On edge-gracefulness of 2-regular graphs

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Abstract. We investigate the edge-gracefulness of 2-regular graphs.

1. Introduction.

We consider finite undirected graphs without loops and multiple edges in this paper.

A graph $G = (V, E)$ is said to be **graceful** if there exists an injection $f: V \to \{0,1,2,\ldots,|E|\}$ such that the induced edge labeling which is defined by $f^*: E \to \{1,2,\ldots,|E|\}$ where $f^*(\{a,b\}) = |f(a) - f(b)|$ is a bijection. The theory of graceful graphs was initiated by A. Rosa [13] in 1966, and became popular after two articles of Golomb [2] and M. Gardner [1].

In 1985, Lo [12] introduced the concept of edge-graceful graphs. A (p,q)-graph $G = (V,E)$, of p vertices and q edges, is said to be **edge-graceful** if there exists a bijection $f: E \to \{1,2,\ldots,q\}$ such that the induced mapping $f^+ \to \{0,1,\ldots,p-1\}$, defined by

$$f^+(v) = \Sigma\,\{f\{u,v\}: \{u,v\} \in E(G)\} \pmod p,$$

is a bijection.

Using a simple counting argument, Lo [12] showed that if G is edge-graceful, then p divides $q^2 + q - [p(p-1)]/2$. For other results on edge-graceful graphs, we refer the readers to [**4, 5, 6, 7, 8, 9, 10, 11, 12**].

Hebbare [3] characterized the graceful cycles. A graph is 2-regular if it is a cycle or disjoint union of cycles. Recently Kotzig and Abraham tried to give graceful labelings for some 2-regular graphs.

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In this paper, we investigate edge-gracefulness of 2-regular graphs. It could also be noted that for 2-regular graphs, edge-gracefulness is equivalent to harmonious. In [12], Lo proved that a cycle C_n is edge-graceful if and only if n is odd. The case of union of disjoint cycles remains open. Lee proposed the following two conjectures in the 1987 Summer Research Institute of American Mathematical Society.

Conjecture 1. *The 2-regular graph $C_{2m} \cup C_{2n+1}$ is edge-graceful for all m and n (except $C_3 \cup C_4$).*

Conjecture 2. *Almost all 2-regular graphs of odd order are edge-graceful.*

We prove some new results related to the above conjectures.

2. Main results.

Let G be a 2-regular graph that is a union of disjoint cycles. Assume $G = C_{m1} \cup C_{m2} \cup \ldots \cup C_{mk}$. We can represent this 2-regular graph by $C(m1, m2, \ldots, mk)$.

For a 2-regular graph G to be edge-graceful, the number of vertices must be odd. This follows from Lo's necessary condition.

If G is a cycle, then the condition of Lo is sufficient. We can find an edge-graceful labeling of C_n simply by labeling the edges by $1, 2, \ldots, n-1, n$ in clockwise direction. We use $(1\ 2\ 3 \ldots n-1\ n)$ to denote this edge labeling.

We note that Lo's condition is not sufficient for disconnected graphs. In particular, an exhaustive computer search for the case of $C_3 \cup C_4$ did not turn up any edge-graceful labelings. The direct proof that $C_3 \cup C_4$ is not edge-graceful requires the consideration of 35 cases. We appreciate the following elegant proof which is suggested by the referee: To any edge-graceful labeling with two edges of C_3 labeled r and s, apply the transformation $f_{a,b}(x) = a(x-b)$ with $b = r$ and $a = (s-r)^{-1}$ to achieve a labeling with these two edges labeled 0 and 1. (Such transformations $f_{a,b}(x)$ alter neither the injectivity of the edge-labeling nor that of the vertex-labeling.) The 3rd edge e_3 of C_3 now cannot be labeled 2, 6, or 4; and a label of 5 can be changed to a 3 by the transformation $f(x) = -x + 1$. So take e_3 to be labeled 3, and the remaining labels for C_4 are 2, 4, 5, and 6. But then the three possible choices for the label opposite 2 are impossible.

However, we know that $C_3 \cup C_6$ and $C_4 \cup C_5$ are edge-graceful with the edge labelings (1 2 4) (3 6 5 8 9 7) and (1 2 5 4) (3 7 6 9 8) (Figure 1).

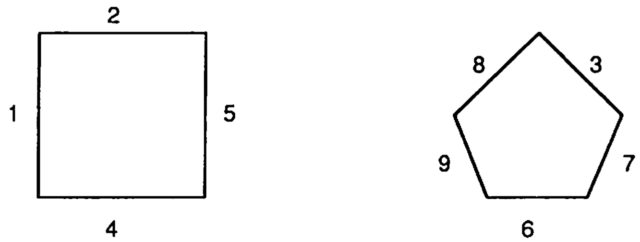

Figure 1

In the following theorem, we present some general results when $k = 2$.

Theorem 1. *For n odd, the 2-regular graph $C_{2n} \cup C_{2n+1}$ is edge-graceful.*

Proof: There are $4n+1$ vertices and $4n+1$ edges. Label the edges as follows (Figure 2):

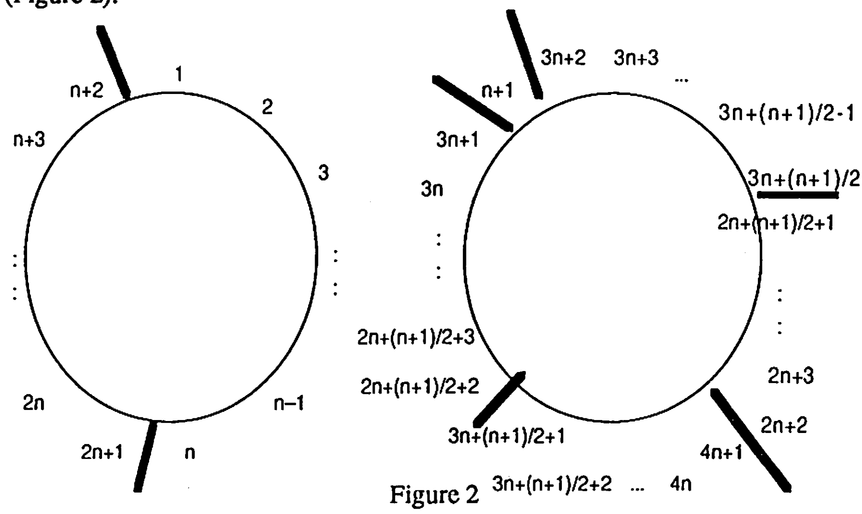

Figure 2

The vertex labels for C_{2n} are:

$$3, 5, \ldots, 2n-1; \quad n+3; \quad 3n+1; \quad 2n+5, 2n+7, \ldots, 4n+1.$$

Note that since n is odd, both $n+3$ and $3n+1$ are even. The set of vertex labels for C_{2n+1} is:

$$(n+1)+(3n+2) \equiv 2 \pmod{4n+1};$$
$$(n+1)+(3n+1) \equiv 1 \pmod{4n+1};$$
$$2n+4, 2n+6, \ldots, \big(3n+(n+1)/2-1\big)+\big(3n+(n+1)/2\big) \equiv 3n-1$$
$$4, 6, \ldots, n+1;$$
$$3n+3, 3n+5, \ldots, 4n;$$
$$n+5, n+7, \ldots, 2n; \quad 2n+2;$$
$$\big(3n+(n+1)/2\big)+\big(2n+(n+1)/2+1\big) \equiv 2n+1 \pmod{4n+1}; \text{ and}$$
$$\big(3n+(n+1)/2+1\big)+\big(2n+(n+1)/2+2\big) \equiv 2n+3 \pmod{4n+1}.$$

It is clear that we have different vertex labels. ∎

Example 1: We give edge-graceful labelings for $C(6,7)$ and $C(10,11)$ under the above labeling scheme (Figure 3):

$$C(6,7) \rightarrow (1\ 2\ 3\ 7\ 6\ 5)(4\ 11\ 9\ 8\ 13\ 12\ 10)$$
$$C(10,11) \rightarrow (1\ 2\ 3\ 4\ 5\ 11\ 10\ 9\ 8\ 7)(6\ 17\ 18\ 14\ 13\ 12\ 21\ 20\ 19\ 15\ 16)$$

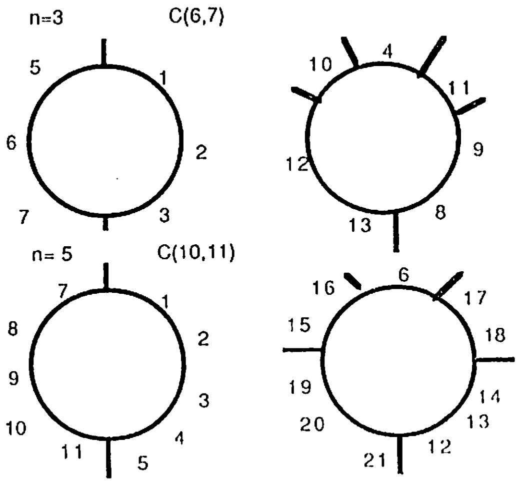

Figure 3

Theorem 2. *For n odd, the 2-regular graph* $C_n \cup C_{2n+2}$ *is edge-graceful.*

Proof: Label the edges as follows (Figure 4):

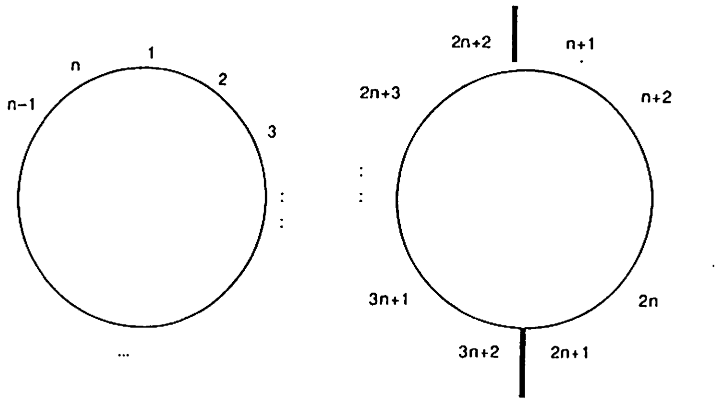

Figure 4

∎

Example 2: The edge-graceful labeling for $C(5,12)$ is given by $C(15,12) \rightarrow (1\ 2\ 3\ 4\ 5)\ (6\ 7\ 8\ 9\ 10\ 11\ 17\ 16\ 15\ 14\ 13\ 12)$.

Theorem 3. *For n odd, $C_n \cup C_{4n}$ is edge-graceful.*

Proof: Label the edges as follows (Figure 5):

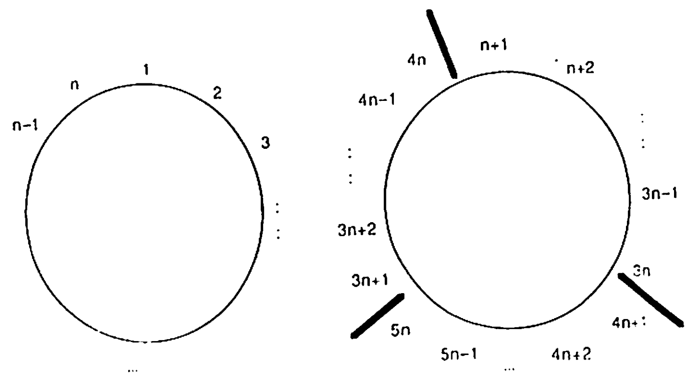

Figure 5

The vertex labeling in C_n is given by:

$$3, 5, 7, 9, \ldots, 2n-1 \text{ and } n+1 \pmod{5n}.$$

The vertex labeling in C_{4n} has

$$2n+3, 2n+5, \ldots, 6n-1, 7n+1 = 2n+1 \pmod{5n},$$
$$3n+3, 3n+5, \ldots, 10n-1 = 5n-1 \pmod{5n},$$
$$8n+1 = 3n+1 \pmod{5n},$$
$$6n+3 = n+3 \pmod{5n},$$
$$6n+5 = n+5 \pmod{5n},$$
$$\ldots, 8n-1 = 3n-1 \pmod{5n},$$
$$5n+1 = 1 \pmod{5n}.$$

The two vertex labelings consist the set of integers $\{0, 1, 2, \ldots, 5n-1\}$. Therefore, the edge labeling is edge-graceful. ∎

Example 3: We see that the 2-regular graphs $C(3, 12)$ and $C(5, 20)$ have the following edge-graceful labelings:

$C(3, 12) \to (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9\ 13\ 14\ 15\ 10\ 11\ 12)$
$C(5, 20) \to (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 21\ 22\ 23\ 24\ 25\ 16\ 17\ 18\ 19\ 20)$

3. The 2-regular graphs with more than two cycles.

In general, it is hard to find the edge-graceful labelings for 2-regular graphs of odd order with different sizes of cycles. For example, it is not trivial to find an edge-graceful labeling of $C_3 \cup C_4 \cup C_4$ (Figure 6).

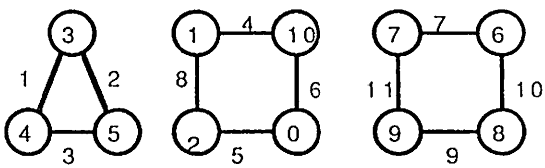

Figure 6

For 2-regular graphs which are disjoint union of the same odd cycle, we have the following result:

Theorem 4. *If $m_1 = m_2 = \ldots = m_k = n$, where n and k are both odd, then G is edge-graceful.*

Proof: In clockwise direction, label the edges in the first cycle by $1, k+1, 2k+1, \ldots, (n-1)k+1$; the edges in the second cycle by $2, k+2, 2k+2, \ldots, (n-1)k+2; \ldots$; the last ($k$th) cycle by $k, 2k, 3k, \ldots, nk$.

The vertex labels in the first C_n are: $k+2, 3k+2, \ldots, (2n-3)k+2, (n-1)k+2$. In the 2nd C_n they are: $k+4, 3k+4, \ldots, (2n-3)k+4, (n-1)k+4$. And finally, the kth C_n they are: $3k, 5k, \ldots, (2n-1)k, (n+1)k$.

It is not difficult to show that the induced vertex labels are distinct modulo nk.

∎

We illustrate the above result by the following example:

Example 3: An edge-graceful labeling of $C(5,5,5)$ (Figure 7).

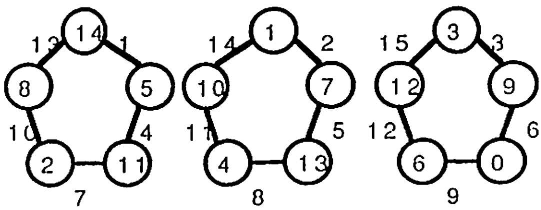

Figure 7

Finally, we give a list of all 2-regular graphs of odd order less than 17 and their edge-graceful labelings.

$p = 3$

$C(3) \rightarrow (1\ 2\ 3)$

$p = 5$

$C(5) \rightarrow (1\ 2\ 3\ 4\ 5)$

$p = 7$

$C(7) \rightarrow (1\ 2\ 3\ 4\ 5\ 6\ 7)$

$C(3,4) \rightarrow$ not edge-graceful

$p = 9$

$C(9) \rightarrow (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$

$C(3,6) \rightarrow (1\ 2\ 4)(3\ 6\ 5\ 8\ 9\ 7)$

$C(4,5) \rightarrow (1\ 2\ 5\ 4)(3\ 7\ 6\ 9\ 8)$

$C(3,3,3) \rightarrow (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)$

$p = 11$

$C(11) \rightarrow (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)$

$C(3,8) \rightarrow (1\ 2\ 3)(4\ 5\ 6\ 7\ 11\ 10\ 9\ 8)$

$C(4,7) \rightarrow (1\ 2\ 3\ 5)(4\ 7\ 6\ 9\ 11\ 10\ 8)$

$C(5,6) \rightarrow (1\ 2\ 3\ 5\ 6)(4\ 8\ 7\ 10\ 11\ 9)$

$C(3,3,5) \rightarrow$ not edge-graceful

$C(3,4,4) \to (1\ 2\ 3)(4\ 6\ 5\ 8)(7\ 10\ 9\ 11)$

$p = 13$

$C(13) \to (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13)$

$C(3,10) \to (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 11\ 9\ 12\ 13\ 10)$

$C(4,9) \to (1\ 2\ 3\ 5)(4\ 6\ 7\ 8\ 9\ 11\ 13\ 12\ 10)$

$C(5,8) \to (1\ 2\ 3\ 4\ 7)(5\ 8\ 6\ 11\ 12\ 13\ 9\ 10)$

$C(6,7) \to (1\ 2\ 3\ 4\ 6\ 7)(5\ 9\ 8\ 11\ 13\ 12\ 10)$

$C(3,3,7) \to (1\ 2\ 3)(4\ 6\ 7)(5\ 9\ 11\ 8\ 13\ 12\ 10)$

$C(3,4,6) \to (1\ 2\ 3)(4\ 6\ 5\ 9)(7\ 8\ 11\ 10\ 12\ 13)$

$C(3,5,5) \to (1\ 2\ 3)(4\ 5\ 9\ 6\ 7)(8\ 11\ 10\ 13\ 12)$

$C(4,4,5) \to (1\ 2\ 3\ 5)(4\ 8\ 6\ 9)(7\ 10\ 12\ 11\ 13)$

$C(3,3,3,4) \to$ not edge-graceful

$p = 15$

$C(15) \to (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15)$

$C(3,12) \to (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9\ 13\ 10\ 11\ 14\ 15\ 12)$

$C(4,11) \to (1\ 2\ 4\ 3)(5\ 6\ 7\ 9\ 8\ 12\ 11\ 13\ 14\ 15\ 10)$

$C(5,10) \to (1\ 2\ 3\ 4\ 5)(6\ 8\ 7\ 9\ 10\ 13\ 15\ 12\ 14\ 11)$

$C(6,9) \to (1\ 2\ 3\ 4\ 5\ 9)(6\ 10\ 7\ 8\ 11\ 12\ 14\ 13\ 15)$

$C(7,8) \to (1\ 2\ 3\ 4\ 5\ 8\ 7)(6\ 10\ 9\ 12\ 13\ 14\ 15\ 11)$

$C(3,3,9) \to (1\ 2\ 3)(4\ 5\ 7)(6\ 9\ 8\ 13\ 15\ 14\ 11\ 12\ 10)$

$C(3,4,8) \to (1\ 2\ 3)(4\ 5\ 6\ 10)(7\ 8\ 9\ 12\ 11\ 14\ 13\ 15)$

$C(3,5,7) \to (1\ 2\ 3)(4\ 5\ 7\ 10\ 12)(6\ 8\ 13\ 15\ 11\ 14\ 9)$

$C(3,6,6) \to (1\ 2\ 3)(4\ 5\ 6\ 8\ 14\ 11)(7\ 9\ 12\ 15\ 13\ 10)$

$C(4,4,7) \to (1\ 2\ 3\ 5)(4\ 6\ 8\ 11)(7\ 9\ 13\ 15\ 12\ 14\ 10)$

$C(4,5,6) \to (12\ 13\ 15\ 14)(6\ 9\ 8\ 11\ 10)(1\ 2\ 3\ 4\ 5\ 7)$

$C(5,5,5) \to (1\ 4\ 7\ 10\ 13)(2\ 5\ 8\ 11\ 14)(3\ 6\ 9\ 12\ 15)$

$C(3,3,3,6) \to (1\ 2\ 3)(4\ 5\ 8)(10\ 12\ 13)(6\ 9\ 7\ 14\ 15\ 11)$

$C(3,3,4,5) \to (1\ 2\ 3)(4\ 5\ 11)(9\ 12\ 14\ 13)(6\ 7\ 10\ 15\ 8)$

$C(3,4,4,4) \to (1\ 2\ 3)(4\ 6\ 5\ 10)(7\ 9\ 8\ 14)(11\ 12\ 15\ 13)$

$C(3,3,3,3,3) \to (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)$

We note that two more examples of disjoint union of cycles that are not edge-graceful are found. It would be interesting to see if there exists an infinite number of such examples.

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