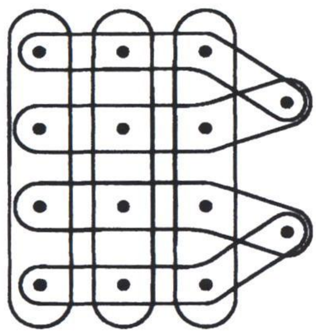
(e) $H_{14,2}$



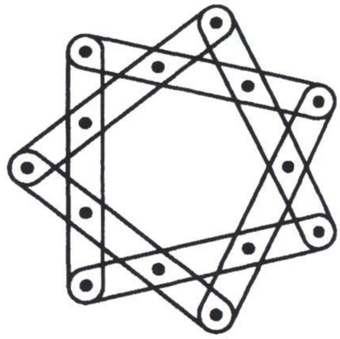
(f) $H_{14,3}$



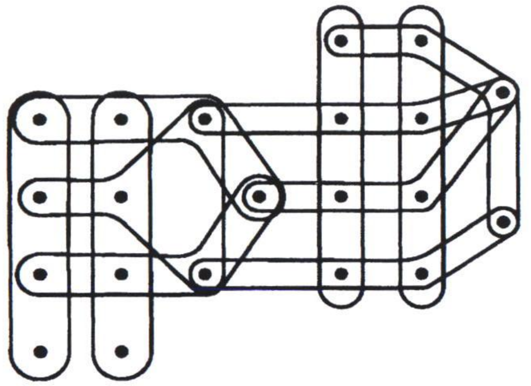
(g) $H_{14,4}$



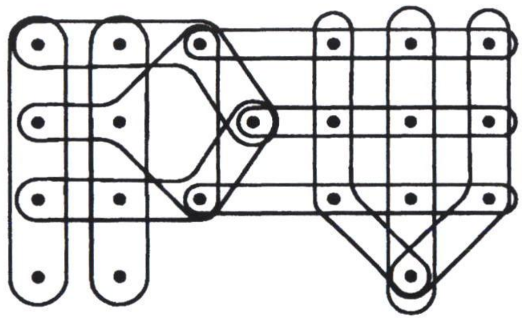
(h) $H_{14,5}$



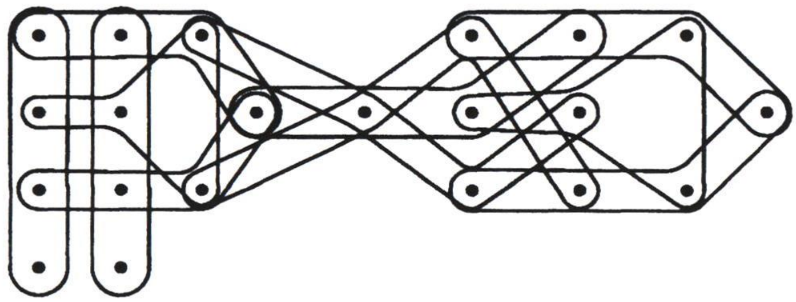
(i) $H_{14,6}$



(j) $H_{21,1}$



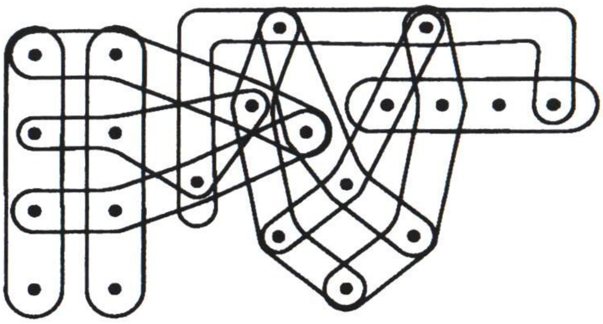
(k) $H_{21,2}$



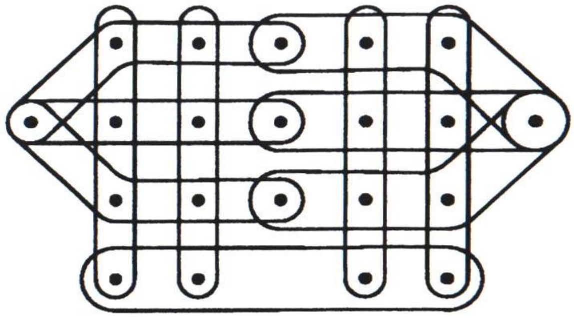
(ℓ) $H_{21,3}$

Observation 2 *If H is a special hypergraph of order n_H and size m_H, then the following holds.*

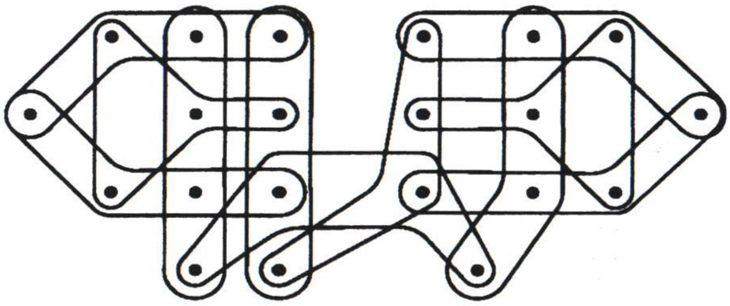
(a) *If $H = H_4$, then $n_H = 4$, $m_H = 1$ and $\tau(H) = 1$.*
(b) *If $H = H_{10}$, then $n_H = 10$, $m_H = 5$ and $\tau(H) = 3$.*
(c) *If $H = H_{11}$, then $n_H = 11$, $m_H = 5$ and $\tau(H) = 3$.*
(d) *If $H = H_{14,i}$ where $i \in [6]$, then $n_H = 14$, $m_H = 7$ and $\tau(H) = 4$.*
(e) *If $H = H_{21,i}$ where $i \in [6]$, then $n_H = 21$, $m_H = 11$ and $\tau(H) = 6$.*



(m) $H_{21,4}$



(n) $H_{21,5}$



(o) $H_{21,6}$

Figure 2: Fifteen special hypergraphs.

(f) *Any given vertex in H belongs to some $\tau(H)$-transversal.*

4.2 The Deficiency of a Set

Let H be a 4-uniform hypergraph. A set X is a *special H-set* if it consists of subhypergraphs of H with the property that every subhypergraph in X is a special hypergraph and further these special hypergraphs are pairwise vertex disjoint. For notational simplicity, we write $V(X)$ and $E(X)$ to denote the set of all vertices and edges, respectively, in H that belong to a subhypergraph $H' \in X$ in the special H-set X. Let X be an arbitrary special H-set.

A set T of vertices in $V(X)$ is an *X-transversal* if T is a minimum set of vertices that intersects every edge from every subhypergraph in X. We define $E^*_H(X)$ to be the set of all edges in H that do not belong to a subhypergraph in X but which intersect at least one subhypergraph in X. Hence if $e \in E^*(X)$, then $e \notin E(H')$ for every subhypergraph $H' \in X$ but $V(e) \cap V(H') \neq \emptyset$ for at least one subhypergraph $H' \in X$. If the hypergraph H is clear from context, we simply write $E^*(X)$ rather than $E^*_H(X)$. We associate with the set X a bipartite graph, which we denote by G_X, with partite sets X and $E^*(X)$, where an edge joins $e \in E^*(X)$ and $H' \in X$ in G_X if and only if the edge e intersects the subhypergraph H' of X in H.

We define a *weak partition* of $X = (X_4, X_{10}, X_{11}, X_{14}, X_{21})$ (where a weak partition is a partition in which some of the sets may be empty) where $X_i \subseteq X$ consists of all subhypergraphs in X of order i, $i \in \{4, 10, 11, 14, 21\}$. Thus, $X = X_4 \cup X_{10} \cup X_{11} \cup X_{14} \cup X_{21}$ and $|X| = |X_4| + |X_{10}| + |X_{11}| + |X_{14}| + |X_{21}|$. As an immediate consequence of Observation 2(a)–(e), we have the following result.

Observation 3 *If X is a special H-set and T is an X-transversal, then*

$$|T| = |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + 6|X_{21}|.$$

We define the *deficiency* of X in H as

$$\mathrm{def}_H(X) = 10|X_{10}| + 8|X_4| + 5|X_{14}| + 4|X_{11}| + |X_{21}| - 13|E^*(X)|.$$

We define the *deficiency* of H by

$$\mathrm{def}(H) = \max \mathrm{def}_H(X)$$

where the maximum is taken over all special H-sets X. We note that taking $X = \emptyset$, we have $\mathrm{def}(H) \geq 0$.

4.3 Known Results and Observations

We shall need the following theorem of Berge [4] about the matching number of a graph, which is sometimes referred to as the Tutte-Berge formulation for the matching number.

Theorem 4 (Tutte-Berge Formula) *For every graph G,*

$$\alpha'(G) = \min_{X \subseteq V(G)} \frac{1}{2} \left(|V(G)| + |X| - \mathrm{oc}(G - X) \right),$$

where $\mathrm{oc}(G - X)$ denotes the number of odd components of $G - X$.

We shall also rely heavily on the following well-known theorem due to König [30] and Hall [16] in 1935.

Theorem 5 (Hall's Theorem) *Let G be a bipartite graph with partite sets X and Y. Then X can be matched to a subset of Y if and only if $|N(S)| \geq |S|$ for every nonempty subset S of X.*

4.4 Key Theorem

Recall that $\mathcal{L}_{4,3}$ is the class of all 4-uniform linear hypergraphs with maximum degree at most 3. We shall need the following key result in order to prove Theorem 3. We remark that some of the more technical details of our proof of Theorem 6 are straightforward but tedious to check and are therefore omitted. However our proof presents all the important and major ideas, taking special care to provide sufficient detail and rigor for the reader to check its correctness without getting side tracked with some of the more technical parts of the proof.[1]

Theorem 6 *If $H \in \mathcal{L}_{4,3}$, then $45\tau(H) \le 6n(H) + 13m(H) + \text{def}(H)$.*

Proof. For a 4-uniform hypergraph H, let $\xi(H) = 45\tau(H) - 6n(H) - 13m(H) - \text{def}(H)$. We wish to show that if $H \in \mathcal{L}_{4,3}$, then $\xi(H) \le 0$. Suppose, to the contrary, that the theorem is false and that $H \in \mathcal{L}_{4,3}$ is a counterexample with minimum value of $n(H) + m(H)$. Thus, $\xi(H) > 0$ but every hypergraph $H' \in \mathcal{L}_{4,3}$ with $n(H') + m(H') < n(H) + m(H)$ satisfies $\xi(H') \le 0$. We show first that the hypergraph H is connected and $\delta(H) \ge 1$. We then prove that given a special H-set, X, there is no X-transversal, T, such that $|T| = |X_4| + 3|X_{10}| + 3|X_{11}| + 4|X_{14}| + |X_{21}|$ and T intersects every edge in $E^*(X)$. From these properties of H, we deduce that H is not a special hypergraph. Among all special non-empty H-sets, let X be chosen so that

(1) $|E^*(X)| - |X|$ is minimum.
(2) Subject to (1), $|X|$ is maximum.

With this choice of the H-set X, we prove next that $|E^*(X)| \ge |X| + 1$ and that if $X' \ne \emptyset$ is a special H-set, then $|E^*(X')| \ge |X'| + 1$. Thereafter, we show that $\text{def}(H) = 0$. For a hypergraph $H' \in \mathcal{L}_{4,3}$, let $\Phi(H') = \xi(H') - \xi(H)$. We show that if $H' \in \mathcal{L}_{4,3}$ satisfies $n(H') + m(H') < n(H) + m(H)$, then $\Phi(H') < 0$. We then show that our earlier property $|E^*(X)| \ge |X| + 1$ can be improved to $|E^*(X)| \ge |X| + 2$. From this property, we deduce that if Y is an arbitrary special H-set, then $|E^*(Y)| \ge |Y| + 2$. Further if $|Y_{10}| \ge 2$, then $|E^*(Y)| \ge |Y| + 3$. In particular, this implies that if $|X_{10}| \ge 2$, then $|E^*(X)| \ge |X| + 3$. We prove next that there is no H_{10}-subhypergraph in H.

Recall that the boundary of a set Z of vertices in a hypergraph H is the set $N_H(Z) \setminus Z$, denoted $\partial_H(Z)$ or simply $\partial(Z)$ if H is clear from context. Let $Z \subseteq V(H)$ be an arbitrary nonempty set of vertices and let $H' = H - Z$. We prove next that either $|E^*_{H'}(Y)| \ge |Y|$ for all special H'-sets Y in H'

[1] A proof with all the technical details can be found in [28].

(and therefore $\mathrm{def}(H') = 0$) or there exists a transversal T' in H', such that $45|T'| \le 6n(H') + 13m(H') + \mathrm{def}(H')$ and $T' \cap \partial(Z) \ne \emptyset$. Now let f be the function defined in Table 1.

i	1	2	3	4	≥ 5
$f(i)$	39	33	27	23	22

Table 1. The function f.

We are now in a position to state the following claim, the proof of which we omit.

Claim A: Let $Z \subseteq V(H)$ be an arbitrary nonempty set of vertices that intersects at least two edges of H, and let $H' = H - Z$. If $\mathrm{def}(H') \le 21$ and $|\partial(Z)| \ge 1$, then there exists a transversal, T', in H', such that $T' \cap \partial(Z) \ne \emptyset$ and the following holds.

(a) $45|T'| \le 6n(H') + 13m(H') + f(|\partial(Z)|)$.
(b) If $|\partial(Z)| \ge 5$ and H' does not contain two intersecting edges e and f, such that
  (i) $\partial(Z) \subseteq (V(e) \cup V(f)) \setminus (V(e) \cap V(f))$,
  (ii) e contains three vertices of degree 1, and
  (iii) $|\partial(Z) \cap V(e)|, |\partial(Z) \cap V(f)| \ge 2$,
then $45|T'| \le 6n(H') + 13m(H') + f(|\partial(Z)|) - 1 = 6n(H') + 13m(H') + 21$.

We call a component of a 4-uniform, linear hypergraph that contains two vertex disjoint copies of H_4 that are both intersected by a common edge and such that each copy of H_4 has three vertices of degree 1 and one vertex of degree 2 a *double-H_4-component*. We call these two copies of H_4 the *H_4-pair* of the double-H_4-component, and the edge that intersects them the *linking edge*. We note that a double-H_4-component contains at least ten vertices, namely eight vertices from the H_4-pair and at least two additional vertices that belong to the linking edge. We are now in a position to state the following claim, the proof of which we omit.

Claim B: If x is an arbitrary vertex of H of degree 3, then one of the following holds.

(a) $\mathrm{def}(H - x) = 8$ and the hypergraph $H - x$ contains an H_4-component that is intersected by all three edges incident with x.
(b) $\mathrm{def}(H - x) = 3$ and the hypergraph $H - x$ contains a double-H_4-component. Further, the H_4-pair in this component is intersected by all three edges incident with x.

Using Claim B, we show that no edge in H contains two vertices of degree 1 of H. Let H' be a 4-uniform, linear hypergraph with no H_{10}-subhypergraph satisfying $\text{def}(H') > 0$. We show that if Y is a special H'-set satisfying $\text{def}(H') = \text{def}_{H'}(Y)$, then $|E^*_{H'}(Y)| = |Y| - i$ for some $i \geq 1$ and $\text{def}(H') \leq 13i - 5|Y_4| - 8|Y_{14}| - 9|Y_{11}| - 12|Y_{21}|$. In particular, $\text{def}(H') \leq 13i - 5|Y|$. Further, if $\text{def}(H') > 8j$ for some $j \geq 0$, then $|E^*_{H'}(Y)| \leq |Y| - (j + 1)$. Using these facts, we are able to establish the following result, the proof of which we omit.

Claim C: The following properties hold in the hypergraph H.

(a) If x is an arbitrary vertex of H of degree 3, then $H - x$ contains an H_4-component.
(b) No edge in H contains two vertices of degree 3.
(c) Every vertex of degree 3 in H has at most one neighbor of degree 1.

By Claim B and Claim C(a), if x is an arbitrary vertex of H of degree 3, then the hypergraph $H - x$ contains an H_4-component that is intersected by all three edges incident with x, and $\text{def}(H - x) = 8$. By Claim C(b), no edge in H contains two vertices of degree 3. By Claim C(c), every vertex of degree 3 in H has at most one neighbor of degree 1. We now define the operation of *duplicating* a vertex of degree 3 x as follows.

Let e_1, e_2 and e_3 be the three edges incident with x. By Claim C(b) and Claim C(c), every neighbor of x has degree 2, except possibly for one vertex which has degree 1. Renaming edges if necessary, we may assume that the edge e_1 contains no vertex of degree 1, and therefore every vertex in e_1 different from x has degree 2. We now delete the edge e_1 from H, and add a new vertex x' and a new edge $e'_1 = (V(e_1) \setminus \{x\}) \cup \{x'\}$ to H. We note that in the resulting hypergraph the vertex x now has degree 2 (and is incident with the edges e_2 and e_3) and the new vertex x' has degree 1 with all its three neighbors of degree 2. We call x' the *vertex duplicated copy* of x.

Let H' be obtained from H by duplicating every vertex of degree 3 as described above. By construction, H' is a linear 4-uniform connected hypergraph with minimum degree $\delta(H') \geq 1$ and maximum degree $\Delta(H') \leq 2$. For $i \in [2]$, let $n_i(H')$ be the number of vertices of degree i in H'. Then, $n(H') = n_1(H') + n_2(H')$ and $4m(H') = 2n_2(H') + n_1(H')$. We are now in a position to state the following claim, the proof of which we omit.

Claim D: The following properties hold in the hypergraph H'.

(a) $\tau(H) = \tau(H')$.
(b) $\text{def}(H') = 0$.

We now consider the multigraph G whose vertices are the edges of H' and whose edges correspond to the $n_2(H')$ vertices of degree 2 in H': if a vertex of H' is contained in the edges e and f of H', then the corresponding edge of the multigraph G joins vertices e and f of G. By the linearity of H', the multigraph G is in fact a graph, called the *dual* of H'. We shall need the following properties about the dual G of the hypergraph H', the proof of which we omit.

Claim E: The following properties hold in the dual, G, of the hypergraph H'.

(a) G is connected, $n(G) = m(H')$ and $m(G) = n_2(H')$.
(b) $\Delta(G) \le 4$, and so $m(G) \le 2n(G)$ and $8n(G) + 6m(G) \le 20n(G)$.
(c) $\tau(H') = m(H') - \alpha'(G)$.

Suppose that x is a vertex of degree 3 in H, and let x' be the vertex duplicated from x when constructing H. Adopting our earlier notation, let e_1, e_2 and e_3 be the three edges incident with x. Further, let e be the edge in the H_4-component of $H - x$, and let y be the vertex of degree 1 in e. Let y_i be the vertex common to e and e_i for $i \in [3]$, and so $y_i = (ee_i)$ and $V(e) = \{y, y_1, y_2, y_3\}$. We note that in the graph G, which is the dual of H', the vertex e has degree 3 and is adjacent to the vertices e_1, e_2 and e_3. Further, we note that in the graph G, the vertex e_1 has degree 3, while the vertices e_2 and e_3 are adjacent and have degree at most 4. Further, the edge y_i in G is the edge ee_i, while the edge x in G is the edge e_2e_3. The vertex e_1 is adjacent in G to neither e_2 nor e_3. This set of four vertices $\{e, e_1, e_2, e_3\}$ in the graph G we call a *quadruple* in G. We illustrate this quadruple in G in Figure 3. We denote the set of (vertex-disjoint) quadruples in G by Q.

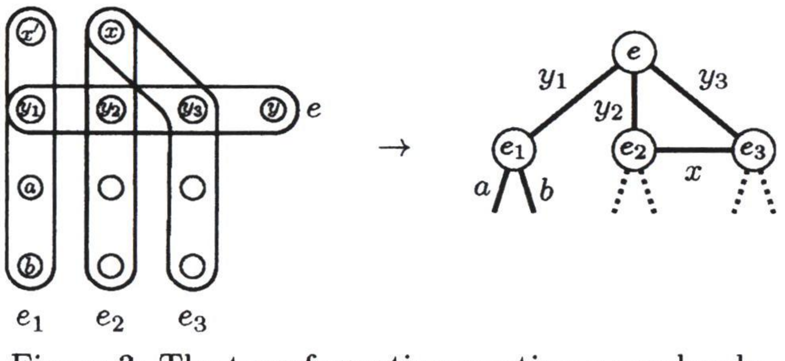


Figure 3: The transformation creating a quadruple.

We shall need the following additional property about the dual G of the hypergraph H', the proof of which we omit.

Claim F: If G is the dual of the hypergraph H', then

$$\begin{aligned} & 45\tau(H') \le 6n(H') + 13m(H') - 6|Q| \\ \Leftrightarrow \quad & 45\alpha'(G) \ge 8n(G) + 6m(G) + 6|Q|. \end{aligned}$$

Let S be a set of vertices in G such that $(n(G) + |S| - \text{oc}(G - S))/2$ is minimum. By the Tutte-Berge Formula,

$$\alpha'(G) = \frac{1}{2}\left(n(G) + |S| - \text{oc}(G - S)\right). \tag{1}$$

We now consider two cases, depending on whether $S = \emptyset$ or $S \neq \emptyset$.

Claim G: If $S \neq \emptyset$, then $45\tau(H') \le 6n(H') + 13m(H') - 6|Q|$.

Proof of Claim G: Suppose that $S \neq \emptyset$. For $i \ge 1$, let $n_i(G - S)$ denote the number of components on $G - S$ of order i. Let $n_5^1(G - S)$ be the number of components of $G - S$ isomorphic to $K_5 - e$ and let $n_5^2(G - S)$ denote all remaining components of $G - S$ on five vertices (with at most eight edges), and so $n_5(G - S) = n_5^1(G - S) + n_5^2(G - S)$. For notational convenience, let $n = n(G)$, $m = m(G)$, $n_5^1 = n_5^1(G - S)$, $n_5^2 = n_5^2(G - S)$, and $n_i = n_i(G - S)$ for $i \ge 1$. Let $\mathbb{Z}^+$ denote the set of all positive integers, and let $\mathbb{Z}^+_{\text{even}}$ and $\mathbb{Z}^+_{\text{odd}}$ denote the set of all even and odd integers, respectively, in $\mathbb{Z}^+$. Further for a fixed $j \in \mathbb{Z}^+$, let $\mathbb{Z}_{\ge j} = \{i \in \mathbb{Z} \mid i \ge j\}$, $\mathbb{Z}^j_{\text{even}} = \{i \in \mathbb{Z}_{\ge j} \mid i \text{ even}\}$, and $\mathbb{Z}^j_{\text{odd}} = \{i \in \mathbb{Z}_{\ge j} \mid i \text{ odd}\}$. We note that

$$n = |S| + \sum_{i \in \mathbb{Z}^+} i \cdot n_i. \tag{2}$$

By Equation (1) and Equation (2), and since

$$\text{oc}(G - S) = \sum_{i \in \mathbb{Z}^+_{\text{odd}}} n_i,$$

we have the equation

$$45\alpha'(G) = 45|S| + \frac{45}{2}\left(\Big(\sum_{i \in \mathbb{Z}^3_{\text{odd}}} (i - 1) \cdot n_i \Big) + \sum_{i \in \mathbb{Z}^2_{\text{even}}} i \cdot n_i \right). \tag{3}$$

Claim G.1: $m \le 4|S| + n_2 + 3n_3 + 6n_4 + 9n_5^1 + 8n_5^2 + \sum_{i \in \mathbb{Z}^6} (2i - 1)n_i - |Q|$.

Proof of Claim G.1: Since G is connected and $\Delta(G) \le 4$, we note that if F is a component of $G - S$ of order i, then $m(F) \le 2i - 1$. Further, every component of $G - S$ of order 5 is either isomorphic to $K_5 - e$ or contains at most eight edges, while every component of $G - S$ of order 2, 3 and 4 contains at most 1, 3 and 6 edges, respectively. The above observations imply that

$$m \leq 4|S| + n_2 + 3n_3 + 6n_4 + 9n_5^1 + 8n_5^2 + \sum_{i \in \mathbb{Z}^6} (2i-1)n_i.$$

We show next that each quadruple in the graph G decreases the count on the right hand side expression of the above inequality by at least 1. Adopting our earlier notation, consider a quadruple $\{e, e_1, e_2, e_3\}$. Recall that the vertices e and e_1 both have degree 3 in G, and there is no vertex in G that is adjacent to both e and e_1. Further, recall that the vertices e_2 and e_3 are adjacent in G. If e or e_1 or if both e_2 and e_3 belong to the set S, then the quadruple decreases the count $4|S|$ by at least 1. Hence, we may assume that e, e_1 and e_2 all belong to a component, C say, of $G - S$. In particular, we note that C has order at least 3. Abusing notation, we say that the component C contains the quadruple $\{e, e_1, e_2, e_3\}$, although possibly the vertex e_3 may belong to S. Since no vertex in G is adjacent to both e and e_1, we note that if the component C has order 3, 4 or 5, then it contains at most 2, 4 and 7 edges, respectively. Further, since we define a component to contain a quadruple if it contains at least three of the four vertices in the quadruple, we note in this case when the component C has order at most 5 that it contains exactly one quadruple. Further, this quadruple decreases the count $3n_3 + 6n_4 + 9n_5^1 + 8n_5^2$ by at least 1.

It remains for us to consider a component F of $G - S$ of order $i \geq 6$ that contains q quadruples, and to show that these q quadruples decrease the count $2i - 1$ by at least q. We note that each quadruple contains a pair of adjacent vertices of degree 3 in G. Further, at least one vertex v in F is joined to at least one vertex of S in G, implying that $d_F(v) < d_G(v)$. These observations imply that $2m(F) = \sum_{v \in V(F)} d_F(v) \leq 4n(F) - 2q - 1 = 4i - 2q - 1$, and therefore that $m(F) \leq 2i - q - 1$. Hence, these q quadruples contained in F combined decrease the count $2i - 1$ by at least q. This completes the proof of Claim G.1. $_{(\Box)}$

By Claim G.1 and by Equation (2), we have

$$\begin{aligned} 8n + 6m + 6|Q| \quad \leq \quad & 32|S| + 8n_1 + 22n_2 + 42n_3 + 68n_4 \\ & + 88n_5^2 + 94n_5^1 + \sum_{i \in \mathbb{Z}^6} (20i - 6)n_i. \end{aligned} \tag{4}$$

Let

$$\Sigma_{\text{even}} = \sum_{i \in \mathbb{Z}^6_{\text{even}}} \left(\frac{5}{2} i + 6 \right) \cdot n_i \quad \text{and} \quad \Sigma_{\text{odd}} = \sum_{i \in \mathbb{Z}^7_{\text{odd}}} \frac{1}{2} (5i - 33) \cdot n_i.$$

We note that every (odd) component in G isomorphic to K_1 corresponds to a subhypergraph H_4 in H', while every (odd) component in G isomorphic to $K_5 - e$ corresponds to a subhypergraph H_{11} in H'. Hence the odd components of G isomorphic to K_1 or isomorphic to $K_5 - e$ correspond to a special H'-set, X say, where $|X| = |X_4| + |X_{11}|$, $|X_4| = n_1$ and $|X_{11}| = n_5^1$. Further, the set S of vertices in G correspond to the set $E^*(X)$ of edges in H', and so $|E^*(X)| \le |S|$. Thus,

$$\mathrm{def}_{H'}(X) = 8|X_4| + 4|X_{11}| - 13|E^*(X)| \ge 8n_1 + 4n_5^1 - 13|S|.$$

By Claim D(b), $\mathrm{def}_{H'}(X) \le \mathrm{def}(H') = 0$, and therefore we have that

$$13|S| \ge 8n_1 + 4n_5^1. \tag{5}$$

By Equation (3), and by Inequalities (4) and (5), and noting that $\Sigma_{\mathrm{even}} \ge 0$ and $\Sigma_{\mathrm{odd}} \ge 0$, the following now holds.

$$\begin{aligned} 45\alpha'(G) &\overset{(3)}{=} 32|S| + 8n_1 + 22n_2 + 42n_3 + 68n_4 + 88n_5^2 + 94n_5^1 + \sum_{i \in \mathbb{Z}^6} (20i - 6)n_i \\ &\quad +13|S| - 8n_1 + 23n_2 + 3n_3 + 22n_4 + 2n_5^2 - 4n_5^1 + \Sigma_{\mathrm{even}} + \Sigma_{\mathrm{odd}} \\ &\overset{(4)}{\ge} (8n + 6m + 6|Q|) + (13|S| - 8n_1 - 4n_5^1) \\ &\overset{(5)}{\ge} 8n + 6m + 6|Q|. \end{aligned}$$

Claim G now follows from Claim F. (□)

Claim H: If $S = \emptyset$, then $45\tau(H) \le 6n(H) + 13m(H) - 6|Q|$.

Proof of Claim H: Suppose that $S = \emptyset$. Then, $\alpha'(G) = (n(G) - \mathrm{oc}(G))/2$. Since G is connected by Claim E, we have the following.

$$\alpha'(G) = \begin{cases} \frac{1}{2}n(G) & \text{if } n(G) \text{ is even} \\ \frac{1}{2}(n(G) - 1) & \text{if } n(G) \text{ is odd.} \end{cases}$$

By Claim E(b), $\Delta(G) \le 4$. As every quadruple in G contains two vertices of degree 3,

$$2m(G) = \sum_{v \in G} d_G(v) \le 4n(G) - 2|Q|,$$

implying that $12n(G) \ge 6m(G) + 6|Q|$. If $n(G)$ is even, then $\alpha'(G) = n(G)/2$, and so

$$45\alpha'(G) = \frac{45}{2}n(G) > 8n(G) + 12n(G) \ge 8n(G) + 6m(G) + 6|Q|.$$

This completes the case when $n(G)$ is even by Claim F. Suppose next that $n(G)$ is odd. In this case $45\alpha'(G) = 45(n(G)-1)/2$, and so

$$90\alpha'(G) = 45n(G)-45 = 21n(G)+24n(G)-45 \geq 21n(G)+12m(G)+12|Q|-45.$$

If $5n(G) \geq 45$, then $90\alpha'(G) \geq 16n(G) + 12m(G) + 12|Q|$, which completes the proof by Claim F. We may therefore assume that $5n(G) < 45$, implying that $n(G) \in \{1, 3, 5, 7\}$. Since every quadruple contains four vertices, we must therefore have $|Q| \leq 1$.

We first consider the case when $|Q| = 1$. In this case $n(G) \geq 6$, as the quadruple contains four vertices and one vertex (called e_1 in the definition of a quadruple) has two neighbours outside the quadruple. Therefore, $n(G) = 7$. Since two vertices in the quadruple have degree at most 3, $2m(G) = \sum_{v\in V(G)} d(v) \leq 4n(G) - 2 = 26$, and so $m(G) \leq 13$. If $m(G) \leq 12$, then $45\alpha'(G) = 45 \cdot 3 > 8 \cdot 7 + 6 \cdot 12 + 6 \geq 8n(G) + 6m(G) + 6|Q|$, and the desired result follows from Claim F. Therefore, we may assume that $m(G) = 13$, for otherwise the case is complete. In this case, all vertices in G have degree 4 except for two vertices in the quadruple which have degree 3. Let $\{e, e_1, e_2, e_3\}$ be the vertices in the quadruple in G, such that $d(e) = d(e_1) = 3$ and e, e_2 and e_3 form a 3-cycle in G. Define u_1 and u_2 such that $N(e_1) = \{e, u_1, u_2\}$ and define w, such that $V(G) = \{e, e_1, e_2, e_3, u_1, u_2, w\}$. As $d(w) = 4$ and w is not adjacent to e or e_1 we have $N(w) = \{e_2, e_3, u_1, u_2\}$. Therefore, e_2 is adjacent to e, e_3 and w. Its fourth neighbour is either u_1 or u_2. Renaming u_1 and u_2 if necessary, we may assume that $N(e_2) = \{e, e_3, w, u_2\}$. This implies that u_1 must be adjacent to u_2 and e_3 and G is the graph shown in Figure 4.

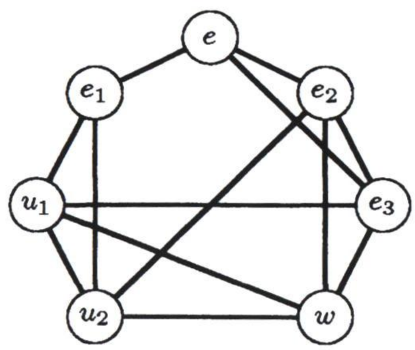


Figure 4: The graph G if $|Q| = 1$.

If we draw the corresponding hypergraph whose dual is the graph G, we note that it is obtained by duplicating the vertex of degree 3 in $H_{14,3}$. However, H is not equal to $H_{14,3}$ by Claim C, implying that G cannot be the graph in Figure 4, a contradiction. This completes the case when $|Q| = 1$. Therefore, we may assume that $|Q| = 0$.

If $n(G) = 1$, then $H = H_4$, a contradiction. Hence, $n(G) \in \{3, 5, 7\}$. Suppose that $n(G) = 3$. Then, $\alpha'(G) = 1$ and $m(G) \leq 3$. In this case, $8n(G) + 6m(G) \leq 8 \cdot 3 + 6 \cdot 3 = 42 < 45 = 45\alpha'(G)$, which by Claim F completes the proof. Hence we may assume that $n(G) = 5$ or $n(G) = 7$.

Suppose that $n(G) = 5$. Then, $\alpha'(G) = 2$ and by Claim E(b), $m(G) \leq 10$. If $m(G) = 10$, then $G = K_5$. In this case, H is a 4-uniform 2-regular linear intersecting hypergraph. However, H_{10} is the unique such hypergraph as shown, for example, in [11, 26]. Thus if $m(G) = 10$, then $H = H_{10}$, a contradiction. Hence, $m(G) \leq 9$. If $m(G) = 9$, then $G = K_5 - e$, where e denotes an arbitrary edge in K_5. In this case, $H = H_{11}$, a contradiction. Hence, $m(G) \leq 8$. Thus, $8n(G)+6m(G) \leq 8 \cdot 5+6 \cdot 8 = 88 < 90 = 45\alpha'(G)$, which by Claim F completes the proof in this case.

Finally suppose that $n(G) = 7$. Then, $\alpha'(G) = 3$ and by Claim E(b), $m(G) \leq 14$. Suppose that $m(G) = 14$. Then, G is a 4-regular graph of order 7. Equivalently, the complement, $\overline{G}$, of G is a 2-regular graph of order 7. If $\overline{G} = C_3 \cup C_4$, then $H = H_{14,2}$. If $\overline{G} = C_7$, then $H = H_{14,4}$. Both cases we produce a contradiction. Hence, $m(G) \leq 13$. Thus, $8n(G) + 6m(G) \leq 8 \cdot 7 + 6 \cdot 13 = 134 < 135 = 45\alpha'(G)$, which by Claim F completes the proof. (□)

Recall that $n(H') = n(H) + |Q|$ and $m(H') = m(H)$. By Claim D(a), Claim G and Claim H, $45\tau(H) = 45\tau(H') \leq 6n(H') + 13m(H') - 6|Q| = 6n(H)+13m(H)$, a contradiction. This completes the proof of Theorem 6. □

4.5 Proof of Theorem 3

We are finally in a position to present a proof of Theorem 3. Recall its statement.

Theorem 3. $q_4 = \frac{1}{5}$.

Proof of Theorem 3. Let $H \in \mathcal{L}_4$ have n vertices and m edges. We show that $\tau(H) \leq (n + m)/5$. We proceed by induction on n. If $n = 4$, then H consists of a single edge, and $\tau(H) = 1 = (n + m)/5$. Let $n \geq 5$ and suppose that the result holds for all hypergraphs in $\mathcal{L}_4$ on fewer than n vertices. Let $H \in \mathcal{L}_4$ have n vertices and m edges.

Suppose that $\Delta(H) \geq 4$. Let v be a vertex of maximum degree in H, and consider the 4-uniform, linear hypergraph $H' = H - v$ on n' vertices with m' edges. We note that $n' = n - 1$ and $m' = m - \Delta(H) \leq m - 4$. Every transversal in H' can be extended to a transversal in H by adding to it the vertex v. Hence, applying the inductive hypothesis to H', we have that $\tau(H) \leq \tau(H')+1 \leq (n'+m')/5+1 \leq (n+m-5)/5+1 = (n+m)/5$.

Hence, we may assume that $\Delta(H) \leq 3$, for otherwise the desired result follows. With this assumption, we note that $4m \leq 3n$. Applying Theorem 6 to the hypergraph H, we have

$$45\tau(H) \leq 6n(H) + 13m(H) + \text{def}(H).$$

If $\text{def}(H) = 0$, then $45\tau(H) \leq 6n(H) + 13m(H) = (9n + 9m) + (4m - 3n) \leq 9(n + m)$, and so $\tau(H) \leq (n + m)/5$. Hence, we may assume that $\text{def}(H) > 0$, for otherwise the desired result follows. Among all special non-empty H-sets, let X be chosen so that $|E^*(X)| - |X|$ is minimum. We note that since $\text{def}(H) > 0$, $|E^*(X)| - |X| < 0$. As in Section 4.2, we associate with the set X a bipartite graph, G_X, with partite sets X and $E^*(X)$, where an edge joins $e \in E^*(X)$ and $H' \in X$ in G_X if and only if the edge e intersects the subhypergraph H' of X in H. Suppose that there is no matching in G_X that matches $E^*(X)$ to a subset of X. By Hall's Theorem, there is a nonempty subset $S \subseteq E^*(X)$ such that $|N_{G_X}(S)| < |S|$. We now consider the special H-set, $X' = X \setminus N_{G_X}(S)$, and note that $|X'| = |X| - |N_{G_X}(S)| > |E^*(X)| - |S| \geq 0$ and $|E^*(X')| = |E^*(X)| - |S|$. Thus, X' is a special non-empty H-set satisfying

$$\begin{aligned}
|E^*(X')| - |X'| &= (|E^*(X)| - |S|) - (|X| - |N_{G_X}(S)|) \\
&= (|E^*(X)| - |X|) + (|N_{G_X}(S)| - |S|) \\
&< |E^*(X)| - |X|,
\end{aligned}$$

contradicting our choice of the special H-set X. Hence, there exists a matching in G_X that matches $E^*(X)$ to a subset of X. By Observation 2(f), there exists a minimum X-transversal, T_X, that intersects every edge in $E^*(X)$. By Observation 2, every special hypergraph F satisfies $\tau(F) \leq (n(F) + m(F))/5$. Hence, letting

$$n(X) = \sum_{F \in X} n(F) \quad \text{and} \quad m(X) = \sum_{F \in X} m(F),$$

we note that

$$|T_X| = \sum_{F \in X} \tau(F) \leq \sum_{F \in X} \frac{n(F) + m(F)}{5} = \frac{n(X) + m(X)}{5}.$$

We now consider the 4-uniform, linear hypergraph $H' = H - V(X)$ on n' vertices with m' edges. We note that $n' = n - n(X)$ and $m' = m - m(X) - |E^*(X)| \leq m - m(X)$. Every transversal in H' can be extended to a transversal in H by adding to it the set T_X. Hence, applying the inductive hypothesis to H', we have that

$$\begin{aligned}\tau(H) &\leq \tau(H') + |T_X| \\ &\leq \tfrac{1}{5}(n' + m') + |T_X| \\ &\leq \tfrac{1}{5}(n + m) - \tfrac{1}{5}(n(X) + m(X)) + |T_X| \\ &\leq \tfrac{1}{5}(n + m).\end{aligned}$$

Thus, $q_4 \leq \frac{1}{5}$. Since the affine plane $AG(2,4)$ exists, we know by Theorem 2(b) that $q_4 \geq \frac{1}{5}$. Consequently, $q_4 = \frac{1}{5}$. □

5 Applications of Theorem 3

In this section, we present a few applications to serve as motivation for the significance of our result given in Theorem 3.

5.1 Application 1

The following conjecture is posed in [22].

Conjecture 1 ([22]) *If H is a 4-uniform, linear hypergraph on n vertices with m edges, then $\tau(H) \leq \frac{n}{4} + \frac{m}{6}$.*

We remark that the linearity constraint in Conjecture 1 is essential. Indeed if H is not linear, then Conjecture 1 is not always true, as may be seen, for example, by taking H to be the complement of the Fano plane, F, shown in Figure 1. The second consequence of our main result proves Conjecture 1.

Theorem 7 *Conjecture 1 is true.*

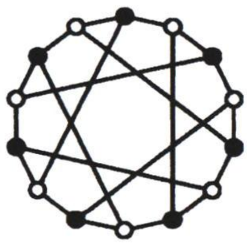
Proof. Let H be a 4-uniform, linear hypergraph on n vertices with m edges. We show that $\tau(H) \leq \frac{n}{4} + \frac{m}{6}$. We proceed by induction on n. If $n = 4$, then H consists of a single edge, and $\tau(H) = 1 < \frac{n}{4} + \frac{m}{6}$. Let $n \geq 5$ and suppose that the result holds for all 4-uniform, linear hypergraphs on fewer than n vertices. Let H be a 4-uniform, linear hypergraph on n vertices with m edges. Suppose that $\Delta(H) \leq 6$. In this case, $2m \leq 3n$. By Theorem 3, $60\tau(H) \leq 12n + 12m = 15n + 10m + 2m - 3n \leq 15n + 10m$, or, equivalently, $\tau(H) \leq \frac{n}{4} + \frac{m}{6}$. Hence, we may assume that $\Delta(H) \geq 7$, for otherwise the desired result follows from Theorem 3. Let v be a vertex of maximum degree in H, and consider the 4-uniform, linear hypergraph

$H' = H - v$ on $n' = n - 1$ vertices with m' edges. We note that $n' = n - 1$ and $m' = m - \Delta(H) \leq m - 7$. Every transversal in H' can be extended to a transversal in H by adding to it the vertex v. Hence, applying the inductive hypothesis to H', we have that $\tau(H) \leq 1 + \tau(H') \leq 1 + \frac{n'}{4} + \frac{m'}{6} \leq 1 + \frac{n-1}{4} + \frac{m-7}{6} < \frac{n}{4} + \frac{m}{6}$. □

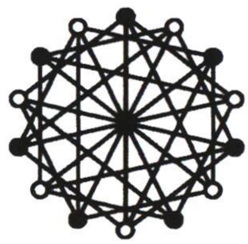
5.2 Application 2

A *total dominating set*, also called a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Total domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in a recent book on this topic that can be found in [24]. A survey of total domination in graphs can be found in [17].

The Heawood graph, shown in Figure 5(a), is the unique 6-cage. The bipartite complement of the Heawood graph, shown in Figure 5(b), is the bipartite graph formed by taking the two partite sets of the Heawood graph and joining a vertex from one partite set to a vertex from the other partite set by an edge whenever they are not joined in the Heawood graph.



(a) The Heawood graph



(b) The bipartite complement

Figure 5: The Heawood graph.

Thomassé and Yeo [37] established the following upper bound on the total domination number of a graph with minimum degree at least 4. Recall that $\delta(G)$ denotes the minimum degree of a graph G.

Theorem 8 ([37]) *If G is a graph of order n with $\delta(G) \geq 4$, then $\gamma_t(G) \leq \frac{3}{7}n$.*

The extremal graphs achieving equality in the Thomassé-Yeo bound of Theorem 8 are given by the following result.

Theorem 9 ([22, 24]) *If G is a connected graph of order n with $\delta(G) \geq 4$ that satisfies $\gamma_t(G) = \frac{3}{7}n$, then G is the bipartite complement of the Heawood Graph.*

We remark that every vertex in the bipartite complement of the Heawood Graph belongs to a 4-cycle. It is therefore a natural question to ask whether the Thomassé-Yeo upper bound of $\frac{3}{7}n$ can be improved if we restrict G to contain no 4-cycles. As a consequence of our main result Theorem 3, this question can now be answered in the affirmative. For a graph G, the *open neighborhood hypergraph*, abbreviated ONH, of G is the hypergraph H_G with vertex set $V(H_G) = V(G)$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V\}$ consisting of the open neighborhoods of vertices in G. As first observed in [37] (see also [24]), the transversal number of the ONH of a graph is precisely the total domination number of the graph; that is, for a graph G, we have $\gamma_t(G) = \tau(H_G)$.

As an application of Theorem 3, we have the following result, which significantly improves the upper bound of Theorem 8 when the graph G contains no 4-cycle.

Theorem 10 *If G is a quadrilateral-free graph of order n with $\delta(G) \geq 4$, then $\gamma_t(G) \leq \frac{2}{5}n$.*

Proof. Let G be a quadrilateral-free graph of order n with $\delta(G) \geq 4$ and let H_G be the ONH of G. Then, each edge of H_G has size at least 4. Since G is contains no 4-cycle, the hypergraph H_G contains no overlapping edges and is therefore linear. Let H be obtained from H_G by shrinking all edges of H_G, if necessary, to edges of size 4. Then, H is a 4-uniform linear hypergraph with n vertices and n edges; that is, $n(H) = m(H) = n(G) = n$. By Theorem 3 we note that $\tau(H) \leq \frac{1}{5}(n(H)+m(H)) = \frac{2}{5}n$. This completes the proof of the theorem since $\gamma_t(G) = \tau(H_G) \leq \tau(H)$. □

That the bound in Theorem 10 is best possible, may be seen by taking, for example, the 4-regular bipartite quadrilateral-free graph G_{30} of order $n = 30$ illustrated in Figure 6 satisfying $\gamma_t(G_{30}) = 12 = \frac{2}{5}n$. We note that the graph G_{30} is the incidence bipartite graph of the linear 4-uniform hypergraph obtained by removing an arbitrary vertex from the affine plane $AG(2,4)$ of order 4.

5.3 Application 3

There has been much interest in determining upper bounds on the transversal number of a 3-regular 4-uniform hypergraph. In particular, as a conse-

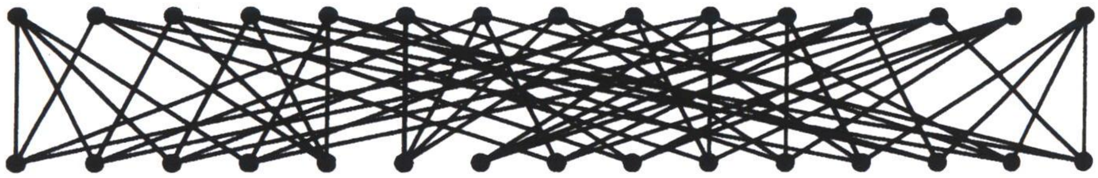


Figure 6: A quadrilateral-free 4-regular graph G_{30} of order $n = 30$ with $\gamma_t(G_{30}) = \frac{2}{5}n$.

quence of more general results we have the Chvátal-McDiarmid bound, the improved Lai-Chang bound, the further improved Thomassé-Yeo bound, and the recent bound given in [25]. These bounds are summarized in Theorem 11.

Theorem 11 *Let H be a 3-regular, 4-uniform hypergraph on n vertices. Then the following bounds on $\tau(H)$ have been established.*

(a) $\tau(H) \le \frac{5}{12}n \approx 0.41667\,n$ (Chvátal, McDiarmid [9]).
(b) $\tau(H) \le \frac{7}{18}n \approx 0.38888\,n$ (Lai, Chang [33]).
(c) $\tau(H) \le \frac{8}{21}n \approx 0.38095\,n$ (Thomassé, Yeo [37]).
(d) $\tau(H) \le \frac{3}{8}n \approx 0.375\,n$ (Henning, Yeo [25]).

The bound in Theorem 11(d) is best possible, due to the (non-linear) hypergraph, H_8, with $n = 8$ vertices and $\tau(H) = 3$ shown in Figure 7.

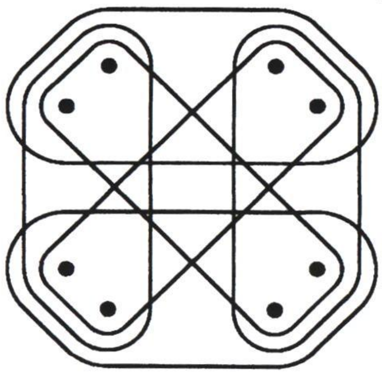


Figure 7: A 3-regular 4-uniform hypergraph, H_8, on n vertices with $\tau(H_8) = \frac{3}{8}n$.

A natural question is whether the upper bound in Theorem 11(d), namely $\tau(H) \le \frac{3}{8}n$, can be improved if we restrict our attention to linear hypergraphs. We answer this question in the affirmative. If H is a 3-regular, 4-uniform, linear hypergraph on n vertices with m edges, then

$m = \frac{3}{4}n$, and so, by Theorem 3, $\tau(H) \le \frac{1}{5}(n+m) = \frac{1}{5}(n + \frac{3}{4}n) = \frac{7}{20}n$. Hence, as an immediate corollary of Theorem 3, we have the following result.

Theorem 12 *If H is a 3-regular, 4-uniform, linear hypergraph on n vertices, then $\tau(H) \le \frac{7}{20}n = 0.35n$.*

5.4 Application 4

Lai and Chang [33] established the following upper bound on the transversal number of a 4-uniform hypergraph.

Theorem 13 ([33]) *If H is a 4-uniform hypergraph with n vertices and m edges, then $\tau(H) \le \frac{2}{9}(n+m)$.*

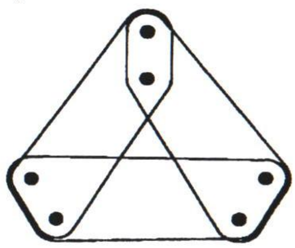


Figure 8: The hypergraph T_4.

The hypergraph T_4, illustrated in Figure 8, shows that the Lai-Chang bound is best possible, even if we restrict the maximum degree to be equal to 2. Our main result, namely Theorem 3, improves this upper bound from $\frac{2}{9}(n+m)$ to $\frac{1}{5}(n+m)$ in the case of linear hypergraphs. As an application of the proof of our main result, we show that the $\frac{1}{5}(n+m)$ bound can be further improved to $\frac{3}{16}(n+m) + \frac{1}{16}$ if we exclude the special hypergraph H_{10}.

For this purpose, we construct a family, $\mathcal{F}$, of 4-uniform, connected, linear hypergraphs with maximum degree $\Delta(H) = 2$ as follows. Let F_0 be the hypergraph with one edge (illustrated in Figure 2(a), but with a different name, H_4). For $i \ge 1$, we now build a hypergraph F_i inductively as follows. Let F_i be obtained from F_{i-1} by adding 12 new vertices, adding three new edges so that each new vertex belongs to exactly one of these added edges, and adding one further edge that contains a vertex in $V(F_{i-1})$ and three additional vertices, one from each of the three newly added edges, in such a way that $\Delta(F_i) = 2$. Let $\mathcal{F}$ be the family of all such hypergraphs, F_i, where $i \ge 0$. A hypergraph, F_6, in the family $\mathcal{F}$ is illustrated in Figure 9.

We proceed further with the following lemma. Let $c(H)$ denote the number of components of a hypergraph H. Recall that if X is a special

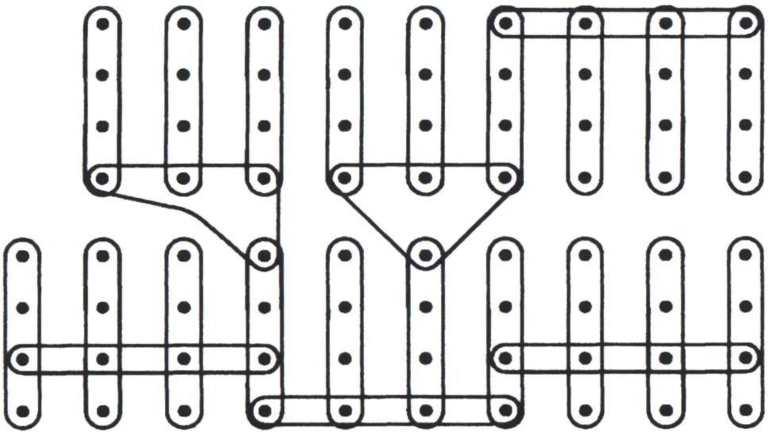


Figure 9: A hypergraph, F_6, in the family $\mathcal{F}$.

H-set, we write $E^*(X)$ to denote the set $E^*_H(X)$ if the hypergraph H is clear from context.

Lemma 14 *If H is a 4-uniform, linear hypergraph and X is a special H-set, then $3|E^*_H(X)| \geq |X| - c(H)$.*

Proof. We proceed by induction on $|E^*_H(X)| = k \geq 0$. If $k = 0$, then $c(H) \geq |X|$, and so $3|E^*(X)| = 0 \geq |X| - c(H)$. This establishes the base case. Suppose $k \geq 1$ and the result holds for special H-sets, X, such that $|E^*_H(X)| < k$. Let X be a special H-set satisfying $|E^*_H(X)| = k$. Let $e \in E^*_H(X)$ and consider the hypergraph $H' = H - e$. We note that H' is a 4-uniform, linear hypergraph, and that $c(H') \leq c(H) + 3$. Further, the set X is a special H'-set satisfying $|E^*_{H'}(X)| = k - 1$. Applying the inductive hypothesis to the hypergraph $H' \in \mathcal{L}_{4,3}$ and to the special H'-set, X, we have $3(|E^*_H(X)| - 1) = 3|E^*_{H'}(X)| \geq |X| - c(H') \geq |X| - (c(H) + 3)$, implying that $3|E^*_H(X)| \geq |X| - c(H)$. □

We are now in a position to state the following result, where H_{10}, $H_{14,5}$ and $H_{14,6}$ are the 4-uniform, linear hypergraphs shown in Figure 2(b), 2(h) and 2(i), respectively. As observed earlier, $H_4 = F_0$, and so $H_4 \in \mathcal{F}$.

Theorem 15 *Let $H \neq H_{10}$ be a 4-uniform, connected, linear hypergraph with maximum degree $\Delta(H) \leq 2$ on n vertices with m edges. Then, $\tau(H) \leq \frac{3}{16}(n + m) + \frac{1}{16}$, with equality if and only if $H \in \{H_{14,5}, H_{14,6}\}$ or $H \in \mathcal{F}$.*

Proof. Let $H \neq H_{10}$ be a 4-uniform, connected, linear hypergraph with maximum degree $\Delta(H) = 2$ on n vertices with m edges. Suppose firstly that H is a special hypergraph. By assumption, $H \neq H_{10}$. Since $\Delta(H) \leq 2$, we note that $H \in \{H_4, H_{11}, H_{14,5}, H_{14,6}\}$. If $H = H_{11}$, then by Observation 2(c), $\tau(H) = \frac{3}{16}(n+m)$. If $H \in \{H_4, H_{14,5}, H_{14,6}\}$, then by Observation 2(a) and 2(d), $\tau(H) = \frac{3}{16}(n+m) + \frac{1}{16}$. Hence, we may assume that H is not a special hypergraph, for otherwise the desired result holds, noting that $H_4 \in \mathcal{F}$.

Since $\Delta(H) \le 2$, we observe that $m \le \frac{1}{2}n$. By Theorem 6, $45\tau(H) \le 6n + 13m + \mathrm{def}(H)$. Suppose that $\mathrm{def}(H) = 0$. Then, since $0 \le \frac{1}{2}n - m$, we have

$$\begin{aligned} 45\tau(H) &\le 6n + 13m \\ &\le 6n + 13m + \tfrac{14}{3}(\tfrac{1}{2}n - m) \\ &= (6 + \tfrac{14}{6})n + (13 - \tfrac{14}{3})m \\ &= \tfrac{25}{3}(n+m), \end{aligned}$$

or, equivalently, $\tau(H) \le \frac{5}{27}(n+m) < \frac{3}{16}(n+m)$. Hence, we may assume that $\mathrm{def}(H) > 0$, for otherwise the desired result follows. Let X be a special H-set such that $\mathrm{def}(H) = \mathrm{def}_H(X)$. If H_{10} belongs to X, then, since $\Delta(H) \le 2$ and H is connected, $H = H_{10}$, a contradiction. If $H_{14,5}$ or $H_{14,6}$ belong to X, then, analogously, $H \in \{H_{14,5}, H_{14,6}\}$, contradicting our assumption that H is not a special hypergraph. Thus, if $F \in X$, then $F \in \{H_4, H_{11}\}$, noting that $\Delta(H) \le 2$. Recall that if F is a hypergraph, we denote by $n_1(F)$ the number of vertices of degree 1 in H. Let

$$n_1(X) = \sum_{F \in X} n_1(F),$$

and note that $n_1(H) \ge n_1(X) - 4|E^*(X)|$, since every edge in $E^*(X)$ contains at most four vertices whose degree is 1 in some subhypergraph $F \in X$. Since $\Delta(H) \le 2$ and H is 4-uniform, we note that $4m = 2n - n_1(H)$, or, equivalently, $n_1(H) = 2n - 4m$. Let $\beta = \frac{73}{64}$. If $F = H_4$, then $n_1(F) = 4$ and $\mathrm{def}(F) = 8 = (8 - 4\beta) + 4\beta = (8 - 4\beta) + n_1(F) \cdot \beta$. If $F = H_{11}$, then $n_1(F) = 2$ and $\mathrm{def}(F) = 4 < 5.71875 = 8 - 2\beta = (8 - 4\beta) + n_1(F) \cdot \beta$. Hence, if $F \in X$, then $\mathrm{def}(F) \le (8 - 4\beta) + n_1(F) \cdot \beta$, with strict inequality if $F = H_{11}$. Therefore,

$$\begin{aligned} \mathrm{def}(H) &= 8|X_4| + 4|X_{11}| - 13|E^*(X)| \\ &= \left(\sum_{F \in X} \mathrm{def}(F) \right) - 13|E^*(X)| \\ &\le (8 - 4\beta)|X| + n_1(X) \cdot \beta - 13|E^*(X)|, \end{aligned}$$

with strict inequality if $X \ne X_4$. By Lemma 14, $|E^*(X)| \ge \frac{1}{3}(|X| - 1)$. We also note that $n \ge 4|X_4| + 11|X_{11}| \ge 4|X|$, and so $|X| \le \frac{n}{4}$. Thus by our previous observations,

$$
\begin{aligned}
45\tau(H) &\le 6n + 13m + \mathrm{def}(H) \\
&\le 6n + 13m + (8-4\beta)|X| + n_1(X)\cdot\beta - 13|E^*(X)| \\
&\le 6n + 13m + (8-4\beta)|X| + (n_1(H) + 4|E^*(X)|)\cdot\beta - 13|E^*(X)| \\
&= 6n + 13m + (8-4\beta)|X| + n_1(H)\cdot\beta - |E^*(X)|(13-4\beta) \\
&\le 6n + 13m + (8-4\beta)|X| + n_1(H)\cdot\beta - \tfrac{1}{3}(|X|-1)(13-4\beta) \\
&= 6n + 13m + (8-4\beta-\tfrac{1}{3}(13-4\beta))|X| + (2n-4m)\cdot\beta + \tfrac{1}{3}(13-4\beta) \\
&\le 6n + 13m + (8-4\beta-\tfrac{1}{3}(13-4\beta))\tfrac{n}{4} + (2n-4m)\cdot\beta + \tfrac{1}{3}(13-4\beta) \\
&= (\tfrac{83}{12} + \tfrac{4}{3}\beta)n + (13-4\beta)m + \tfrac{1}{3}(13-4\beta) \\
&= (13-4\beta)(n+m) + \tfrac{1}{3}(13-4\beta) \\
&= \tfrac{135}{16}(n+m) + \tfrac{135}{48},
\end{aligned}
$$

or, equivalently, $\tau(H) \le \frac{3}{16}(n+m) + \frac{1}{16}$. This establishes the desired upper bound.

Recall that H is a 4-uniform, connected, linear hypergraph with maximum degree at most 2. Suppose that $\tau(H) = \frac{3}{16}(n+m) + \frac{1}{16}$. Then we must have equality throughout the above inequality chain. This implies that $X = X_4$, $V(H) = V(X)$, $n = 4|X|$, $E(H) = E(X) \cup E^*(X)$, and $|E^*(X)| = \frac{1}{3}(|X|-1)$. We show by induction on $n \ge 4$ that these conditions imply that $H \in \mathcal{F}$. When $n = 4$, $|X| = 1$ and $|E^*(X)| = 0$, and so $H = H_4 \in \mathcal{F}$. This establishes the base case. Suppose that $n > 4$. Thus, $|X| \ge 2$ and $n = 4|X| \ge 8$. We now consider the bipartite graph, G_X, with partite sets X and $E^*(X)$, where an edge joins $e \in E^*(X)$ and $H' \in X$ in G_X if and only if the edge e intersects the subhypergraph H' of X in H. Since H is 4-uniform and linear, each vertex in $E^*(X)$ has degree 4 in G_X. Let $n_1 = n_1(G)$, and so n_1 is the number of vertices of degree 1 in G. Counting the edges in G, we note that $\frac{4}{3}(|X|-1) = 4|E^*(X)| = m(G) \ge n_1 + 2(|X| - n_1)$, implying that $n_1 \ge \frac{1}{3}(2|X|+4)$. By the Pigeonhole Principle, there is a vertex of $E^*(X)$ adjacent in G to at least

$$
\frac{n_1}{|E^*(X)|} = \frac{\left(\frac{2|X|+4}{3}\right)}{\left(\frac{|X|-1}{3}\right)} = \frac{2|X|+4}{|X|-1} = 2 + \frac{6}{|X|-1}
$$

vertices of degree 1 (that belong to X). Thus, since $|X| \ge 2$ here, some vertex $e \in E^*(X)$ in G_X is adjacent to three vertex of degree 1, say x_1, x_2 and x_3. Let x_4 be the remaining neighbor of e in G_X. We now consider the hypergraph H' obtained from H by deleting the 12 vertices from the three special H_4-subhypergraphs, say F_1, F_2, and F_3, corresponding to x_1, x_2 and x_3, respectively, and deleting the hyperedge corresponding to e. Since

H is connected and linear, so too is H'. Let $X' = X \setminus \{F_1, F_2, F_3\}$, and so $|X'| = |X| - 3$. We note that $|E^*_{H'}(X')| = |E^*_H(X)| - 1 = \frac{1}{3}(|X| - 1) - 1 = \frac{1}{3}(|X'| - 1)$. Further, $X' = X_4$, $V(H') = V(X')$, $n' = 4|X'|$, and $E(H') = E(X') \cup E^*_{H'}(X')$. Applying the inductive hypothesis to H', we deduce that $H' \in \mathcal{F}$. The original hypergraph H can now be reconstructed from H' by adding back the three deleted edges and 12 deleted vertices in $F_1 \cup F_2 \cup F_3$, and adding back the deleted edge e that contains the vertex $x_4 \in V(H')$ and contains one vertex from each edge in F_1, F_2 and F_3. Thus, $H \in \mathcal{F}$. This completes the proof of Theorem 15. □

6 Closing Comments

For small $k \in \{2, 3, 4\}$, we now know that $q_k = \frac{1}{k+1}$. The asymptotic behaviour of q_k as k grows is of the order $\ln(k)/k$. It would be extremely interesting to determine precisely the value of q_k for every $k \geq 5$, and we pose this as an open problem.

Problem 1 *Determine the precise value of q_k for any $k \geq 5$.*

The amount of work to show that $q_4 = \frac{1}{5}$ suggests that Problem 1 may be difficult, even in the special case when $k = 5$. Recall that by Observation 1, $q_k \geq \frac{1}{k+1}$ for all $k \geq 2$. By Theorems 2 and 3, we have $q_k = \frac{1}{k+1}$ for $k \in \{2, 3, 4\}$. It is shown in [27] that for $k \geq 60$, we have $q_k > \frac{1}{k+1}$. We pose the following open problem.

Problem 2 *Determine the smallest value, $k_{\min}$, of k for which $q_k > \frac{1}{k+1}$ holds.*

From our earlier observations and results, we note that $5 \leq k_{\min} \leq 60$. We conjecture that $k_{\min} \geq 6$. Equivalently, we conjecture that $q_5 = \frac{1}{6}$.

References

[1] N. Alon, Transversal number of uniform hypergraphs. *Graphs Combin.* **6** (1990), 1–4.

[2] N. Alon, G. Kalai, J. Matousek and R. Meshulam, Transversal numbers for hypergraphs arising in geometry. *Advances Applied Math.* **29** (2002), 79–101.

[3] J. Balogh, R. Morris, and W. Samotij, Independent sets in hypergraphs. *J. Amer. Math. Soc.* **28** (2015), 669–709.

[4] C. Berge, *C. R. Acad. Sci. Paris Ser. I Math.* **247,** (1958) 258–259 and *Graphs and Hypergraphs* (Chap. 8, Theorem 12), North-Holland, Amsterdam, 1973.

[5] V. Blinovsky and C. Greenhill, Asymptotic enumeration of sparse uniform linear hypergraphs with given degrees. *Electronic J. Combin.* **23**(3) (2016), #P3.17.

[6] C. Bujtás, M. A. Henning and Zs. Tuza, Transversals and domination in uniform hypergraphs. *European J. Combin.* **33** (2012), 62–71.

[7] C. Bujtás, M. A. Henning, Zs. Tuza and A. Yeo, Total transversals and total domination in uniform hypergraphs. *Electronic J. Combin.* **21**(2) (2014), #P2.24.

[8] D. Caen and Z. Füredi, The maximum size of 3-uniform hypergraphs not containing a Fano plane. *J. Combin. Theory B* **78** (2000), 78–80.

[9] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs. *Combinatorica* **12** (1992), 19–26.

[10] E. J. Cockayne, S. T. Hedetniemi and P. J. Slater, Matchings and transversals in hypergraphs, domination and independence in trees. *J. Combin. Theory B* **27** (1979), 78–80.

[11] M. Dorfling and M. A. Henning, Linear hypergraphs with large transversal number and maximum degree two. *European J. Combin.* **36** (2014), 231-–236.

[12] P. Erdős and Zs. Tuza, Vertex coverings of the edge set in a connected graph. In: *Graph Theory, Combinatorics, and Applications* (Y. Alavi and A. J. Schwenk, eds.), John Wiley & Sons, 1995, pp. 1179–1187.

[13] P. Frankl and Z. Füredi, Finite projective spaces and intersecting hypergraphs. *Combinatorica* **6** (1986), 335–354.

[14] Z. Füredi, Linear trees in uniform hypergraphs. *European J. Combin.* **35** (2014), 264–272.

[15] K. Gundersona and J. Semeraro, Tournaments, 4-uniform hypergraphs, and an exact extremal result. *J. Combin. Theory B* **126** (2017), 114–136.

[16] P. Hall, On representation of subsets. *J. London Math. Soc.* **10** (1935), 26–30.

[17] M. A. Henning, Recent results on total domination in graphs: A survey. *Discrete Math.* **309** (2009), 32–63.

[18] M. A. Henning and C. Löwenstein, Hypergraphs with large transversal number and with edge sizes at least four. *Central European J. Math.* **10**(3) (2012), 1133–1140.

[19] M. A. Henning and C. Löwenstein, A characterization of the hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem. *Discrete Math.* **323** (2014), 69—75.

[20] M. A. Henning and A. Yeo, Hypergraphs with large transversal number and with edge sizes at least three. *J. Graph Theory* **59** (2008), 326–348.

[21] M. A. Henning and A. Yeo, Strong transversals in hypergraphs and double total domination in graphs. *SIAM J. Discrete Math.* **24**(4) (2010), 1336–1355.

[22] M. A. Henning and A. Yeo, Hypergraphs with large transversal number. *Discrete Math.* **313** (2013), 959–966.

[23] M. A. Henning and A. Yeo, Transversals and matchings in 3-uniform hypergraphs. *European J. Combin.* **34** (2013), 217–228.

[24] M. A. Henning and A. Yeo, *Total domination in graphs (Springer Monographs in Mathematics)*. ISBN-13: 978-1461465249 (2013).

[25] M. A. Henning and A. Yeo, Transversals in 4-uniform hypergraphs. *Electronic J. Combin.* **23**(3) (2016), #P3.50.

[26] M. A. Henning and A. Yeo, Transversals and independence in linear hypergraphs with maximum degree two. *Electronic J. Combin.* **24**(2) (2017), #P2.50.

[27] M. A. Henning and A. Yeo, Uniform linear hypergraphs with large transversal number, manuscript.

[28] M. A. Henning and A. Yeo, Transversals in linear uniform hypergraphs. Developments in Mathematics, vol 63. Springer, Cham. (2020), 229 pp. DOI: 10.1007/978-3-030-46559-9

[29] S. Khan and B. Nagle, A hypergraph regularity method for linear hypergraphs: With applications. Lap Lambert Academic Publishing (2011), ISBN-10: 3844388397.

[30] D. König, Graphen und Matrizen. *Math. Riz. Lapok* **38** (1931), 116–119.

[31] Y. Kohayakawa, B. Nagle, V. Rödl, and M. Schacht, Weak hypergraph regularity and linear hypergraphs. *J. Combin. Theory B* **100** (2010), 151–160.

[32] A. Kostochka, D. Mubayi, and J. Verstaëte, On independent sets in hypergraphs. *Random Structures & Algorithms* **44** (2014), 224–239.

[33] F. C. Lai and G. J. Chang, An upper bound for the transversal numbers of 4-uniform hypergraphs. *J. Combin. Theory Ser. B* **50** (1990), 129–133.

[34] Z. Lonc and K. Warno, Minimum size transversals in uniform hypergraphs. *Discrete Math.* **313**(23) (2013), 2798–2815.

[35] Y. Metelsky and, R. Tyshkevich, On line graphs of linear 3-uniform hypergraphs. *J. Graph Theory* **25** (1997), 243–251.

[36] R. N. Naik, S. B. Rao, S. S. Shrikhande, and N. M. Singhi, Intersection graphs of k-uniform linear hypergraphs. *European J. Combin.* **3** (1982), 159–172.

[37] S. Thomassé and A. Yeo, Total domination of graphs and small transversals of hypergraphs. *Combinatorica* **27** (2007), 473–487.

[38] I. Tomescu, Chromatic coefficients of linear uniform hypergraphs. *J. Combin. Theory B* **72** (1998), 229–235.

[39] Zs. Tuza, Covering all cliques of a graph. *Discrete Math.* **86** (1990), 117–126.