Vertex arboricity of cographs*

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Dedicated to Gary MacGillivray on the occasion of his milestone birthday.

Abstract

Arboricity is a graph parameter akin to chromatic number, in that it seeks to partition the vertices into the smallest number of sparse subgraphs. Where for the chromatic number we are partitioning the vertices into independent sets, for the arboricity we want to partition the vertices into cycle-free subsets (i.e., forests). Arboricity is NP-hard in general, and our focus is on the arboricity of cographs. For arboricity two, we obtain the complete list of minimal cograph obstructions. These minimal obstructions do generalize to higher arboricities; however, we no longer have a complete list, and in fact, the number of minimal cograph obstructions grows exponentially with arboricity. We obtain bounds on their size and the height of their cotrees.

More generally, we consider the following common generalization of colouring and partition into forests: given non-negative integers p and q, we ask if a given cograph G admits a vertex partition into p forests and q independent sets. We give a polynomial-time dynamic programming algorithm for this problem. In fact, the algorithm solves a more general problem which also includes several other problems such as finding a maximum q-colourable subgraph, maximum subgraph of arboricity-p, minimum feedback vertex set and minimum q of a q-colourable feedback vertex set.

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1 Introduction

The *vertex-arboricity* of a graph G is the minimum p such that the vertices of G can be partitioned into p subsets each of which *induces a forest.* We contrast this with the *chromatic number* of G, which is the minimum number q such that the vertices of G can be partitioned into q subsets each of which is *independent.* Like the chromatic number, determining the vertex arboricity of graphs is NP-hard in general [2]. We focus our attention on the class of cographs, where both problems are polynomial-time solvable. We define a common generalization as follows. We say that a graph G is (p, q)-*partitionable* if the vertex set of G can be partitioned into p forests and q independent sets. This problem is NP-hard in general as well, as long as $2p + q \geq 3$ (and is polynomial-time solvable otherwise). So, the study of this problem for restricted graph families comes as a natural problem. For instance, it is easy to verify whether a chordal graph is (p, q)-partitionable. A chordal graph G is (p, q)-partitionable if and onl if it does not contain K_{2p+q+1} as an induced subgraph. This follows directly from the results in [8].

Cographs turn out to be more interesting than chordal graphs. It follows from [5] that this problem has a polynomial-time algorithm for any p, q, when restricted to cographs. Moreover, it follows from [5] that the number of minimal obstructions for cograph (p, q)-partitionability is finite, for any p, q. We investigate such minimal obstructions for $(p, 0)$-partitionability of cographs, i.e., for arboricity p. We give a complete answer only for arboricity 2, and give some useful information for general p. We also give a concrete dynamic programming algorithm to decide whether a cograph is (p, q)-partitionable after the deletion of at most r vertices. This last problem, allowing the deletion of vertices, is natural for the dynamic programming algorithm, but it is an interesting problem which can be formulated as follows.

Let p, q and r be non-negative integers and let G be a graph. A (p, q, r)-*partition* of G is a partition (P, Q, R) of its vertex set such that the subgraph induced on P has vertex-arboricity p, the subgraph induced on Q is q-colourable, and R has at most r vertices. We say that G is (p, q, r)-*partitionable* if it admits a (p, q, r)-partition, and we say that G is a minimal (p, q, r)-obstruction if it is not (p, q, r)-partitionable but every induced subgraph is. (When $r = 0$, we simplify $(p, q, 0)$ to (p, q) in all the notation.) Note that finding the minimum r such that G is $(0, q, r)$-partitionable is the well-known problem of finding the maximum q-colourable subgraph; finding the minimum r such that G is $(p, 0, r)$-partitionable is the problem of finding the maximum subgraph of arboricity p; finding the minimum r

such that G is $(1, 0, r)$-partitionable is the minimum feedback vertex set[1] problem; finding the minimum q such that G is $(1, q, 0)$-partitionable is the problem of finding the smallest q such that G has a q-colourable feedback vertex set.

If G and H are graphs, then we denote the disjoint union of G and H by $G+H$, and so, if n is a positive integer, the disjoint union of n different copies of G will be denoted by nG. The join of G and H will be denoted by $G \oplus H$.

A *cograph* is a graph than can be obtained recursively from the following rules

- K_1 is a cograph.
- If G is a cograph, then $\overline{G}$ is a cograph.
- If G and H are cographs, then $G + H$ is a cograph.

There are many interesting characterizations of the family of cographs [3], but there are two that are particularly useful when dealing with minimal obstructions for a hereditary property. A graph is a cograph if and only if it is P_4-free, if and only if the complement of any of its nontrivial connected subgraphs is disconnected. Notice also that the complement operation can be replaced by the join of two graphs $(G \oplus H)$.

Recall that a *star* is a complete bipartite graph $K_{1,s}$ for some natural number s. A *star forest* is a forest where every connected component is a star.

The rest of the article is organized as follows. In Section 2, cographs that are $(2, 0, 0)$-partitionable are characterized in terms of 7 minimal obstructions; some families of cograph minimal obstructions for $(p, 0, 0)$-partitions are studied. In Section 3 we consider minimal obstructions for $(1, q, 0)$-partitions, and notice how these partitions are related to the independent feedback vertex set problem. Although finite, the cograph minimal obstructions for $(p, 0, 0)$-partition can be very large, both in size and in number, Section 4 is devoted to present lower and upper bounds for these parameters, as well as an upper bound on the height of the cotree of a minimal obstruction. A polynomial algorithm to determine the arboricity of a cograph is presented in Section 5. In Section 6 we present conclusions and related open problems.

[1]Given a graph G, a feedback vertex set of G is a subset S of V such that $G - V$ is acyclic.

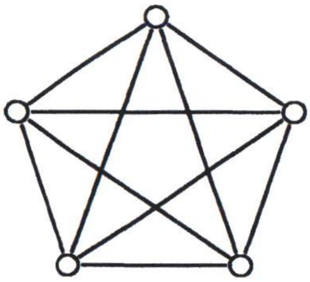
2 All Minimal Cograph Obstructions for Arboricity 2

Note that a graph has arboricity one if and only if it has no cycles. Thus there are precisely two cograph minimal obstructions for arboricity one, the cycles $C_3 = K_3$ and $C_4 = \overline{2K_2}$.

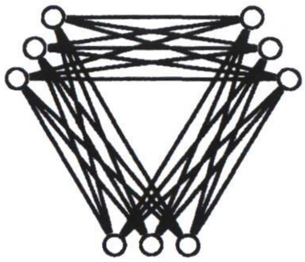
We now introduce a family of cographs $\mathcal{A}_2$ consisting of

$$\{K_5, \overline{3K_3}, 2K_3 \oplus \overline{K_2}, 2\left(\overline{2K_2}\right) \oplus \overline{K_3}, \overline{2K_2} \oplus (K_1 + K_2), \left(\overline{2K_2} + K_3\right) \oplus \overline{K_2}, \overline{3K_2 + K_1}\}.$$

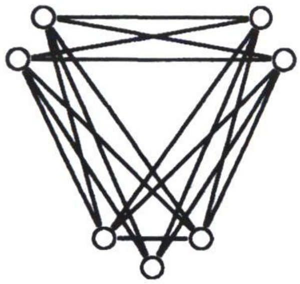
These graphs are depicted in Figure 1.



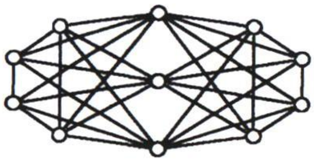
(a) K_5



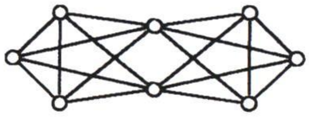
(b) $\overline{3K_3}$



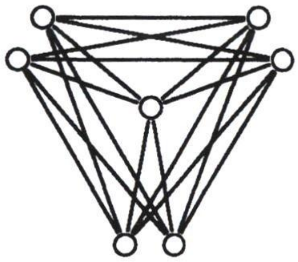
(c) $\overline{2K_2} \oplus (K_1 + K_2)$



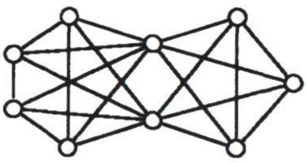
(d) $2\left(\overline{2K_2}\right) \oplus \overline{K_3}$



(e) $2K_3 \oplus \overline{K_2}$



(f) $\overline{3K_2 + K_1}$



(g) $\left(\overline{2K_2} + K_3\right) \oplus \overline{K_2}$

Figure 1: The family $\mathcal{A}_2$.

Lemma 1. *Each graph in $\mathcal{A}_2$ is a cograph minimal obstruction to arboricity 2.*

Proof. It is clear from the descriptions that each graph in $\mathcal{A}_2$ is a cograph.

We claim that each of them is not partitionable into two forests, but whenever a vertex is deleted, it becomes so partitionable.

We prove the first, fourth and last cases (K_5, $2(\overline{2K_2}) \oplus \overline{K_3}$, $\overline{3K_2 + K_1}$); the rest of the cases can be handled similarly. Consider first $G = K_5$. It is clear that in a complete graph each (acyclic) colour class has at most 2 vertices, therefore G is a $(2,0,0)$-obstruction. To check the minimality, remove any vertex of G, the remaining graph is a K_4 which is easily 2-colourable. Therefore, G is actually a minimal $(2,0,0)$-obstruction.

Let us assume that $G = 2(\overline{2K_2}) \oplus \overline{K_3}$. First notice that to colour $\overline{K_3}$ by using only 2 colours, at least two vertices receive the same colour, and then we can use that colour on at most one other vertex outside $\overline{K_3}$. Since we cannot use this colour anywhere else, without loss of generality we can assume that all vertices of $\overline{K_3}$ have the same colour and we are using this colour on one other vertex as well. So we have coloured (at most) 4 vertices using one colour. There are two disjoint copies of $\overline{2K_2}$ (minus one vertex) still uncoloured. On each of the $\overline{2K_2}$ copies we can use one colour for at most 3 vertices. Therefore, at most 6 vertices can be coloured, and thus at most $4 + 6 = 10$ vertices can be coloured with 2 colours. Since G has 11 vertices, we conclude that G is a $(2,0,0)$-obstruction. To verify minimality, first consider the case where $v \in \overline{K_3}$. Then we can colour one vertex of $\overline{K_3}$ along with one copy of $\overline{K_2}$ in each of the two $\overline{2K_2}$ parts to colour $G - v$. Now consider the case where v belongs to $2(\overline{2K_2})$. Let v' be the duplicate of v in the other copy of $\overline{2K_2}$ and colour v' along with all vertices of $\overline{K_3}$ using one colour. The remaining vertices induce an acyclic graph and can be coloured with one colour. This shows that $G - v$ is 2 (acyclic) colourable. Therefore G is a minimal $(2,0,0)$-obstruction.

For our final case, let $G = \overline{3K_2 + K_1}$. Notice that G has 7 vertices and each (acyclic) colour class has size at most 3 (two vertices of a $\overline{K_2}$ and another vertex). Therefore G is a $(2,0,0)$-obstruction. By removing any vertex of G we will have 6 vertices and at least 2 copies of $\overline{K_2}$. Considering each of these $\overline{K_2}$'s along with some other vertex (maybe in another $\overline{K_2}$) will give us an acyclic colouring where each colour class has size exactly 3. Thus, G is actually a minimal $(2,0,0)$-obstruction. □

Theorem 2. *A cograph has vertex arboricity* 2 *if and only if it is* $\mathcal{A}_2$*-free.*

Proof. Let G be a cograph. If the vertex arboricity of G is at most 2, then it is clearly $\mathcal{A}_2$-free. So, suppose that G is $\mathcal{A}_2$-free. We may assume without loss of generality that G is connected.

Since G is a connected cograph, there exist cographs G_1 and G_2 such that $G = G_1 \oplus G_2$. If G_1 and G_2 are forests, we are done. So, at least one of them must contain an induced cycle. Without loss of generality suppose that it is G_1. Since G is a cograph, this cycle should be a triangle or a 4-cycle. Suppose that G_1 is triangle free, then it must contain a 4-cycle. Since G is K_5-free, then G_2 is triangle-free, and, since G is $(\overline{3K_2 + K_1})$-free, then G_2 is P_3-free, i.e., G_2 is a disjoint union of K_1's and K_2's; but G is $(C_4 \oplus (K_1 + K_2))$-free, then G_2 is $(K_1 + K_2)$-free, and thus, it is either a K_2, or an empty graph. As a triangle-free cograph, G_1 is bipartite, with bipartition (X, Y). If $|V(G_2)| \leq 2$, colour red one vertex in G_2, together with all the vertices in X, and colour blue the other vertex in G_2 (possibly none), together with the vertices in Y, this colouring of G realizes the vertex arboricity 2. If $|V(G_2)| \geq 3$, as G is $(2C_4 \oplus 3K_1)$-free, we have that every component of G_1, different from the one containing the induced 4-cycle, is a star. The component of G_1 containing the induced 4-cycle is a bipartite connected cograph, and thus, it is a complete bipartite graph; moreover, since G is $\overline{3K_3}$-free, one of the two parts of this component has less than three vertices. colour red one of the vertices in this small part, together with all the vertices in G_2, and colour blue all the remaining vertices of G. Clearly, the red vertices induce a star, and the blue vertices induce a disjoint union of stars.

Now, suppose that G_1 contains an induced triangle. Using again that G is K_5-free, we conclude that G_2 is an empty graph. Let the set $\{v_1, v_2, v_3\}$ induce a triangle in G_1, and let B be the component of G_1 containing it. Now, B is a connected cograph, and there are cographs B_1 and B_2 such that $B = B_1 \oplus B_2$. Recall that G_1 is K_4-free, and thus, both B_1 and B_2 are triangle-free, so, we assume without loss of generality that $v_1, v_2 \in V(B_1)$, and $v_3 \in V(B_2)$; moreover, B_2 must be an independent set.

We will consider two cases; suppose first that G_2 has at least two vertices. Then, since G is $((K_3 + C_4) \oplus 2K_1)$ and $(2K_3 \oplus 2K_1)$-free, we have that G_1 has precisely one component, namely B, which is not acyclic. Again, we have two cases. First, suppose that B_2 has at least two vertices. From the fact that G is $(C_4 \oplus (K_1 + K_2))$-free, we obtain that B_1 is connected, and, since G is $(\overline{3K_2 + K_1})$-free, then B_1 is a path on two vertices, x and y. So, we can colour x, together will all the vertices in G_2 red, and the rest of the vertices in G blue; it is not hard to verify that each colour class induces a forest. So, we may now suppose that B_2 has only one vertex. Since B_1 is triangle-free, it is bipartite with bipartition (X, Y). Again, as G is $(\overline{3K_2 + K_1})$-free, we have that B_1 is acyclic, and we can colour the only vertex in B_2 together with all the vertices in G_2 red, and the rest of the vertices in G blue; again, each colour class induces a forest.

As a second case, suppose that G_2 has only one vertex v. Since G is $\{K_5, \overline{3K_2 + K_1}\}$-free, and v is a universal vertex in G, we have that G_1 is

$\{K_4, \overline{3K_2}\}$-free. It follows from Theorem 4 with $q = 1$ that G_1 contains an independent feedback vertex set, S. Thus, colouring v together with the vertices in S red, and the rest of the vertices of G blue, clearly yields acyclic colour classes. □

The family $\mathcal{A}_2$ has a natural generalization for higher arboricity. Let p be an integer, $p \geq 2$, and denote by $\mathcal{A}_p$ the following family of cographs.

- K_{2p+1}
- $\overline{(p+1)K_{p+1}}$
- $2K_{2p-1} \oplus \overline{K_2}$
- $2\overline{pK_p} \oplus \overline{K_{p+1}}$
- $\overline{pK_2} \oplus (K_1 + K_p)$
- $(\overline{pK_p} + K_{2p-1}) \oplus \overline{K_2}$
- $\left\{\overline{(p+1+i)K_2 + (p-1-2i)K_1}\right\}_{0 \leq i \leq \lfloor \frac{p-1}{2} \rfloor}$

Lemma 3. *Let p be an integer, $p \geq 2$. Each graph in the family $\mathcal{A}_p$ is a cograph minimal obstruction for arboricity p.*

Proof. The proof is analogous to that of Lemma 1. □

There is however, no analogue to Theorem 2. In fact, in Section 4 we prove that the number of cograph minimal obstructions for arboricity p grows exponentially with p.

3 Minimal Cograph Obstructions for q-Colourable Feedback Vertex Set

By analogy with the independent feedback vertex set problem, we say that a cograph G has a *q-colourable feedback vertex set* if it has a $(1, q)$-partition. It turns out there are exactly two minimal cograph obstructions for $(1, q)$-partition, namely, the complete graph K_{q+3}, and the complete $(q+2)$-partite graph with two vertices in each part, $\overline{(q+2)K_2}$.

Theorem 4. *Let q be a non-negative integer. A cograph G has a q-colourable feedback vertex set if and only if it is $\left\{K_{q+3}, \overline{(q+2)K_2}\right\}$-free.*

Proof. Clearly, a $(1, q, 0)$-partitionable cograph is $\left\{K_{q+3}, \overline{(q+2)K_2}\right\}$-free. We prove the converse by induction on q. The base case $q = 0$ follows from the simple fact that a cograph is a forest is and only if it is $\{K_3, \overline{2K_2}\}$-free. Suppose that the claim holds for all $\ell < q$, and let G be a $\left\{K_{q+3}, \overline{(q+2)K_2}\right\}$-free cograph. Without loss of generality, we may assume G is connected. The fact that G is a cograph and K_{q+3}-free implies $\chi(G) \leq q + 2$, and the claim holds by taking an independent set as the forest if $\chi(G) \leq q + 1$, so we may assume $\chi(G) = q + 2$.

Since G is a connected cograph, there exists a family of cographs $\{G_i\}_{i=1}^{s}$ such that $G = \bigoplus_{i=1}^{s} G_i$. Notice that in any $(q + 2)$-colouring of G, each colour class is contained in $V(G_i)$ for some $i \in \{1, 2, \ldots, s\}$. If there is a $(q + 2)$-colouring of G with a colour class with a single vertex v, then v together with any other colour class induces a forest. By taking this forest and the remaining q colour classes we obtain a $(1, q, 0)$-partition of G, so we may assume that every colour class has at least two vertices.

Since $\sum_{i=1}^{s} \chi(G_i) = q + 2$, if G_i has an induced $\overline{\chi(G_i)K_2}$ for every $i \in \{1, \ldots, s\}$, then G has an induced copy of $\overline{(q+2)K_2}$, thus G_ℓ is $\left(\overline{\chi(G_j)K_2}\right)$-free for some $\ell \in \{1, \ldots, s\}$. By induction hypothesis, G_ℓ has a $(1, \chi(G_\ell) - 2, 0)$-partition (notice that if $\chi(G_\ell) = 1$, the fact that each colour class has at least two vertices would imply the existence of an induced $\overline{K_2}$, and so we must have that $\chi(G_\ell) \geq 2$). Since $G' = G - V(G_\ell)$ has chromatic number $q + 2 - \chi(G_\ell)$, a proper colouring of G', together with the $(1, \chi(G_\ell) - 2, 0)$-partition of G_ℓ gives us a $(1, q, 0)$-partition of G. □

We can use the Theorem to derive a min-max relationship. For the purposes of its statement, we shall call K_s a *thin s-clique* and $\overline{sK_2}$ a *thick s-clique*. The *strength* of a thin s-clique is defined to be s, and the strength of a thick s-clique is defined as $s + 1$. We let $s(G)$ denote the maximum strength of a (thin or thick) clique in G. We also let $q(G)$ denote the minimum number of colours q such that G admits a q-colourable feedback vertex set.

Corollary 5. *If G is a cograph, then $q(G) = s(G) - 2$.*

We note that the maximum strength of a clique in a cograph can be computed by a cotree bottom-up procedure analogous to the well-known algorithm for computing the size of a maximum complete subgraph of a cograph [3].

4 Bounds on Minimal Cograph Obstructions for Arboricity p

As we mentioned at the end of Section 2, we do not have a complete description for all cograph minimal obstructions for arboricity p, $p > 2$. In this section we illustrate the fact that there are exponentially many. We will construct this obstructions as joins of star forests.

Proposition 6. *Let p be an integer, $p \geq 2$. There are at least $\frac{e^{2\cdot\sqrt{p}}}{14}$ minimal obstructions for arboricity p.*

Proof. Let us first observe that if we add some edges to a minimal obstruction, the resulting graph is still an obstruction but might not be minimal. Consider the complete multipartite graph $\overline{(p+1)K_{p+1}}$ (which is a minimal obstruction for arboricity p), and add edges to i of the parts to make them non-empty forests.

Let $\mathcal{F}_p$ be the set of all non-empty forests on p vertices which are cographs. Notice that a tree cograph on a fixed number of vertices is unique (it is a star). Therefore, $|\mathcal{F}_p| = \pi(p) - 1$, where $\pi(p)$ is the is the partition number of p (the number of possible partitions of p) . In [7] the following lower bound is proved for π

$$\frac{e^{2\cdot\sqrt{p}}}{14} < \pi(p).$$

Let i be an integer, $0 \leq i \leq p$, and define the graph $\mathcal{O}_i(f_1, \ldots, f_i)$ as $\mathcal{O}_i(f_1, \ldots, f_i) = \overline{(p+1-i)K_{p+1-i}} \oplus f_1 \oplus f_2 \ldots \oplus f_i$, where $f_j \in \mathcal{F}_{p+2-i}$ for $j \in \{1, \ldots, i\}$. Notice that when $i = 0$, $\mathcal{O}_i$ does not receive arguments, and we obtain $\overline{(p+1)K_{p+1}}$, and when $i = p$ we have that f_j is isomorphic to K_2 for every $j \in \{1, \ldots, p\}$, and thus, the only possible graph that can be obtained is K_{2p+1}. We denote the set of all graphs $\mathcal{O}_i(f_1, \ldots, f_i)$ for all selections of $f_j \in \mathcal{F}_{p+2-i}$ by $\mathcal{O}_i$.

Claim 1. *For every $i \in \{0, \ldots, p\}$, each $G \in \mathcal{O}_i$ is not p-colourable.*

Let G be a member of $\mathcal{O}_i$. The number of vertices of G is $(p+1-i)^2 + (i)(p+2-i) = p^2 + 2p - ip + 1 = p(p+2-i) + 1$. On the other hand, notice that we can use each colour class in at most two parts of G (otherwise we will get a monochromatic cycle). Also if we want to use a colour in two parts, in one of them we are using it at most once (or we will get a monochromatic C_4). Since each f_j is non-empty, if we colour it with just one colour, then we cannot use that colour for any other vertices (or we will get a monochromatic triangle). Therefore, in each colour class we can have at most $p+2-i$ vertices. Hence, using p colours we can colour at most $p(p+2-i)$ vertices of G. Therefore G is not p-colourable for $i \in \{0, \dots, p\}$. ■

Claim 2. *For every $i \in \{0, \dots, p\}$, every $G \in \mathcal{O}_i$ is a minimal obstruction for arboricity p.*

We have proved that G is an obstruction, now we need to show that it is minimal. If we remove a vertex from f_j, then we can use each colour for one vertex of f_j and one of the parts of $\overline{(p+1-i)K_{p+1-i}}$, which is an independent set (using $p+1-i$ colours). Also we can use one colour for each of the remaining f_l (using $i-1$ colours).

If we remove a vertex from a part of $\overline{(p+1-i)K_{p+1-i}}$, then we can colour the remaining graph by using only p colours; use one colour for each vertex of this part together with all the vertices in another of the parts, i.e., use each vertex of this part as the center of a star having all the vertices in some other part as leaves (using $p-i$ colours) and use one colour for each f_j (using i colours). ■

Now we need to calculate the number of different elements in $\mathcal{O}_i$. For a fixed value of i, we defined $\mathcal{O}_i$ to be the set of all graphs $\mathcal{O}_i(f_1, \dots, f_i)$ for all selections of $f_j \in \mathcal{F}_{p+2-i}$. The number of such selections is $(\pi(p+2-i)-1)^i$, but this formula counts all the different permutations of a set of i elements in $\mathcal{F}_{p+2-i}$, which result in isomorphic graphs. We can overcome this problem by dividing the previous formula by $i!$. It is easy to see that for two different sets of forests we will get different obstructions. So we have

$$|\mathcal{O}_i| = \frac{(\pi(p+2-i)-1)^i}{i!}.$$

Notice that if $i \neq j$, then members of $\mathcal{O}_i$ and $\mathcal{O}_j$ have different number of vertices and therefore they are different. So the total number of different minimal obstruction that we will get from this structure is

$$\sum_{i=0}^{p} |\mathcal{O}_i| = \sum_{i=0}^{p} \frac{(\pi(p+2-i)-1)^i}{i!} \geq \sum_{i=0}^{p} \frac{e^{2i\sqrt{p+2-i}}}{14^i i!} > \frac{e^{2\sqrt{p}}}{14}.$$

□

Next we focus on upperbound for minimal cograph obstructions to arboricity p. First, we consider the cotree height.

Theorem 7. *Let p be an integer, $p \geq 2$. If G is a minimal cograph obstruction for arboricity p with cotree T, then the height of T is at most $4p+1$.*

Proof. We may assume that G is connected and therefore the root vertex of the cotree is a join vertex, J_0. Notice that to have a unique co-tree all children of a join vertex should be union or single vertices and all children of union vertices should be join or single vertices. Recall that K_{2p+1}, whose cotree has height two, is a minimal obstruction and therefore no other minimal obstruction can contain it. For simplicity, in the following we will use $\omega(X)$ instead of $\omega(G[X])$ to denote the clique number of the subgraph of G induced by the vertex set X.

Let J be a join vertex of the co-tree with degree d whose children are $U_1, U_2, \ldots, U_d$. It is easy to see that $\omega(J) = \sum_{i=1}^{d} \omega(U_i)$. In particular, $d \geq 2$ implies that $\omega(J) \geq \omega(U_i) + 1$ for any U_i that is a child of J. This implies that any path from J_0 to a leaf of T contains at most $2p$ join vertices, hence the height of the co-tree is at most $4p + 1$. □

Corollary 8. *Let p be an integer, $p \geq 2$. Let G be a minimal cograph obstruction for arboricity p and let T be its cotree.*

If $G \neq K_{2p+1}$, then every join vertex in T has at most $2p$ children.

Theorem 9. *Let G_1 and G_2 be minimal cograph obstructions for p-vertex arboricity such that $\rho(G_i) = \chi(G_i) = p + 1$ for $i \in \{1, 2\}$. Let $T(G)$ denote the height of the cotree of a cograph G. If S is an independent set of size $p + 2$, then the cograph $H = (G_1 + G_2) \oplus S$ satisfies:*

(a) $\rho(H) = \chi(H) = p + 2$.

(b) *H is a cograph minimal obstruction for $(p + 1)$-arboricity.*

(c) $T(H) = \max\{T(G_1), T(G_2)\} + 1$

Proof. It is easy to see that $\chi(H) = p+2$, and $\rho(H) \leq p+2$. If $\rho(H) \leq p+1$, then there exists a partition $\mathcal{F}$ of $V(H)$ into $p + 1$ induced forests, and hence, at least one forest F in $\mathcal{F}$ contains two distinct vertices of S. This implies that there exists $i \in \{1, 2\}$ such that $V(F) \cap V(G_i) = \varnothing$, and so the restriction of $\mathcal{F}$ to $V(G_i)$ is a partition of $V(G_i)$ into $\rho(G_i) - 1 = p$ forests, which is a contradiction, and so (a) holds.

To show (b), let $v \in V(H)$. If $v \in S$, let S' be $S' = \{v_1, \ldots, v_{p+1}\} = S - \{v\}$, and take a $(p + 1)$-colouring, $f_i \colon V(G_i) \to S'$, of G_i for $i \in \{1, 2\}$. Notice that for every $r \in \{1, \ldots, p + 1\}$, the set $\{v_r\} \cup f_1^{-1}(v_r) \cup f_2^{-1}(v_r)$ induces a forest in H, which shows $\rho(H - v) \leq p + 1$. Suppose now that $v \in V(G_1)$, and let $w \in V(G_2)$ and take $G'_1 = G_1 - v$ and $G'_2 = G_2 - w$. Let $f'_i \colon V(G'_i) \to \{1, \ldots, p\}$ be a partition of $V(G'_i)$ into p forests for $i \in \{1, 2\}$. (Such partitions exist due to the minimality of G_1 and G_2.) Let $f \colon V(H - v) \to \{1, \ldots, p + 1\}$ be given by

$$f(x) = \begin{cases} f_i(x) & \text{if } x \in V(G_i') \text{ for } i \in \{1,2\} \\ p+1 & \text{if } x \in S \cup \{w\} \end{cases}$$

It is easy to see that f induces a partition of $V(H)$ into $p+1$ forests, which shows (b).

Part (c) follows directly from the construction of H. □

We did not succeed to obtain an analog to Corollary 8 in terms of union vertices, which would yield an upper bound on the size of a cograph minimal obstruction for arboricity p. Instead, we derive, from the algorithm in the next section, the following result.

Theorem 10. *Each minimal cograph obstruction for arboricity p has at most $O((2p)!^2)$ vertices.*

5 A polynomial time algorithm

The following simple observation describes the recursive structure of (p,q,r)-partitions in cographs. The only thing to remember is that for the second statement (join of two cographs), a forest cannot intersect both sides in more than one vertex.

Proposition 11. *1. Let $G = G_u + G_d$ be a cograph with non-empty subgraphs G_u and G_d. Then G has a (p,q,r)-partition if and only if there exist integers $r_u, r_d \geq 0$ such that $c = r_u + r_d$ and G_u and G_d have a (p,q,r_u)-partition and a (p,q,r_d)-partition, respectively.*

2. Let $G = G_u \oplus G_d$ be a cograph with non-empty subgraphs G_u and G_d. Then G has a (p,q,r)-partition if and only if there exist integers $p_u, p_d, q_u, q_d, r_u, r_d, t_u, t_d \geq 0$ such that $p = p_u + p_d + t_u + t_d$, $q = q_u + q_d$, $r = r_u + r_d$ and G_u and G_d have a $(p_u, q_u + t_d, r_u + t_u)$-partition and $(p_d, q_d + t_u, r_d + t_d)$-partition, respectively.

Notice that the integers $p_u, p_d, q_u, q_d, r_u, r_d, t_u, t_d$ in Proposition 11 are not necessarily unique. If G is a cograph, then there exists cographs G_u and G_d such that either $G = G_u + G_d$ or $G = G_u \oplus G_d$. Suppose G_u and G_d have an (x,y,z)-partition and an (x',y',z')-partition, respectively. Then, the triples (p,q,r) such that G has a (p,q,r)-partition as the one described in Proposition 11 are said to be *derived* from (x,y,z) and (x',y',z').

This structure can be used to obtain an efficient algorithm to solve the (p,q,r)-partition problem in cographs. To this end, define the *weight* of a triple (p,q,r) to be $p+q+r$.

Proposition 12. *Given two triples T_1 and T_2 with weights at most m, the set of all triples derived from T_1 and T_2 can be generated in $O(m)$ time.*

Proof. If $G = G_u + G_d$ then according to the first item in Proposition 11, we have that $(p, q, r) = (\max\{x, x'\}, \max\{y, y'\}, z + z')$ as the only option. But if $G = G_u \oplus G_d$ then we may have more options for (p, q, r). In this case, by setting $t = t_d + t_u$, any triple $(x + x' + t, y + y' - t, z + z' - t)$ can be produced, given that all the components of the triple are non-negative. □

Theorem 13. *Given a cograph G with n vertices, there exists an algorithm that computes in $O(np^7)$ all the triples T with weight at most p such that G admits a T-partition.*

Proof. We build an algorithm ALG recursively as follows. The algorithm is trivial when G is a clique or an independent set. So suppose either $G = G_u + G_d$ or $G = G_u \oplus G_d$. Then apply ALG to G_1 and G_2 to obtain the lists L_u and L_d. Then for each pair $(T_u, T_d) \in L_u \times L_d$, add all the triples derived from T_u and T_d to some list L, which is our final answer. Let $f(n)$ be the running time of this algorithm on a cograph with n vertices. Suppose G, G_u and G_d have n, s and $n - s$ vertices, respectively. Considering Observation 12 and the fact that lists L_u and L_d each have at most $O(p^3)$ members, we get the following recursion:

$$f(n) = f(s) + f(n - s) + O(p^7)$$

which implies $f(n) = O(np^7)$. □

Finally, we use a similar approach, based on triples, to prove Theorem 10. Let $\mathcal{T}$ be a set of triples of non-negative integers. We say a graph G is a *minimal cograph obstruction for the set* $\mathcal{T}$ if the following conditions hold:

1. G does not admit a T-partition for any $T \in \mathcal{T}$.
2. For any vertex $v \in V(G)$, there exists a triple $T \in \mathcal{T}$ such that $G - \{v\}$ admits a T-partition.

Given a triple $T = (p, q, r)$, by the *weight* $w(T)$ *of* T now we mean $2p + q + r$ (and not $p + q + r$). The following observation is immediate.

Observation 14. *Suppose a triple T is derived from triples T_u and T_d. Then $w(T_u), w(T_d) \leq w(T)$. Furthermore, if $G = G_u + G_d$, G is T-partitionable, and G_u and G_d are T_u-partitionable and T_d-partitionable, respectively, then $w(T) = w(T_u) + w(T_d)$.*

For integers k, m, let $f(k,m)$ be the smallest integer with the following property: any minimal obstruction with respect to a set of triples with weight at most k and at most m triples with weight exactly k has size at most $f(k,m)$. Now we embark on estimating $f(k,m)$ using recursion. Note that $m = O(k^2)$. For the sake of convenience, we assume $f(k,0) = f(k-1, O(k^2))$.

Theorem 15. $f(k,m) = O(k!^2)$

Proof. Let $\mathcal{T}$ be a set of triples with weight at most k and at most m triples having weight k. Let G a minimal obstruction with respect to $\mathcal{T}$. Ignoring trivial cases, we may assume that either $G = G_u + G_d$ or $G = G_u \oplus G_d$, where both G_u and G_d are non-empty. For $i \in \{u, d\}$, denote by L_i the set of triples X with weight at most k such that G_i admits an X-partition. We say a triple X is *dangerous* for G_u (G_d, respectively) if there exists a triple $X' \in L_d$ ($X' \in L_u$, respectively) such that from X and X' we can derive a triple in $\mathcal{T}$. Let D_u (D_d, respectively) be the set of all triples dangerous for G_u. Note that D_u is non-empty if G_u has at least two vertices. To see this, let $v \in V(G_u)$ be an arbitrary vertex. Then $G - \{v\}$ has a T-partition for some $T \in \mathcal{T}$. Applying Observation 11, there must be triples X and X' such that T is derived from X and X' and $G_u - \{v\}$ and G_d admit an X-partition and an X'-partition, respectively. This means $X' \in L_d$ and $X \in D_u$. A similar argument shows that G_u must be a minimal obstruction with respect to D_u if it has at least two vertices. Note that since G_d is non-empty so any triple in L_d has non zero weight. This implies the weight of triples in D_u is at most k according to Observation 14. In fact, if $G = G_u \oplus G_d$, the weight of the triples in D_u (and D_d) are at most $k-1$. So in this case, we have $|V(G_u)|, |V(G_d)| \leq f(k-1, O(k^2))$, and thus:

$$f(k,m) \leq 2f(k-1, O(k^2)). \tag{1}$$

Now suppose $G = G_u + G_d$. Suppose D_u contains a triple $X = (x, y, z)$ with weight k. This implies G_d admits an $(x, y, 0)$-partition. So $X \in \mathcal{T}$, which means $X \notin D_d$ (otherwise G would admit an X-partition). This means that for some integer t, $0 \leq t \leq m$ we have

$$f(k,m) \leq f(k, m-t) + f(k,t). \tag{2}$$

Now (1) and (2) imply that $f(k,m) = O(k!^2)$.

□

6 Concluding remarks

We have already observed that the (p, q, r)-partition problem can be considered as a general framework that includes interesting problems, e.g., q-colouring, arboricity p, or independent feedback vertex set. In these cases, the value of r is 0. Notice that r can be used as an additional input value to state some classic decision problems arising from optimization problems. We discuss two examples.

Recall that the vertex cover optimization problem asks, given a graph G, to find the size of a minimum vertex cover of G. There is a decision problem associated to this optimization problem. The problem VERTEX COVER takes as input a graph G and a non-negative integer r, and asks whether G contains a vertex cover with at most r vertices. Now, notice that a $(0, 1, r)$-partition of a graph G is a partition into an independent set, and a set with at most r vertices C, this is, all the edges of G must have at least one end in the set C. From here, it is easy to conclude that G has a vertex cover with at most r vertices if and only if G admits a $(0, 1, r)$-partition.

The odd cycle transversal problem asks to find the minimum set of vertices having a non-empty intersection with every odd cycle in a graph G. Again, this optimization problem has an associated decision problem. Consider the BIPARTIZATION problem, with input (G, r), where G is a graph and r is a non-negative integer, and where we have to decide whether or not there is a subset X of at most r vertices of G such that $G - X$ is a bipartite graph. Notice that alternatively, we could ask whether G admits a $(0, 2, r)$-partition. It was proved in [6] that BIPARTIZATION is NP-complete, even when restricted to planar graphs.

In this setting, it is easy to notice that $(1, 0, r)$-partition corresponds to the Feedback Vertex Set problem. We think that (p, q, r)-partitions represent a nice framework where many seemingly unrelated problems converge.

As a final remark, we make a simple observation about the structure of minimal obstructions to the $(0, q, r)$-partition problem.

Theorem 16. *Let q and r be non-negative integers. Every disconnected cograph minimal obstruction for the $(0, q, r)$-partition problem is of the form $\bigcup_{i \in I} G_i$ where G_i is a cograph minimal obstruction for the $(0, q, r_i)$-partition problem and $|I| - 1 + \sum_{i \in I} r_i = r$.*

Proof. Let G be a minimal obstruction for the $(0, q, r)$-partition problem, and let $\{G_i\}_{i \in I}$ be the set of components of G. Since G is a minimal obstruction, we know that for every $x \in V(G)$ there exists $L_x \subseteq V(G)$ such that $x \in L_x$, $|L_x| \leq r + 1$, and $G - L_x$ is q-colourable.

Claim 1. *For every $x, y \in V(G)$, every choice of L_x and L_y, we have $|L_x \cap V(G_i)| = |L_y \cap V(G_i)|$ for each $i \in I$.*

Suppose otherwise and let $x, y \in V(G)$, $L_x, L_y \subseteq V(G)$ and $i \in I$ be such that $|L_y \cap V(G_i)| < |L_x \cap V(G_i)|$. Since G is a minimal obstruction for $(0, q, r)$-partition, then $|L_v| = r + 1$ for every $v \in V(G)$. This means that $L'_x = (L_x - V(G_i)) \cup (L_y \cap V(G_i))$ satisfies $|L'_x| \leq r$, and so $G - L'_x$ is not q-colourable. Since $G_j - L'_x = G_j - L_x$ for every $j \in I$, $j \neq i$, it follows that $\chi(G_i - L'_x) > q$, but $G_i - L'_x = G_i - L_y$, contradicting the choice of L_y. ■

Claim 2. *Let $x \in V(G)$ and take $r_i = |L_x \cap V(G_i)| - 1$. For every $i \in I$, the cograph G_i is a minimal obstruction for the $(0, q, r_i)$-partition problem.*

Due to the choice of r_i and Claim 1, G_i is an obstruction to $(0, q, r_i)$-partition. To see that it is minimal, let $x \in V(G_i)$ and $L_x \subseteq V(G)$. Since $G - L_x$ is q-colourable and $|(V(G_i) \cap L_x) - x| = r_i$, then G is a minimal obstruction for $(0, q, r_i)$-partition. ■

Since $\sum_{i \in I} r_i = |L_x| - |I|$, we get that $|I| - 1 + \sum_{i \in I} r_i = |L_x| - 1 = r$, completing the proof of the theorem. □

It seems that varying the three parameters p, q and r for a (p, q, r)-partition, makes it difficult to find the minimal obstructions to this problem. So, we propose the problem of finding the minimal obstructions to (p, q, r)-partition when two of the parameters remain fixed.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, Berlin, 2008.

[2] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman & Co., New York, NY, USA, 1979.

[3] D. G. Corneil, H. Lerchs and L. Stewart Burlingham, Complement reducible graphs, Discrete Applied Mathematics 3(3) (1981) 163–174.

[4] B. Courcelle, Graph Operations and Monadic Second-Order Logic: A Survey, in Voronkov A. (eds), Logic for Programming and Automated Reasoning, LPAR 2000, 20-24.

[5] P. Damaschke, Induced subgraphs and well-quasi-ordering, Journal of Graph Theory 14(4) (1990) 427–435.

[6] J. M. Lewis and M. Yannakakais, The node-deletion problem for hereditary properties is NP-complete, Journal of Computer and System Sciences 20(2) (1980) 219–230.

[7] A. Maróti, On elementary lower bounds for the partition function, Electronic Journal of Combinatorial Number Theory 3 (2003) #A10.

[8] A. N. Trenk, Generalized perfect graphs: Characterizations and inversion. Discrete Applied Mathematics 60 (1995) 359–387.