The Effects of Graph Modifications on Edge Independence and Edge Covering Numbers

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Abstract. We consider the changing and unchanging of the edge covering and edge independence numbers of a graph when the graph is modified by deleting a node, deleting an edge, or adding an edge. In this paper we present characterizations for the graphs in each of these classes and some relationships among them.

1. Introduction

Any invariant of a graph can change or remain the same when the graph is altered by deleting a node, deleting an edge, or adding an edge. For several invariants this problem of changing and unchanging has recently been investigated ([1], [2], [3], [4], [5], [6], [9], [11], [12], [15], [16], [17], [19]). Results from such studies are important in applications of graph theory where a property of a system's graphical model changes or remains intact when the system experiences a component failure or a link addition ([7], [13], [15], [16], [18]). The unchanging of a graphical invariant upon such modifications is in some sense a measure of the stability of the system for which the graph is a model. For example, when the graph is a model for a computer network where an edge removal represents a link failure, the unchanging of an invariant implies the network is fault tolerant with respect to that invariant.

Let $G = (V, E)$ be an undirected graph with node set V and edge set E where $|V| = p$. A set of edges, C, such that every node of G is incident to an edge in C is an edge cover for G. The minimum cardinality among the sets of edge coverings is the edge covering number of G and is denoted by $\alpha_1(G)$, or simply by α_1 when G is clear from the context. The edge covering number is undefined when G has isolated nodes. A collection of edges is independent if no two edges in the collection are adjacent. The maximum cardinality among the sets of independent edges is the edge independence number of the graph G and is denoted by $\beta_1(G)$ or just β_1. In this paper we abbreviate "minimum edge covering" ("maximum independent set") as MEC (MIS). The notation and terminology not defined here may be found in [10].

Gallai's well-known result [8] states that for graphs with no isolated nodes

$$\alpha_1 + \beta_1 = p.$$

This relationship made possible the parallel study of the changing and unchanging of α_1 and β_1.

Using the general approach and terminology first suggested to us by Frank Harary, we consider the changing and unchanging of the invariants α_1 and β_1 under three different graph modifications: deleting an arbitrary node, deleting an arbitrary edge, and adding an arbitrary edge. Formally, we characterize those graphs G for which

$$\begin{aligned}
&Class\ 1: \alpha_1(G-v) \neq \alpha_1(G),\ \ \beta_1(G-v) = \beta_1(G) \text{ for all } v \in V;\\
&Class\ 2: \alpha_1(G-v) = \alpha_1(G),\ \ \beta_1(G-v) \neq \beta_1(G) \text{ for all } v \in V;\\
&Class\ 3: \alpha_1(G-e) \neq \alpha_1(G),\ \ \beta_1(G-e) \neq \beta_1(G) \text{ for all } e \in E;\\
&Class\ 4: \alpha_1(G-e) = \alpha_1(G),\ \ \beta_1(G-e) = \beta_1(G) \text{ for all } e \in E;\\
&Class\ 5: \alpha_1(G+e) \neq \alpha_1(G),\ \ \beta_1(G+e) \neq \beta_1(G) \text{ for all } e \in \overline{E};\\
&Class\ 6: \alpha_1(G+e) = \alpha_1(G),\ \ \beta_1(G+e) = \beta_1(G) \text{ for all } e \in \overline{E}.
\end{aligned}$$

This report is organized as follows. Section 2.1 contains the results for graphs in Class 1 where the edge independence number, β_1, is unchanged and the edge covering number, α_1, is changed when any arbitrary node is removed. In Section 2.2 we present the results for Class 2 graphs, that is, those graphs for which β_1 is changed and α_1 is unchanged when any arbitrary node is removed. Both sections 2.1 and 2.2 contain material useful in the construction of families of graphs with these properties as well as examples of such graphs. We consider edge removal in Section 3 and present the characterizations of graphs in Classes 3 and 4. The last type of modification, adding an arbitrary edge, is the subject of Section 4 where we present characterizations of graphs in Classes 5 and 6. We establish some relationships among the six classes in Section 5 and make concluding remarks in Section 6.

2. Node Removal

When a node is removed from a graph neither α_1 nor β_1 can increase, hence $\alpha_1(\beta_1)$ decreases or remains the same. Observe that if α_1 or β_1 changes it must decrease by exactly one. That is, if $\alpha_1(G) \neq \alpha_1(G-v)$, then $\alpha_1(G-v) = \alpha_1(G) - 1$. A corresponding statement can be made for β_1.

2.1 The graphs for which the edge independence number, β_1, is unchanged and the edge covering number, α_1, is changed when any arbitrary node is removed. [Class 1 graphs]

We consider the characterization of graphs where α_1 changes upon node removal, followed by the characterization of graphs with unchanging β_1.

Theorem 1. *For any graph G, $\alpha_1(G-v) = \alpha_1(G) - 1$ for all $v \in V$ if and only if for any v there exists at least one MEC, C, and an edge $e = uv \in C$ where e is only in C to cover v.*

Proof: For sufficiency let v be an arbitrary node in G, and suppose there exists a MEC, C, such that for some u, $uv \in C$ and u is covered in C by another edge. Clearly $C-uv$ covers $G-v$, hence $\alpha_1(G-v) = |C-uv| = \alpha_1(G)-1$. Suppose next that $\alpha_1(G-v) = \alpha_1(G) - 1$ for every $v \in V$. Let C' be a MEC for $G-v$ where v is an arbitrary node. Since v is not isolated there exists a node $u \in N(v)$ covered by C'. But $C'+uv = C$ is an edge cover of G whose cardinality is $\alpha_1(G)$. Hence C is a MEC for G and the edge uv is in C only to cover v. ∎

Observe that any minimum edge covering of a graph consists of a maximum collection of stars and that from any covering a collection of independent edges can be constructed by choosing one edge from each star ([10]). We note that the following two conditions are equivalent: (1) for any v there exists at least one MEC, C, and an edge $e = uv \in C$ where e is only in C to cover v and (2) for any v there exists at least one MIS, I, that does not saturate v. Hence, assuming that G and $G-v$ can have no isolated nodes we can apply the relationship $\alpha_1 + \beta_1 = p$ and the above theorem to obtain results for β_1 as follows.

Theorem 2. *For any graph G, $\beta_1(G-v) = \beta_1(G)$ for all $v \in V$ if and only if for any v there exists a MIS, I, that does not saturate v for all $v \in V$.*

Since the removal of an isolated node does not change β_1, the graphs characterized in Theorem 2 may have any number of isolated nodes. Assuming no isolated nodes, Theorems 1 and 2 are equivalent and characterize the graphs of Class 1. The following two theorems establish properties of and give a method for constructing infinite families of Class 1 graphs. Although these results were presented for β_1 in [12], they are restated here in terms of β_1 and α_1 for completeness. Two nodes are 'identified' when they are replaced by a single node with a neighborhood which is the union of the neighborhoods of the two removed nodes.

Theorem A ([12]). *Any Class 1 graph with $p \geq 3$ has no bridges.*

Theorem B ([12]). *Let G be constructed from G_1 and G_2 by identifying a node of G_1 with one of G_2. Then G is a Class 1 graph if and only if G_1 and G_2 are Class 1 graphs.*

In view of Theorem B, components are blocks (connectivity greater than one). Hence we need only be concerned with a component. A stronger characterization than the one in Theorem 2 for connected graphs with unchanging β_1 is given in [14] and is restated here.

Theorem C ([14]). *Graph G is connected and $\beta_1(G-v) = \beta_1(G)$ for all $v \in V$ if and only if for any v, $G-v$ has a perfect matching.*

Assuming G is not the trivial graph, an analogous statement can be made for α_1. Hence connected Class 1 graphs have $\beta_1 = (p-1)/2$ and $\alpha_1 = (p+1)/2$. Note that the union of Class 1 graphs is a Class 1 graph. Thus, for each component

G_i in a Class 1 graph, $G_i - v$ has a perfect matching for all $v \in V$. We make the following observations.

Observations. *If G is a Class 1 graph, then*

(a) *every node has degree greater than or equal to 2 and must lie on a cycle,*
(b) $\beta_1(G - v - u) = \beta_1(G - v) - 1$ *for all* $u, v \in V$,
(c) $\alpha_1(G - v - u) = \alpha_1(G - v) = \alpha_1(G) - 1$ *for all* $u,v \in V$.

The infinite families K_{2n+1} and C_{2n+1} for $n \geq 1$ are examples of Class 1 graphs.

2.2 The graphs for which the edge independence number β_1, is changed and the edge covering number α_1, is unchanged when any arbitrary node is removed. [Class 2 graphs]

The following Theorem reproduced here from [12] for completeness gives the characterization for graphs with changing β_1.

Theorem 3 ([12]). *For any graph G, $\beta_1(G - v) = \beta_1(G) - 1$ for all $v \in V$ if and only if G has a perfect matching.*

Note that there can be no isolated nodes in the graphs G from Theorem 3, so the following corresponding characterization follows directly from the fact that $\alpha_1 + \beta_1 = p$.

Theorem 4. *For any graph G where for any v, $G - v$ has no isolated nodes $\alpha_1(G) = \alpha_1(G - v)$ for all $v \in V$ if and only if G has a perfect matching.*

Corollary. *A graph G is a Class 2 graph if and only if G has a perfect matching and for all $v \in V$, G and $G - v$ have no isolated nodes.*

Examples of infinite families of Class 2 graphs include K_{2n} for $n \geq 2$, C_{2n} for $n \geq 2$, $K_{n,n}$ for $n \geq 2$, $G \times K_2$, for connected graphs G with two or more nodes. Note that paths with an even number of nodes satisfy the characterization of graphs of Theorem 3, but fail in Theorem 4 since $G - v$ may have an isolated node.

We complete this section with a Theorem that is useful in the construction of arbitrarily large Class 2 graphs. The proof is immediate from the fact that G has a perfect matching when G_1 and G_2 have perfect matchings.

Theorem 5. *Let G be constructed from G_1 and G_2 by connecting an arbitrary node of G_1 with an arbitrary node of G_2 with a bridge. If G_1 and G_2 are Class 2 graphs, then G is a Class 2 graph.*

3. Edge Removal

Observe that the removal of an edge cannot increase β_1, so $\beta_1(G - e) \leq \beta_1(G)$. Furthermore, the removal of an edge can decrease β_1 by at most one. Hence, if

β_1 changes when an edge e is removed, $\beta_1(G-e) = \beta_1(G) - 1$. Since both G and $G-e$ are the same order p and $p = \alpha_1 + \beta_1$, when there are no isolated nodes in G or $G-e$, $\beta_1(G-e) = \beta_1(G) - 1$ implies $\alpha_1(G-e) = \alpha_1(G) + 1$.

3.1. The graphs for which the edge independence number, β_1, is changed and the edge covering number, α_1, is changed when any arbitrary edge is removed. [Class 3 graphs]

Theorem 6 gives a characterization for graphs whose edge independence number changes upon the removal of an arbitrary edge.

Theorem 6. *In a graph G, $\beta_1(G-e) = \beta_1(G) - 1$ for all $e \in E$ if and only if $G = mK_2 \cup nK_1$.*

Proof: Suppose $\beta_1(G-e) = \beta_1(G) - 1$ for all $e \in E$. Then e must be in every MIS for all $e \in E$. Thus $G = mK_2 \cup nK_1$. Sufficiency is obvious. ∎

Again when there are no isolated nodes, changing β_1 implies changing α_1. Thus only the graphs $G = nK_2$ in Theorem 6 can be considered. However, $G-e$ has isolated nodes so $\alpha_1(G-e)$ is not defined and Theorem 7 follows directly.

Theorem 7. *Class 3 graphs do not exist.*

3.2. The graphs for which the edge independence number, β_1, is unchanged and the edge covering number, α_1, is unchanged when any arbitrary edge is removed. [Class 4 graphs]

The next two characterizations are for the families of graphs that have unchanging α_1 and hence unchanging β_1 when any arbitrary edge is removed. The proofs of Theorems 8 and 9 are straightforward.

Theorem 8. *A graph G has $\alpha_1(G-e) = \alpha_1(G)$ for all $e \in E$ if and only if for any e there exists a MEC, C, of G such that $e \notin C$.*

Theorem 9. *A graph G has $\beta_1(G-e) = \beta_1(G)$ for all $e \in E$ if and only if for any e there exists a MIS, I, of G such that $e \notin I$.*

Notice that Theorem 9 allows the existence of isolated nodes while Theorem 8 does not. Assuming no isolated nodes in either G or any $G-e$, Theorems 8 and 9 characterize Class 4 graphs. Examples of infinite families of Class 4 graphs include C_p, K_p for $p \geq 3$, and $K_{m,n}$ where $m \geq 2$ and $n \geq 2$. Paths of length two or more satisfy conditions for Theorem 9, although they are not Class 4 graphs since $G-e$ may have an isolated node.

4. Edge Addition

Consider the modification of the graph G by the addition of an edge from the complement of G. Here we assume that $\overline{E}$ is not empty. Again we use the relationship $p = \alpha_1 + \beta_1$ to establish that α_1 changes (stays the same) when β_1 changes (stays the same).

4.1. The graphs for which the edge independence number, β_1 is changed and the edge covering number, α_1, is changed when any arbitrary edge from the complement is added. [Class 5 graph]

The next two Theorems characterize the Class 5 graphs where β_1 and α_1 change when an arbitrary edge is added.

Theorem 10. *In a graph G, $\beta_1(G+e) = \beta_1(G)+1$ for all $e \in \overline{E}$ if and only if for every pair of non-adjacent nodes u and v, $\beta_1(G-u-v) = \beta_1(G)$.*

Proof: Let u and v be non-adjacent nodes of G. Since $\beta_1(G+uv) = \beta_1(G)+1$, a MIS of $G+uv$ must include uv. Thus there are $\beta_1(G)$ independent edges not incident to either u or v so $\beta_1(G-u-v) = \beta_1(G)$. Suppose a set of independent edges of size $\beta_1(G)$ can be found in $G-u-v$ for an arbitrary pair of non-adjacent nodes u and v. Then $\beta_1(G+uv) = \beta_1(G)+1$. ∎

To characterize the class of graphs where α_1 changes when an arbitrary edge is added, we assume that G has no isolated nodes and restate Theorem 10 in terms of α_1.

Theorem 11. *In a graph G, $\alpha_1(G+e) = \alpha_1(G)-1$ for all $e \in \overline{E}$ if and only if for every pair of non-adjacent nodes u and v, $\alpha_1(G-u-v) = \alpha_1(G)-2$.*

Theorem 11 implies that $G-u-v$ also has no isolated nodes for any pair of nonadjacent nodes u and v.

Corollary. *In a Class 5 graph, for any component on p nodes*

(a) *any degree 1 node must be adjacent to a degree $p-1$ node;*
(b) *any degree 2 node must be in a triangle.*

Proof: Suppose there is a degree 1 node u with neighbor v in a component of a Class 5 graph G. Then there exists a node $w \notin N(v)$. But $G-w-v$ will leave u isolated. Hence the degree of v is $p-1$. For the proof of (b) suppose u is a degree 2 node with neighbors v and w in a component of a Class 5 graph G. Suppose v is not adjacent to w. Then u is isolated in $G-v-w$, a contradiction. Hence v must be adjacent to w. ∎

Assuming no isolated nodes, Theorems 10 and 11 characterize Class 5 graphs. Lovàsz and Plummer, [14], present a characterization of saturated non-factorizable graphs, that is, a graph G which has no perfect matching, but $G+uv$ does for all $uv \in \overline{E}$. We note that these graphs are a subset of Class 5 graphs. We restate their characterization in Theorem D.

Theorem D ([14]). *A graph G is saturated non-factorizable if and only if it has the following structure: either p is odd and G is complete, or p is even and G consists of point-disjoint complete subgraphs $S, G_1, \ldots, G_k$ such that $k = |S|+2$,* *the number of nodes in each G_i is odd and every point of every G_i is adjacent to every point of S.*

Next we present a generalization of Theorem D that characterizes the structure of graphs with changing β_1. The proof to this characterization is based on the following lemma.

Lemma. *If $\beta_1(G+e) = \beta_1(G) + 1$ for all $e \in \overline{E}$ and $S \subset G$ is the subgraph containing the nodes of G that are adjacent to every other node, then every component of $G - S$ is complete.*

Proof: Since $\overline{E} \neq \phi$, $S \neq G$, hence $G - S \neq \phi$. If each component in $G - S$ is either an isolated node or a K_2, we are finished. Hence consider three nodes in any component of $G - S$, say u,v, and w where uv and vw are edges. We claim that u is adjacent to w. Suppose not. Since $v \notin S$, there exists an $x \in V$ where x is not adjacent to v. Since $\beta_1(G+uw) = \beta_1(G) + 1$ there exists a MIS, M_1, of $G + uw$ where $uw \in M_1$ and $|M_1| = \beta_1(G) + 1$. Similarly there exists a MIS, M_2, of $G + vx$ where $vx \in M_2$ and $|M_2| = |M_1|$. Notice $uw \notin M_2$, $vx \notin M_1$, and uv is not in either M_1 or M_2. Let H be the symmetric difference of M_1 and M_2, that is $H = M_1 \oplus M_2$. The components of H are isolated points, even paths, and even cycles (since M_1 and M_2 are the same size). Since $uw \in M_1$ and $uw \notin M_2$ and $vx \in M_2$ and $vx \notin M_1$ we know u, v, w, and x are not isolated nodes in H. Thus, uw and vx are either on even paths or cycles. Suppose that uw and vx are in the same component of H. Then there exists a path containing both vx and uw and hence a path exists from v to u that may or may not include x. For the first case, let P be the path from v to u (or w, if w is encountered before u) containing x that is, $P = v, x, \ldots, u$. Then $M_2 \oplus (P + uv)$ (or $M_2 \oplus (P + vw)$) is a MIS of G of size $|M_2|$, a contradiction. Otherwise let P be the path from v to u (or w) where x is not on the path. Then $M_1 \oplus (P + vu)$ (or $M_1 \oplus (P + vw)$) is a MIS of G of size $|M_1|$, a contradiction. Hence uw and xv cannot lie in the same component of H. Now we suppose that uw is on an even path, P, or an even cycle C. Here we let $M = M_1 \oplus C$ or $M = M_1 \oplus P$ and M is a MIS of G with size $\beta_1(G) + 1$, a contradiction. Hence the node u must be adjacent to the node w and the components of $G - S$ must be complete. ▮

Using this lemma we now give the following structural characterization.

Theorem 12. *For a graph G, $\beta_1(G+e) = \beta_1(G) + 1$ for all $e \in \overline{E}$ if and only if G consists of node disjoint subgraphs $S, G_1, G_2, \ldots, G_k$ where no node of G_i is adjacent to any node of G_j for $i \neq j$, every node in S is adjacent to every node in V, each G_i, is complete with an odd number of nodes, and $k \geq |S| + 2$ when p is even and $k \geq |S| + 3$ for p odd. Moreover, $\beta_1(G) = (p - k + |S|)/2$.*

Proof: The proof of sufficiency is straightforward. To show necessity suppose that $\beta_1(G+e) = \beta_1(G) + 1$ for all $e \in \overline{E}$. The lemma implies that the components,

G_i, of $G - S$ are complete. If there exists a component with an even number of nodes then any edge from a node in this component to any other node could not increase β_1. Hence no component has an even number of nodes. If $k \le |S| + 1$ then G has a perfect matching if p is even or there exists a $v \in V$ for which $G - v$ has a perfect matching if p is odd. In either case $\beta_1(G + e) \ne \beta_1(G) + 1$. Hence $k > |S| + 1$. Then there are $k - |S|$ nodes not matched in any MIS of G, and $p - (k - |S|)$ is even. Thus, for p even, k and $|S|$ have the same parity and for p odd, k and $|S|$ have opposite parity. It follows that for p even, $k \ge |S| + 2$ and for p odd, $k \ge |S| + 3$. Moreover, $\beta_1(G) = (p - k + |S|)/2$. ∎

Assuming there are no isolated nodes in G, Theorem 12 gives a complete characterization of the structure of Class 5 graphs. See Figure 1 for an example. Other examples of Class 5 graphs include bipartite graphs $K_{1,n}$, $n \ge 3$, the union of any arbitrary number of odd complete graphs on at least three nodes, and $K_{1,n} \cup K_{2m+1}$, $n \ge 3$, $m \ge 1$.

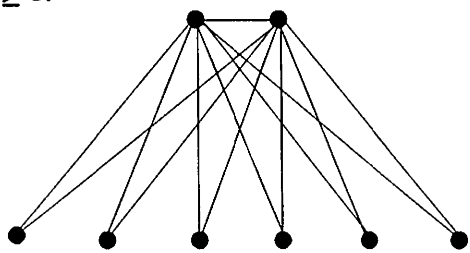

Figure 1: An example of a Class 5 graph.

4.2. The graph for which the edge independence number, β_1, is unchanged and the edge covering number, α_1, unchanged when any arbitrary edge from the complement is added. [Class 6 graphs]

Now consider the classes of graphs where α_1 and β_1 stay the same when an arbitrary edge is added.

Theorem 13. *A graph G has $\beta_1(G + e) = \beta_1(G)$ for all $e \in \overline{E}$ if and only if $\beta_1(G) = \lceil (p - 1)/2 \rceil$.*

Proof: Assume $\beta_1(G) = \lceil (p-1)/2 \rceil$. Hence $\beta_1(G)$ is at its maximum possible value for a graph on p nodes. Suppose $\beta_1(G + e) = \beta_1(G)$ for all $e \in \overline{E}$ and $\beta_1(G) < \lceil (p - 1)/2 \rceil$. For any MIS, I, of G there exist at least two unsaturated nodes, say u and v. However, u and v must be nonadjacent, otherwise β_1 would be larger. But $\beta_1(G+ uv) \ge \beta_1(G)$, a contradiction. Hence $\beta_1(G) \ge \lceil (p-1)/2 \rceil$. ∎

We now assume no isolated nodes and restate the Theorem in terms of α_1.

Theorem 14. *A graph G has $\alpha_1(G + e) = \alpha_1(G)$ for all $e \in \overline{E}$ if and only if $\alpha_1(G) = \lfloor (p - 1)/2 \rfloor$.*

Assuming no isolated nodes, Theorems 13 and 14 are equivalent and characterize Class 6 graphs. Examples of Class 6 graphs include C_n, $K_{n,n}$, for $n \geq 2$, paths, wheels, and $H \times G$ where G is any connected graph on $p \geq 2$ nodes and H is any graph with a perfect matching.

5. Relationships Among The Classes Of Graphs

Considering the results from the previous sections we obtain some interesting relationships among the six classes of graphs. Assume $\overline{E} \neq \phi$ when Class 5 and Class 6 graphs are discussed. Note that assuming $\overline{E}$ is not empty removes from Class 1 complete graphs on an odd number of nodes and from Class 2 complete graphs on an even number of nodes. (The only graphs in both Class 1 and 2 are complete graphs.) It follows directly from the characterizations that both Class 1 and Class 2 graphs are Class 6 graphs. However, the converse is not true as can be seen in Figure 2. The following two Theorems relate Class 1 graphs and Class 4 graphs.

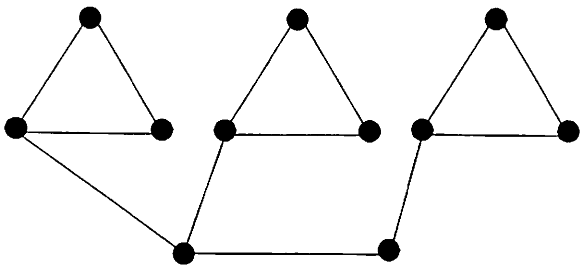

Figure 2: Example of a graph in Class 6, but not in Classes 1 or 2.

Theorem 15. *If* $\beta_1(G - v) = \beta_1(G)$ *for all* $v \in V$ *then* $\beta_1(G - e) = \beta_1(G)$ *for all* $e \in E$.

Proof: Choose an arbitrary node v. Then $\beta_1(G) = \beta_1(G - v) \leq \beta_1(G - uv) \leq \beta_1(G)$ since removing v also removes its incident edges. ∎

Theorem 15 can be restated in terms of α_1 as follows.

Theorem 16. *In a graph* G *with no isolated nodes, if* $\alpha_1(G - v) = \alpha_1(G) - 1$ *for all* $v \in V$ *then* $\alpha_1(G - e) = \alpha_1(G)$ *for all* $e \in E$.

Theorems 15 and 16 show that a Class 1 graph is also a Class 4 graph. However, the converse is not true. For a counterexample, an even cycle is a Class 4 graph but not a Class 1 graph. Next we show that any Class 5 graph is also a Class 4 graph when α_1 is defined for both.

Theorem 17. *If* $\beta_1(G + e) = \beta_1(G) + 1$ *for all* $e \in \overline{E}$ *then* $\beta_1(G - e) = \beta_1(G)$ *for all* $e \in E$.

Proof: Let $e_1 = wv \in E$ where $e = uv \in \overline{E}$. Now $\beta_1(G + uv) = \beta_1(G) + 1$ so $\beta_1(G - u - v) = \beta_1(G)$. Hence $\beta_1(G) = \beta_1(G - wv) = \beta_1(G - e_1)$. Now suppose that the degrees of w and v are $p-1$. Let I be a MIS of G. If $e_1 \notin I$, $\beta_1(G-e_1) = \beta_1(G)$. Suppose $e_1 \in I$. If $G = K_3$, clearly $\beta_1(G-e) = \beta_1(G)$. If $p \geq 4$ there is second edge $e_2 = rs \in I$. Notice that w, v, r, s induces a K_4 and e_1 and e_2 saturate the four nodes. Another MIS of G occurs by replacing e_1 and e_2 in I by wr and vs. Thus in this last case we also have $\beta_1(G-e_1) = \beta_1(G)$. ∎

Theorem 17 can be restated in terms of α_1 as follows.

Theorem 18. *Assume G and $G-e$ for all $e \in E$ have no isolated nodes. Then $\alpha_1(G+e) = \alpha_1(G) - 1$ for all $e \in \overline{E}$ implies that $\alpha_1(G-e) = \alpha_1(G)$ for all $e \in E$.*

Theorems 17 and 18 show that Class 5 graphs with no pendant edges are also in Class 4. A star is a Class 5 graph, but not a Class 4 graph since removing any edge leaves an isolated node. Again the converses of Theorems 17 and 18 are not true. The cycle C_4 provides a counterexample.

Theorem 19. *For connected graphs, Class 5 $\cap$ Class 1 $= \phi$.*

Proof: Let G be a connected Class 5 graph. If $\Delta < p-1$, then $|S| = 0$ where S is the set of nodes adjacent to all nodes in G. By Theorem 12, $G - S$ consists of components $G_1, \ldots, G_k$, $k \geq |S|+2$. This implies G has at least two components, a contradiction. Thus every connected Class 5 graph G has $\Delta(G) = p-1$. Since $\overline{E} \neq \phi$, there exists two nonadjacent nodes u and v such that $\beta_1(G-u-v) = \beta_1(G)$. By Theorem C, G is not a Class 1 graph. ∎

Figure 3 gives an example of a Class 4 graph G where G is not in Classes 1 or 5.

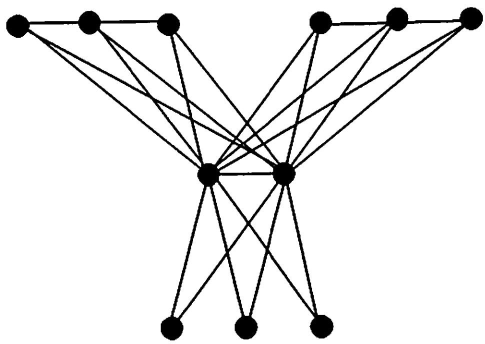

Figure 3: Example of a graph in Class 4, but not in Classes 1 or 5.

6. Concluding Remarks

We have given characterizations of the six classes of graphs for which α_1 and β_1 change or remain the same under node and edge removal and edge addition and have explored some relationships among these classes of graphs. The characterization of graphs in each of the six classes for many other graphical invariants is still an open problem. It is an ongoing project of the authors to study this problem for several invariants.

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