Possible Automorphism Groups For An S(3, 5, 26)

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Abstract. Let G be the automorphism group of an $S(3,5,26)$. We show the following: (i) if 13 divides $|G|$ then G is a subgroup of $Z_2 \times Fr_{13 \cdot 12}$, where $Fr_{13 \cdot 12}$ is the Frobenius group of order $13 \cdot 12$; (ii) if 5 divides $|G|$ then $G \simeq Z_5$ or $G \simeq D_{10}$; and (iii) otherwise, either $|G|$ divides $3 \cdot 2^3$ or 2^4.

1. Introduction.

A Steiner system $S(t,k,v)$ is an ordered pair $(X, \mathcal{B})$ where X is a v-set of points and $\mathcal{B}$ a collection of k-subsets of X, called blocks, such that any t-subset from X appears exactly once among the blocks of $\mathcal{B}$. For any fixed $x \in X$, define $\mathcal{B}_x = \{B \backslash \{x\} \mid x \in B \in \mathcal{B}\}$ and $X_x = X \setminus \{x\}$. Then $(X_x, \mathcal{B}_x)$ is an $S(t-1, k-1, v-1)$ called a *derived design* of the $S(t,k,v)$. Equivalently, we say that $(X, \mathcal{B})$ is an *extension* of $(X_x, \mathcal{B}_x)$. If G is a group acting on X, then G is said to be an automorphism group of $(X, \mathcal{B})$ if G also preserves $\mathcal{B}$. For more details and basic facts on Steiner systems, t-designs, and groups see [1] and [3]. Throughout this paper G will be the automorphism group of an $S(3,5,26)$ unless specifically stated otherwise. If p is a prime and $s \mid p-1$, we denote by $Fr_{p \cdot s}$ the Frobenius group of order ps.

Table 1
Summary of $S(2,4,25)$ designs and their automorphism groups

Design number	$\|G\|$	G
1	504	$PSL_2(7) \times Z_3$
2	63	$Z_3 \times Fr_{7 \cdot 3}$
3,4,5	9	$Z_3 \times Z_3$
6	150	$(Z_5 \times Z_5) \cdot S_3$
7	21	$Fr_{7 \cdot 3}$
8	6	S_3
9 – 16	3	Z_3

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Table 2
Types of elements in $G = Aut(DES_1) = PSL_2(7) \times Z_3$.

Order	Type	Number	Order	Type	Number
1	1^{25}	1	6	$1^1 6^4$	42
2	$1^1 2^{12}$	21	7	$1^4 7^3$	48
3	$1^4 3^7$	112	12	$1^1 12^2$	48
3	$1^1 3^8$	58	21	$1^1 3^1 21^1$	36
4	$1^1 4^6$	42		Total	504

2. Bounding the order of G.

Because the sixteen $S(2,4,25)$'s with nontrivial automorphism groups have been completely determined (see [6]) we can often deduce information about the automorphism groups of their extensions. To aid the reader we summarize some of the information from [6] in Tables 1 and Table 2. Further, as in [6], we let DES_n, for $1 \leq n \leq 16$, denote the sixteen $S(2,4,25)$'s listed in [6].

The cycle type of an element of order 3 in $PSL_2(7)$ is $1^4 3^7$ and in Z_3 it is $1^1 3^8$.

Theorem 2.1. *Let G be the automorphim group of an $S(3,5,26)$. Then the order of G divides $2^4 3 \cdot 5 \cdot 13$. Also, $5 \cdot 13$ does not divide the order of G.*

Theorem 2.1 will follow from the Lemmas below:

Lemma 2.1. *The only primes that can divide the order of G are $2, 3, 5, 7$, and 13.*

Proof: If an element g of prime order fixes no points then it is of order 2 or 13. If the element g fixes at least one point then the derived $S(2,4,25)$ through the fixed point has g as an automorphism (on the restricted point set) and all $S(2,4,25)$'s with nontrivial automorphism groups have been determined [6]. ∎

Lemma 2.2. *Let H be a subgroup of G where $|H| = 2^a$. Then a is at most 4.*

Proof: Since $v = 26$, H must have a point orbit of length at most 2. If $\{x, y\}$ is an orbit then the point stabilizer H_x of x is isomorphic to the point stabilizer H_y of y. Then $[H: H_x] = 2$ so that $|H_x|$ is $2^{(a-1)}$. But the biggest 2-group on an $S(2,4,25)$ is of order $2^3 = 8$. So $2^{(a-1)}$ is at most 8 and a is at most 4. ∎

Lemma 2.3. *If $|H| = p^a$, where p is 3 or 5, then $a \leq 2$.*

Proof: If $|H| = p^3$, where p is 3 or 5, then H fixes a point by orbit length argument. But there is no group of order p^3, for $p = 3$ or 5, on an $S(2,4,25)$. Thus, a is at most 2. ∎

Lemma 2.4. *Let* $(X,\mathcal{B})$ *be an* $S(2,4,25)$ *system and* ξ *an automorphism of order* 3 *of* $(X,\mathcal{B})$*, fixing* 4 *points,* x_1, x_2, x_3, x_4*. Then* ξ *fixes exactly five blocks of* $\mathcal{B}$*, of the form* $B_i = \{x_i\} \cup \{$ *a* 3*-cycle of* $\xi\}$*, for* $1 \le i \le 4$*, and* $B_5 = \{x_1, \ldots, x_4\}$.

Proof: An $S(2,4,25)$ system $\mathcal{D} = (X,\mathcal{B})$ admitting an automorphism of order 3 fixing 4 points must be isomorphic to one of DES_1, DES_2, DES_3, DES_4, DES_5, DES_7, DES_{15}, and DES_{16}. Since an element of order 3 belongs to some Sylow-3 subgroup of $Aut(\mathcal{D})$, and Sylow-3 subgroups are conjugate, it suffices to prove the assertion of the lemma for the elements of order 3 fixing 4 points in a particular Sylow-3 subgroup T of $Aut(\mathcal{D})$. In the cases of $DES_1, \ldots, DES_5$, the Sylow-3 is elementary abelian of order 9, and is conjugate to $\langle\widehat{\alpha}, \widehat{\beta}\rangle$, where $\widehat{\alpha} = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)\ (16\ 17\ 18)\ (19)(20)(21)\ (22\ 23\ 24)(25)$, and $\widehat{\beta} = (1\ 4\ 7)(2\ 5\ 8)\ (3\ 6\ 9)(10\ 13\ 16)(11\ 14\ 17)\ (12\ 15\ 18)(19\ 20\ 21)\ (22)(23)(24)(25)$. The elements of order 3 fixing 4 points in T are $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\alpha}^2$, $\widehat{\beta}^2$, and the statement of the lemma holds for these elements and for designs $DES_1, \ldots, DES_5$. In the case of designs DES_7, DES_{15}, DES_{16}, the Sylow-3 subgroup is conjugate to $<\beta>$ and the statement of the lemma is verified for β and β^2 for each of the designs DES_7, DES_{15}, DES_{16}. ∎

Lemma 2.5. *Let* ξ *be an automorphism of order* 3 *of an* $S(3,5,26)$ *system* $(X,\mathcal{B})$*. Then* ξ *has exactly two fixed points on* X.

Proof: Any element of order 3 on a derived $S(2,4,25)$ design has either 1 or 4 fixed points, so ξ has either 2 or 5 fixed points. Suppose ξ has the five fixed points x_1, x_2, x_3, x_4, x_5. Let $\mathcal{D}_i$ be the derived $S(2,4,25)$ through x_i. There are 5 fixed blocks as described in the previous lemma inside each $\mathcal{D}_i$. Thus, $\mathcal{B}$ contains fixed blocks of the type $\{x_i, x_j\} \cup \{$ a 3-cycle of $\xi\}$ for each of the ten unordered 2-subsets of $\{x_1, \ldots, x_5\}$. But ξ has only seven 3-cycles and any 3-set determines a unique block of $\mathcal{B}$, a contradiction. Thus, ξ can not fix 5 points of X. ∎

Lemma 2.6. *Let* G *be an automorphism group of an* $S(3,5,26)$ *design* $(X,\mathcal{B})$*. Then,* 3^2 *does not divide the order of* G.

Proof: Suppose there is an automorphism group H of order 3^2 for an $S(3,5,26)$. Then H must fix at least 2 points of X, and H is an automorphism group of a derived $S(2,4,25)$. But such a group of order 9 contains elements of order 3 fixing 4 points out of the remaining 25, a contradiction. ∎

Lemma 2.7. *Let* G *be an automorphism group of an* $S(3,5,26)$*. Then* 5^2 *does not divide* $|G|$.

Proof: Assume $|G| = 25$. Then G fixes a point and the derived design is DES_6. Thus, we need to find $210 = (260 - 50)$ 5-sets that cover triples not containing the fixed point and not covering triples in DES_6. There are $92 = (84 + 8)$ orbits of 3-sets under G where exactly 8 are covered by the 4-sets of DES_6. There are $53,130 = 2125 \cdot 25 + 6 \cdot 5$ 5-sets where G decomposes them into 2125 orbits of length 25 and 6 orbits of length 5. Only 1770 of the 5-set orbits cover 3-sets at most once, and if a design preserved by G exists, exactly 2 short orbits must be used. We sieve out rows and columns corresponding to the 3-sets covered in DES_6 and a complete search is negative. ∎

Lemma 2.8. *The prime 7 does not divide the order of G.*

Proof: Let $o(g) = 7$ so g has 3 seven cycles and fixes 5 points. Derived designs through any fixed points are $S(2,4,25)$'s that have an automorphism of order 7. Without loss of generality we can assume that any of these five $S(2,4,25)$'s have the automorphism $(1,2,\ldots,7)\ (8,\ldots,14)(15,\ldots,21)\ (22)(23)(24)\ (25)\ (26)$. Now there are exactly 3 nonisomorphic $S(2,4,25)$'s with an automorphism of order 7, namely, DES_1, DES_2, and DES_7 (see [6]). In each of these there is a cyclic Fano plane on the point set $\{1,\ldots,7\}$, that is, if we intersect $\{1,2,3,4,5,6,7\}$ with the blocks of B_x, for each fixed $x \in \{22,23,24,25,26\}$, we must either get the seven 3-sets generated by $\{1,2,4\}$ or the seven 3-sets generated by $\{1,3,4\}$. This obviously forces a repeated 3-set in the original $S(3,5,26)$. ∎

Lemma 2.9. *Let H be a sbgroup of G. Then $|H|$ is not 13^2.*

Proof: Assuming the contrary, then H is elementary abelian which requires elements of order 13 that fix 13 points. However, no $S(2,4,25)$ has an automorphism of order 13, so this is clearly impossible. ∎

Lemma 2.10. *The primes 5 and 13 cannot both divide $|G|$.*

Proof: Let $o(g) = 13$, $o(h) = 5$ where g, h are both elements of G. The cycle structure of g is 13^2 and that of h is $5^5 \cdot 1$ so G is clearly transitive and, therefore, 130 divides $|G|$. Since 5 divides $|G|$, any derived design must be of type DES_6 and, therefore, $|G|$ would divide $26 \cdot 150$. If G were non-solvable, then, by order considerations, the only non-solvable simple group that could be involved in G would be A_5. Therefore, since 5^2 does not divide $|G|$, we have $|G| = 2^2 3 \cdot 5 \cdot 13$. Consequently, a Sylow-13 subgroup would be normal, hence, would be centralized by an element of order 5, a contradiction. If G were solvable, then, by P. Hall's theorem, G would contain a subgroup H of order 65 having a normal subgroup F of order 5 fixing a point of X. Then an element, of order 13, normalizing F would have to fix a point of X, a contradiction. ∎

3. When 5 divides $|G|$.

In this section we prove the following theorem:

Theorem 3.1. *If 5 divides the order of G, then either $G \simeq Z_5$ or $G \simeq D_{10}$.*

This will follow from the Lemmas below. Throughout this section $g_1 \in G$ will be an element of order 5, whose cycle structure must be $5^5 \cdot 1$, and x will be the point fixed by g_1.

Lemma 3.1. *Let $g_1 \in G$ and $H = \langle g_1 \rangle$. Then $N_G(H) \simeq Z_5$ or D_{10} and consequently, the stabilizer G_x of the point x is either Z_5 or D_{10}.*

Proof: Since the subgroup H fixes the point x, $N_G(H)$ also fixes x. It follows that $N_G(H)$ is in the automorphism group of the derived design $D_x = (X_x, \mathcal{B}_x)$. Because only DES_6 has 5 dividing its order, $N_G(H) \leq G_x \leq (Z_5 \times Z_5) \cdot S_3$. But 5^2 can not divide $|G|$ so 5^2 can not divide $|N_G(H)|$ or $|G_x|$. So $|G_x|$ divides $5 \cdot 3 \cdot 2$. But there is no subgroup of order 15 in $(Z_5 \times Z_5) \cdot S_3$, and no cyclic group of order 10. Our result follows. ∎

Lemma 3.2. *If 5 divides the order of G and $10 < |G|$ then $60 \leq |G|$.*

Proof: Let g_1 be an element of G as indicated above. Suppose $|G| < 60$. None of $5^2, 7$, or 11 divide $|G|$ so the possible orders for G are $15, 20, 30, 40$, or 45. By Sylow's theorem if $|G|$ is $15, 20, 40$ or 45 then $\langle g_1 \rangle$ is normal, a contradiction to Lemma 3.1. If $|G| = 30$ then by P. Hall's theorem G has a subgroup of order 15 in which $\langle g_1 \rangle$ is normal, again a contradiction to Lemma 3.1. ∎

Lemma 3.3. *If 5 divides the order of G and $|G| \geq 60$ then $|G| = 60$ and $G \simeq A_5$.*

Proof: Let $g_1 \in G$ be an element as indicated above. Since $|G_x| \leq 10$ and $|X| = 26$, $|G| \leq 260$. Further $|G|$ divides $2^4 3 \cdot 5 \cdot 13$ and $5 \cdot 13$ does not divide $|G|$. Assume $|G| > 60$. So $|G| = 5 \cdot 16, 5 \cdot 24$, or $5 \cdot 48$. Let $H = \langle g_1 \rangle$ be our subgroup of order 5. Now $[G{:}\, N_G(H)] = 1 + 5k = 1, 6$, or 16. So if $|G| = 5 \cdot 24$, or $5 \cdot 48$ we would have $|N_G(H)| \geq 15$, a contradiction to Lemma 3.1. If $|G| = 5 \cdot 16$ then we would have $N_G(H) = H$. Suppose $|G| = 5 \cdot 16$. Let N be a minimal normal subgroup in G, then by Lemma 3.1 N is an elementary abelian group of order 16. Hence, $G \simeq E_{16} \cdot Z_5$. Since $N_G(H) = H$, by Lemma 3.1, $|G_x| = 5$, so we have $|x^G| = 16$. Let y be a point not in the orbit of x. Then $|y^G| = 5$ or 10. But then $|G_y| = 16$ or 8 and the derived $S(2,4,25)$ through the point y would have an automorphism group containing an elementary abelian subgroup of order 16 or 8. But the only $S(2,4,25)$ with an automorphism group of order divisible by 8 is DES_1. However, a Sylow-2 subgroup for DES_1 is not elementary abelian. So $|G| \neq 5 \cdot 16$. Thus, we have shown that if $|G| \geq 60$ then $|G| = 60$. We see that G must be nonsolvable, otherwise by P. Hall's theorem G would have a cyclic subgroup of order 15, which is a contradiction of Lemma 3.1. Hence, $G \simeq A_5$. ∎

Lemma 3.4. *Assume $G \simeq A_5$. Then the orbit structure of G on X, $|X| = 26$, is one of the following: [5, 6, 15], [6, 10, 10], or [6, 20].*

Proof: From [6] we see that an automorphism of order 5 fixes exactly one point of X. Similarly, automorphisms of order 2 fix 0, 2 or 6 points of X. Moreover, by Lemma 2.5 automorphisms of order 3 fix 2 points. An application of the Cauchy-Frobenius lemma yields that an A_5 must have 2 or 3 orbits on X. The possible transitive representations of A_5 of degree < 26 are of degrees $1, 5, 6, 10, 12, 15$, or 20. Furthermore, since none of the 16 designs [6] have an A_5 in their automorphism group, A_5 cannot fix a point of X. It easily follows that the ways of decomposing 26 as the sum of 2 or 3 integers from among $\{5, 6, 10, 12, 15, 20\}$ is $20 + 6$, $15 + 6 + 5$, and $10 + 10 + 6$. ∎

Lemma 3.5. *If 5 divides $|G|$, then G cannot be isomorphic to A_5.*

Proof: From Lemma 3.4 we have 3 cases:

Case 1. A_5 has point-orbits of lengths $15, 6, 5$. Here G is generated by $g_1 = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15)(16\ 17\ 18\ 19\ 20)(21\ 22\ 23\ 24\ 25)(26)$ and $g_2 = (1\ 11\ 9)(2\ 13\ 4)(3\ 7\ 10)(5\ 8\ 15)(6\ 14\ 12)(16\ 17\ 18)(19)(20)(21\ 25\ 23)(22\ 24\ 26)$. Now there is a special orbit on 3-sets of length 5, namely, $\{1,9,11\}$, $\{2,10,12\}$, $\{3,6,13\}$, $\{4,7,14\}$, and $\{5,8,15\}$. The set stabilizer G_S of $S = \{1,9,11\}$ is isomorphic to A_4, of order 12, and is generated by g_2 and g_3 where $g_3 = (1)(2\ 5)(3\ 4)(6\ 7)(8\ 10)(9)(11)(12\ 15)(13\ 14)(16\ 19)(17\ 18)(20)(21)(22\ 25)(23\ 24)(26)$. Note that the above orbit containing S consists of blocks of imprimitivity. If an element of G stabilizes a 5-set that contains S, then it must also stabilize S. But G_S stabilizes no 2-set so the stabilizer of a 5-set containing S has order less than 12 and, hence, the lengths of 5-set orbits covering S are greater than 5. Hence, S is covered at least twice in any such orbit. Thus, an $S(3,5,26)$ using this A_5 is not possible.

Case 2. A_5 has point orbit lengths $10, 10, 6$. Here G is generated by $g_1 = (1)(2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11)(12\ 13\ 14\ 15\ 16)(17\ 18\ 19\ 20\ 21)(22\ 23\ 24\ 25\ 26)$ and $g_2 = (1\ 3\ 5)(2\ 6\ 4)(7)(8\ 12\ 15)(9\ 10\ 13)(11\ 16\ 14)(17)(18\ 22\ 25)(19\ 20\ 23)(21\ 26\ 24)$. We examine 5-set orbits that cover the 3-set $S = \{7,8,12\}$. This S is in an orbit of 3-sets whose length is 20. Let Δ be an orbit of 5-sets that cover S. If Δ has length greater than 20 then it is easily seen that S is covered more than once. Scrutiny of the elements of G reveal that the only elements stabilizing a 5-set F, containing S, are elements of orders 2 or 3. In fact, the shortest such orbit of 5-sets is of length 20 and there is exactly one — namely, the orbit containing $F = \{7, 8, 12, 15, 17\}$. But this F contains four 3-sets in the orbit of S, namely $\{7,8,12\}$, $\{7,8,15\}$, $\{7,12,15\}$, and $\{8,12,15\}$. Thus, S is covered $(20 \times 4)/20 = 4$ times in F. Hence, there is no $S(3,5,26)$ using this A_5.

Case 3. A_5 has point orbit lengths 20, 6. In this case A_5 is generated by $g_1 = (1)(2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11)\ (12\ 15\ 16\ 17\ 13)\ (14\ 19\ 20\ 21\ 18)\ (22\ 23\ 24\ 25\ 26)$ and $g_2 = (1\ 2\ 3)(4\ 6\ 5)\ (7)(8\ 11\ 12)(9\ 13\ 14)(10\ 18\ 15)(16\ 21\ 23)\ (17\ 22\ 19)\ (20\ 26\ 24)(25)$. An exhaustive computer run easily rules out this permutation group. Our lemma and main theorem is proved. ∎

4. When 13 divides $|G|$.

Lemma 4.1. *If* 13 *divides* $|G|$, *then* $|G| = 2^a3^b13$ *with* $0 \leq a \leq 3$ *and* $0 \leq b \leq 1$.

Proof: From Theorem 2.1 $|G| = 2^a3^b13$ where $0 \leq a \leq 4$ and $0 \leq b \leq 1$. Now let $Z_{13} = \langle g \rangle$ be a Sylow-13 subgroup of G. By Sylow's theorem, Z_{13} is normal in G. Suppose that $a = 4$. Since $Aut(Z_{13}) \simeq Z_{12}$, 2^2 divides $|C_G(Z_{13})|$ and a Sylow-2 subgroup of $C_G(Z_{13})$ has order at least 2^2. But this implies that there is an involution $t \in C_G(Z_{13})$ which fixes points of X. Hence, by [6], the number of fixed points of t is either 2 or 6, which leads to a contradiction, because the element of order 13 must fix the fixed points of t. Our lemma follows.

Theorem 4.1. *If* 13 *divides* $|G|$, *then* G *is isomorphic to a sugroup of* $Z_2 \times Fr_{13\cdot 12}$.

Proof: Now $|G| = 2^a3^b13$, $0 \leq a \leq 3$, $0 \leq b \leq 1$. Also G is solvable with Z_{13} normal in G. An element which commutes with an element of order 13 must act fixed-point-free on X, hence, the order of the Sylow-2 subgroup of $C_G(Z_{13})$ is at most 2 (see the argument in the preceeding Lemma), and G is a subgroup of $Z_2 \times Fr_{13\cdot 12}$.

Note that the $S(3,5,26)$ designs found independently by Hanani [5], Denniston [2], and Grannell, Griggs, and Phelan [4] are isomorphic. Such a design has $Z_2 \times Fr_{13\cdot 12}$ as its full automorphism group. ∎

5. The case $|G| = 2^a3^b$.

Here we assume that neither 5 nor 13 divide $|G|$ so that $|G| = 2^a3^b$ with $0 \leq a \leq 4$, $0 \leq b \leq 1$. We show in what follows that if $b = 1$ then $0 \leq a \leq 3$.

Theorem 5.1. *If* $|G| = 3 \cdot 2^a$, *then* $|G|$ *divides* $3 \cdot 2^3$.

Proof: Suppose that G is an automorphism group of order $3 \cdot 2^4$ for an $S(3,5,26)$. Recall that the only $S(2,4,25)$'s which admit an automorphism of order 2 are DES_1, DES_6, and DES_8. Moreover, the Sylow-2 subgroups of these designs have orders 2^3, 2, and 2, respectively. Also recall that an automorphism of order 2 for DES_1 fixes exactly one point. Further, an automorphism of order 3 of an $S(3,5,26)$ fixes exactly 2 points.

Consider the action of G on points. Easily, the possible orbit lengths are $\{24, 16, 12, 8, 6, 4, 3, 2, 1\}$. Let H be a Sylow-2 subgroup of G, so that $|H| = 2^4$. If

G has an orbit of length 1, then G fixes a point and G would be an automorphism group of the derived design through that point, a contradiction since 2^4 divides $|G|$. If G had an orbit Δ of length 3, then the stabilizer $G_x, x \in \Delta$, would have order 2^4, a contradiction to the fact that 2^4 does not divide the order of any of the automorphism groups of $S(2,4,25)$'s.

Suppose G has an orbit Δ of length 4. Then the stabilizer G_x for any $x \in \Delta$, would have order 12, hence, the derived designs are isomorphic to DES_1. Now, G is represented as a subgroup of S_4 on Δ. By order consideration the kernel of this representation is non-trivial. Furthermore, since elements of order 3 can not fix 4 points, the kernel can not be of order divisible by 3. Hence, the kernel has order divisible by 2, and so there is an element of order 2 in $Aut(DES_1)$ fixing at least four points, a contradiction.

Suppose that G has an orbit $\{x, y\}$ of length 2. Then the stabilizer $G_x = G_y$ would have order 24, hence, the derived designs through x and y are isomorphic to DES_1. Hence, $G_x \leq PSL_2(7) \times Z_3$. Up to conjugacy there are exactly two subgroups of order 24 in $PSL_2(7) \times Z_3$. One of these subgroups is isomorphic to $S_4 \leq PSL_2(7)$, but elements of order 3 in $PSL_2(7)$ fix 4 of the 25 points. By Lemma 2.5 such an S_4 can not be a group of automorphisms of an $S(3,5,26)$. The other subgroup of order 24 in G_x is $L \simeq D_8 \times Z_3$, where D_8 is a Sylow-2 subgroup of $PSL_2(7)$. Computationally it is verified that L is regular on the 24 points, and fixes x and y pointwise. Hence, if G has an orbit of length 2, then the other orbit has length 24.

Easily, the only orbit length combinations of G are $\{24,2\}$, $\{12,8,6\}$, and $\{8,6,6,6\}$.

Suppose G has an orbit Δ of length 6. Let H be a Sylow-2 subgroup of G. Then H has an orbit of length at most 2 on Δ, so H has a subgroup K of order 8 that fixes two points x and y. Moreover, the derived designs through either x or y are isomorphic to DES_1. If G has another orbit of length 6 (or 12), then K has an orbit of length at most 2 (or 4) and, hence, an element of order 2 fixing at least two more points. But then there is an element of order 2 in $Aut(DES_1)$ fixing at least 3 points, a contradiction. Thus, G must have $\{24,2\}$ as its orbit lengths.

Suppose now that $L = D_8 \times Z_3$ is a subgroup of G. Using a computer we have ruled out such a group being the automorphism group of an $S(3,5,26)$. Our proof is complete and the following is clear. ∎

Corollary 5.1. *If G is the automorphism group of an $S(3,5,26)$ and neither 5 nor 13 divides the order of G, then $|G|$ divides $3 \cdot 2^3$ or 2^4.*

In closing note that if $|G| = 3 \cdot 2^3$ or 2^4 then G cannot fix a point of X and must have exactly one point-orbit of length 2. Moreover, if $|G| = 2^4$ then G contains $D_8 \leq PSL_2(7)$ as a normal subgroup.

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