# Spreads Of Lines and Regular Group Divisible Designs

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**Abstract.** A 1-spread of a BIBD $\mathcal{D}$ is a set of lines of maximal size of $\mathcal{D}$ which partitions the point set of $\mathcal{D}$. The existence of infinitely many non-symmetric BIBDs which (i) possess a 1-spread, and (ii) are not merely a multiple of a symmetric BIBD, is shown. It is also shown that a 1-spread $S$ gives rise to a regular group divisible design $\mathcal{G}(S)$. Necessary and sufficient conditions that the dual of such a group divisible design $\mathcal{G}(S)$ be a group divisible design are established and used to show the existence of an infinite class of symmetric regular group divisible designs whose duals are not group divisible.

## 1. Introduction.

A line of a BIBD $\mathcal{D}$ is the intersection of all the blocks on two points of $\mathcal{D}$. There is a well-known upper bound on the number of points in a line of a BIBD. A line whose number of points meets this upper bound is said to be of "maximal length". A 1-spread of a BIBD $\mathcal{D}$ is a set of lines of maximal length of $\mathcal{D}$ which partitions the point set of $\mathcal{D}$.

In Section 3, we determine the form of the parameters of a BIBD possessing a 1-spread and also show the existence of an infinite class of non-symmetric BIBDs each member of which

(i) possesses a 1-spread; and
(ii) is not a multiple of a symmetric BIBD.

In Section 4 we show that any 1-spread of a BIBD give rise to a regular group divisible design. We then give a proof (alternative to the one given in [**10**]) of the following result: If a $(48\delta + 15, 24\delta + 7, 12\delta + 3)$-design $\mathcal{D}$ has a 1-spread $S$, then there is an affine BIBD with four blocks in each affine resolution class and with each pair of non-disjoint blocks meeting in $3\delta + 1$ points.

In Section 5 a major aim is to give necessary and sufficient conditions that the dual of the regular group divisible design obtainable from a 1-spread of a BIBD is also a group divisible design. We also construct an infinite class of self-dual regular group divisible designs using "geometric" 1-spreads in $PG(2d+1, q)$. Symmetric regular group divisible designs whose duals are not group divisible seem to be rare. In Section 5 we also show the existence of an infinite class of symmetric regular group divisible designs

(i) whose duals are not group divisible; and
(ii) whose parameters are not the same as those of the group divisible designs of this type given by Jungnickel and Vedder [7].

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## 2. Preliminaries.

We denote the set of points incident with a block $B$ of an incidence structure by $(B)$. Let $\mathcal{F} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure and consider $\overline{\mathcal{P}} \subseteq \mathcal{P}$ and $B = \mathcal{B}$. We say that $\overline{\mathcal{P}}$ and $B$ are *disjoint* if $\overline{\mathcal{P}} \cap (B) = \phi$ and that $\overline{\mathcal{P}}$ is *incident with* $B$ if $\overline{\mathcal{P}} \subseteq (B)$. The *intersection* of blocks $B_1, \ldots, B_m$ of $\mathcal{F}$ is $\cap_{i=1}^{m}(B_i)$. The *multiplicity* of $B$ is $|\{C \in \mathcal{B}: (C) = (B)|$.

A finite incidence structure with $v$ points, $b$ blocks, $\tau$ blocks on each point and $k$ points on each block is called a $(v, b, \tau, k)$*-configuration*. For a $(v, b, \tau, k)$-configuration we must have $v\tau = bk$. An incidence structure $\mathcal{F} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is said to be *2-balanced* if every 2-subset of $\mathcal{P}$ is incident with the same number ($\lambda$, say) of blocks of $\mathcal{B}$. The number $\lambda$ is called the *index* of $\mathcal{F}$. For the purposes of this paper, a *balanced incomplete block design* (BIBD) is a 2-balanced $(v, b, \tau, k)$-configuration with positie index and $v > k \geq 2$. A $(v, b, \tau, k)$-configuration which is a BIBD with index $\lambda$ is called a $(v, b, \tau, k, \lambda)$*-design*. For a $(v, b, \tau, k, \lambda)$-design we must have $v\tau = bk$, $\lambda(v-1) = \tau(k-1)$ and $b \geq v$. A $(v, \omega b, \omega\tau, k, \omega\lambda)$-design where $b \geq v$, is said to be an $\omega$*-quasimultiple of a* $(v, b, \tau, k, \lambda)$*-design*.

For further basic notions, definitions and facts concerning incidence structures [1], [2] or [5] could be consulted.

## 3. Spreads.

A line of a BIBD $\mathcal{D}$ is the intersection of all the blocks of $\mathcal{D}$ on two distinct points of $\mathcal{D}$. If $\mathcal{D}$ is a $(v, b, \tau, k, \lambda)$-design and $L$ is a line of $\mathcal{D}$, then $|L| \leq \frac{b-\lambda}{\tau-\lambda}$ ([5, p. 78]). Also, for any block $B$ of $\mathcal{D}$ we have $L \subseteq (B)$ or $|L \cap (B)| \leq 1$. If $|L| = \frac{b-\lambda}{\tau-\lambda}$, then $L$ is said to be of *maximal length* and we have $L \subseteq (B)$ or $|L \cap (B)| = 1$ for all blocks $B$ of $\mathcal{D}$. A *1-spread* of a BIBD $\mathcal{D}$ is a set of lines of maximal length which partitions the point set of $\mathcal{D}$. A 1-spread $\mathcal{S}$ is said to be *uniform* if the number of blocks containing a pair of distinct lines of $\mathcal{S}$ is independent of the pair of lines chosen.

The following lemma will prove of use.

**Lemma 1.** *Every 1-spread of a BIBD is uniform.*

Proof: Let $\mathcal{S} = \{L_1, \ldots, L_m\}$ be a 1-spread of a $(v, b, \tau, k, \lambda)$-design $\mathcal{D}$. Now a block of $\mathcal{D}^c$ (the complement of $\mathcal{D}$) meets $L_i$ in $\frac{b-\lambda}{\tau-\lambda} - 1$ points or in no points. So $L_i$ is a maximal $(\frac{b-\tau}{\tau-\lambda})$-arc ([9]) of '$\mathcal{D}^c$. Thus, $\{L_1, \ldots, L_m\}$ a maximal arc partition of $\mathcal{D}^c$. But the number of blocks not meeting each of a pair of maximal arcs in a maximal arc partition of a BIBD is independent of the pair of maximal arcs chosen (see [11]). The result follows. ∎

Suppose $\mathcal{D}$ is a $(v, b, \tau, k, \lambda)$-design and $\mathcal{S}$ is a 1-spread of $\mathcal{D}$. From $v\tau = bk$

and $\lambda(v-1) = \tau(k-1)$ we obtain

$$\lambda(bk - \tau) = \tau^2(k-1)$$

whence

$$\lambda(b-\lambda)k = (\tau^2 - \lambda^2)k - \tau(\tau - \lambda)$$

and so

$$\lambda\left(\frac{b-\lambda}{\tau-\lambda}\right)k = (\tau+\lambda)k - \tau.$$

Clearly, we have $\tau = \rho k$ for some positive integer $\rho$. But then

$$\lambda\left(\frac{b-\lambda}{\tau-\lambda}\right) = \rho(k+1) + \lambda$$

and so $\lambda \mid \rho(k-1)$. Let $\rho(k-1) = \alpha\lambda$. We then have that $\mathcal{D}$ is a $\left(k\alpha+1, \rho(k\alpha+1), \rho k, k, \frac{\rho(k-1)}{\alpha}\right)$-design and $|L| = \frac{b-\lambda}{\tau-\lambda} = \alpha + 1$ for each line $L$ of $\mathcal{S}$. Note that $\alpha \geq 1$.

Now $\alpha + 1 \mid v = k\alpha + 1$ and so $\alpha + 1 \mid k - 1$. Let $k = \sigma(\alpha + 1) + 1$. Immediately we have $\alpha \mid \rho\sigma$. We note that there are $\sigma\alpha + 1$ lines in $\mathcal{S}$.

Consider the incidence structure $\mathcal{D}(\mathcal{S})$ whose points are the lines of $\mathcal{S}$ and whose blocks are the blocks of $\mathcal{D}$, incidence being defined by $L$ is on $B$ if and only if $L \subseteq (B)$. Clearly, $\mathcal{D}(\mathcal{S})$ has $\sigma\alpha + 1$ points and $\rho(\alpha + 1)(\sigma\alpha + 1)$ blocks. Also, each line of $\mathcal{D}$ is in $\lambda$ blocks of $\mathcal{D}$ and so each point of $\mathcal{D}(\mathcal{S})$ is on $\lambda = (\alpha + 1)\left(\frac{\rho\sigma}{\alpha}\right)$ blocks of $\mathcal{D}(\mathcal{S})$.

Next let $x$ be the number of lines of $\mathcal{S}$ in a block $B$ of $\mathcal{D}$ and $y$ be the number of lines in $\mathcal{S}$ meeting $B$ in a unique point. Then we have

$$x + y = \sigma\alpha + 1$$

and

$$(\alpha+1)x + y = \sigma(\alpha+1) + 1.$$

Clearly, $x = \frac{\sigma}{\alpha}$. Let $\sigma = \tau\alpha$. We now have that $\mathcal{D}$ is an $\left((\alpha+1)(\tau\alpha^2+1), \rho(\alpha+1)(\tau\alpha^2+1), \rho(\tau\alpha^2+\tau\alpha+1), \tau\alpha^2+\tau\alpha+1, \rho\tau(\alpha+1)\right)$-design and that $\mathcal{D}(\mathcal{S})$ is a $\left(\tau\alpha^2+1, \rho(\alpha+1)(\tau\alpha^2+1), \rho\tau(\alpha+1), \tau\right)$-configuration. But every 1-spread of a BIBD is uniform (Lemma 1). So $\mathcal{D}(\mathcal{S})$ is a BIBD or $\tau = 1$.

But then $\overline{\lambda}(\tau\alpha^2) = \rho\tau(\alpha + 1)\ (\tau - 1)$, where $\overline{\lambda}$ is the index of $\mathcal{D}(S)$. We then infer that $\alpha^2 \mid \rho(\tau - 1)$ and so $\tau = \frac{\delta}{\rho}\alpha^2 + 1$ for some $\delta \geq 0$. We then have that the parameters of $\mathcal{D}$ are given by

$$\begin{aligned} v &= (\alpha + 1)(\Delta\alpha^2 + 1) \\ b &= \rho(\alpha + 1)(\Delta\alpha^2 + 1) \\ r &= \rho(\Delta\alpha^2 + \Delta\alpha + 1) \\ k &= \Delta\alpha^2 + \Delta\alpha + 1 \\ \lambda &= \rho(\alpha + 1)\Delta, \end{aligned} \tag{1}$$

where $\Delta = \frac{\delta}{\rho}\alpha^2$. Also $\mathcal{D}(S)$ is a $(\Delta\alpha^2 + 1,\ \rho(\alpha+1)\ (\Delta\alpha^2 + 1), \rho(\alpha+1)\Delta, \Delta, (\alpha+1)\ \delta)$-design or an $(\alpha^2 + 1, \rho(\alpha^2 + 1), \rho(\alpha + 1), 1)$-configuration as $\delta > 0$ or $\delta = 0$.

Remarks:

(a) If $\rho = 1$, $\delta > 0$ and $\mathcal{D}(S)$ has a block $B$ of multiplicity $\alpha + 1$, then $\mathcal{D}(S)$ is (in the terminology of [13]) a "generalized symmetric design". The substructure of $\mathcal{D}(S)$ defined by the points of $\mathcal{D}(S)$ not on $B$ and the blocks of $\mathcal{D}(S)$ other than $B$ and its repeats is a "generalized residual design" with parameters $v' = \alpha^2(\Delta - \delta)$, $b' = (\alpha+1)\ \alpha^2\Delta$, $r' = (\alpha+1)\Delta$, $k' = \Delta - \delta$ and $\lambda' = (\alpha + 1)\delta$. Such a block $B$ also yields a "generalized derived design".

(b) If $\rho = 1$ and $\mathcal{D}(S)$ is an $(a + 1)$-multiple of a $(\Delta\alpha^2 + 1, \Delta, \delta)$-design, then we say, for brevity, that $\mathcal{D}(S)$ is an "$(\alpha + 1)$-multiple". We also refer to a BIBD with parameters (1) as a "$[\delta, \rho, \alpha]$-design".

Let $n$ and $t$ be integers such that $n \geq 3$ and $1 \leq t < n$. Also, let $q$ be a prime power and $PG(n, q)$ be the $n$-dimensional projective geometry over $GF(q)$. A *$t$-spread* of $PG(n, q)$ is a set of $t$-dimensional subspaces of $PG(n, q)$ which partitions the point set of $PG(n, q)$. If $S$ is a $t$-spread of $PG(n, q)$ and $U \in S$, then we say $U$ is a *component* of $S$. If $S$ is a $t$-spread of $PG(n, q)$ and $V$ is a subspace of $PG(n, q)$ such that the components of $S$ in $V$ form a $t$-spread of $V$, then we say that $S$ *induces a $t$-spread* on $V$. It is well-known that $PG(n, q)$ possesses a $t$-spread if and only if $t + 1 \mid n + 1$ ([4, p.p. 72-3]). Let $d \geq 1$ and $PG_{2d}(2d + 1, 2)$ be the symmetric BIBD formed by the points and hyperplanes of $PG(2d + 1, q)$. We can identify the lines of $PG_{2d}(2d + 1, q)$ (which are all of maximal length) with the 1-dimensional subspaces of $PG(2d + 1, q)$. Any 1-spread of $PG(2d + 1, q)$, thus, yields a 1-spread of $PG_{2d}(2d + 1, q)$. Note that $PG_{2d}(2d + 1, q)$ has parameters given by (1) with $\rho = 1$, $\alpha = q$ and $\delta = 0$ or $\sum_{i=0}^{d-2} q^{2i}$ as $d = 1$ or $d \geq 2$.

Consider a 1-spread $S$ of $PG(2d+1, q)$. Each hyperplane $H$ of $PG(2d+1, q)$ contains $\sum_{i=0}^{d-1} q^{2i}$ lines of $S$ which cover $\sum_{i=0}^{2d-1} q^i$ points of $PG(2d + 1, q)$.

So these lines generate a $(2d-1)$-dimensional subspace or a $(2d)$-dimensional subspace of $PG(2d+1,q)$. Thus, the lines of $S$ in $H$ either generate a hyperplane of $H$ or $H$ itself. In the former case $H$, as a block of $\mathcal{D}(S)$, where $\mathcal{D} = PG_{2d}(2d+1,q)$, has multiplicity $q+1$ and, in the latter case, $H$ is a nonrepeated block of $\mathcal{D}(S)$. If the lines of $S$ in each hyperplane of $PG(2d+1,q)$ generate a $(2d-1)$-dimensional subspace, then $\mathcal{D}(S)$ is a $(q+1)$-multiple of a $(\sum_{i=0}^{d} q^{2i}, \sum_{i=0}^{d-1} q^{2i} \sum_{i=0}^{d-2} q^{2i})$-design, when $d \geq 2$. (If $d = 1$, then $\mathcal{D}(S)$ is a $(q+1)$-multiple of a $(q^2+1, q^2+1, 1, 1)$-configuration.)

Now suppose $S$ is a 1-spread of $PG(4e+3,q)$, where $e \geq 1$. Here $\mathcal{D}(S)$ is a $(q+1)$-quasimultiple of a $(\sum_{i=0}^{2e+1} q^{2i}, \sum_{i=0}^{2e} q^{2i} \sum_{i=0}^{2e-1} q^{2i})$-design. The parameters of $\mathcal{D}(S)$ can be obtained from (1) by putting $\alpha = q^2$, $\rho = q+1$ and $\delta = 0$ or $(q+1)\sum_{i=0}^{e-2} q^{4i}$ as $e = 1$ or $e \geq 2$. So $\mathcal{D}(S)$ might also possess a 1-spread. In fact, the number of points on a line of maximal length of $\mathcal{D}(S)$ is $q^2+1$. Also, the number of lines in a 1-spread of $\mathcal{D}(S)$ is $\sum_{i=0}^{e} q^{4i}$ which equals the number of 3-spaces in a 3-spread of $PG(4e+3,q)$.

Suppose $\overline{L}$ is a line of $\mathcal{D}(S)$ of maximal length with lines $L_1, \ldots, L_{q^2+1}$ of $S$ as points of $\overline{L}$. A little thought shows that $L_1, \ldots, L_{q^2+1}$ generate a 3-space $U$ of $PG(4e+3,q)$ and form a 1-spread of $U$. Conversely, if there is a 3-space $U$ on which $S$ induces a 1-spread, then the $q^2+1$ lines of $S$ in $U$ form a line of maximal length of $\mathcal{D}(S)$. It follows that a 1-spread of $\mathcal{D}(S)$ corresponds to a 3-spread of $PG(4e+3,q)$ upon each of the components of which $S$ induces a 1-spread, and vice versa. However, $PG(4e+3,q)$ possesses 3-spreads and each component 3-space of a 3-spread must contain 1-spreads. So we can easily construct many 1-spreads $S$ of $PG(4e+3,q)$ which induce a 1-spread on each of the components of a 3-spread. We, thus, infer the existence of $(q+1)$-quasimultiples of a $(\sum_{i=0}^{2e+1} q^{2i}, \sum_{i=0}^{2e} q^{2i} \sum_{i=0}^{2e-1} q^{2i})$-design which possess a 1-spread. This is not particularly interesting when such a BIBD is a $(q+1)$-multiple. So we next proceed to show that 1-spreads $S$ exist in $PG(4e+3,q)$, $e \geq 1$, which are such that

(i) $\mathcal{D}(S)$ possesses a 1-spread, and
(ii) $\mathcal{D}(S)$ is not a $(q+1)$-multiple.

Let $S' = \{U_i\colon i = 1, \ldots, \sigma = \sum_{i=0}^{e} q^{4i}\}$ be a 3-spread of $PG(4e+3,q)$. Choose 1-spreads $S_i$ in each of $U_i$ ensuring that $S_1$ contains a 1-regulus ([2, pp. 220-1]) $\mathcal{R}$ of $U_1$. Let $\mathcal{R}'$ be the opposite regulus of $\mathcal{R}$ and $\overline{S}_1$ be the 1-spread of $U_1$ obtained from $S_1$ by replacing $\mathcal{R}$ in $S_1$ by $\mathcal{R}'$. Also, let $S$ and $\overline{S}$ be the 1-spreads $\cup_{i=1}^{\sigma} S_i$ and $\overline{S}_1 \cup (\cup_{i=2}^{\sigma} S_i)$ of $PG(4e+3,q)$. Further, let $L \in \mathcal{R}$, $L' \in \mathcal{R}'$, $V$ be the 2-space generated by $L$ and $L'$ and $H$ be a hyperplane of $PG(4e+3,q)$ containing $V$ but not $U_1$. If the lines of $S$ in $H$ generate $H$, then $\mathcal{D}(S)$ is not a $(q+1)$-multiple. On the other hand, if the lines of $S$ in $H$ generate a hyperplane $H^-$ of $H$, then the lines of $\overline{S}$ in $H$ are those of $S$ in $H$ apart from a replacement of $L$ by $L'$. The lines of $\overline{S}$ in $H$ excluding $L'$ still generate $H^-$. But $L'$ is not in $H^-$ and so the lines of $\overline{S}$ in $H$ generate $H$. Thus, $\mathcal{D}(\overline{S})$ is not a $(q+1)$-multiple.

## 4. Regular group divisible designs.

A $(v, b, r, k)$-configuration $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is said to be a *group divisible design* (GDD) if there is a partition of $\mathcal{P}$ into "groups" $\mathcal{P}_1, \ldots, \mathcal{P}_{\overline{m}}$ where $\overline{m} \geq 2$, such that there are integers $\overline{n} \geq 2$ and $\overline{\lambda}_1$ and $\overline{\lambda}_2$ such that

(a) $|\mathcal{P}_i| = \overline{n}$ for all $i = 1, \ldots, \overline{m}$,
(b) any two points common to a group are on $\overline{\lambda}_1$ blocks of $\mathcal{B}$,
(c) any two points in different groups are on $\overline{\lambda}_2$ blocks of $\mathcal{B}$, and
(d) $\overline{\lambda}_1 \neq \overline{\lambda}_2$.

$\mathcal{P}_1, \ldots, \mathcal{P}_{\overline{m}}$ is called a *group division* of $\mathcal{G}$.

The parameters of a GDD satisfy $\overline{v}\,\overline{r} = \overline{b}\,\overline{k}$, $\overline{v} = \overline{m}\,\overline{n}$ and $(\overline{n}-1)\overline{\lambda}_1 + \overline{n}\,(\overline{m}-1)\overline{\lambda}_2 = \overline{r}(\overline{k} - 1)$.

Let $A$ be an incidence matrix of a GDD with parameters $\overline{v}, \overline{b}, \overline{r}, \overline{k}, \overline{n}, \overline{m}, \overline{\lambda}_1$ and $\overline{\lambda}_2$. (We adopt the convention that points correpond to rows of $A$.) The eigenvalues of $AA^t$ are $\overline{r}\,\overline{k}$, $\overline{r} - \overline{\lambda}_1$ and $\overline{r}\,\overline{k} - \overline{v}\,\overline{\lambda}_2$. It is well-known that group divisions can be exhaustively classified into the following mutually exclusive types:

(1) Singular for which $\overline{r} = \overline{\lambda}_1$.
(2) Semiregular for which $\overline{r} > \overline{\lambda}_1$ and $\overline{r}\,\overline{k} = \overline{v}\,\overline{\lambda}_2$.
(3) Regular for which $\overline{r} > \overline{\lambda}_1$ and $\overline{r}\,\overline{k} > \overline{v}\,\overline{\lambda}_2$.

Since a GDD has a unique group division we can apply the terms "singular", "semiregular" and "regular" to GDDs as well as to group divisions.

Clearly, for a semiregular GDD we must have

$$\overline{r} - \overline{\lambda}_1 = \overline{n}(\overline{\lambda}_2 - \overline{\lambda}_1). \qquad (2)$$

Also, a regular GDD is of rank $\overline{v}$ ( [5, p. 4]) and so we must have

$$\overline{b} \geq \overline{v} \qquad (3)$$

for regular GDDs.

A GDD $\mathcal{G}$ is said to be *self-dual* if $\mathcal{G}^d$ (the dual of $\mathcal{G}$) is a GDD with the same parameters as $\mathcal{G}$.

Suppose $\mathcal{S}$ is a 1-spread of a $(v, b, r, k, \lambda)$-design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ with parameters (1). Define an incidence structure $\mathcal{G}(\mathcal{S}) = (\mathcal{P}, \mathcal{B}, \mathcal{I}\backslash I')$, where $(P, B) \in \mathcal{I}'$ if and only if $L_p \subseteq (B)$. (Here $L_p$ denotes the line of $\mathcal{S}$ on $P$.) Then $\mathcal{G}(\mathcal{S})$ is a group divisible design with parameters

$$\begin{aligned}
\overline{v} &= (\alpha + 1)(\Delta \alpha^2 + 1) \\
\overline{b} &= \rho(\alpha + 1)(\Delta \alpha^2 + 1) \\
\overline{r} &= \rho\alpha^2 \left(\Delta - \frac{\delta}{\rho}\right) \\
\overline{k} &= \alpha^2 \left(\Delta - \frac{\delta}{\rho}\right) \\
\overline{n} &= \alpha + 1 \\
\overline{m} &= \Delta \alpha^2 + 1 \\
\overline{\lambda}_1 &= 0 \\
\overline{\lambda}_2 &= \rho(\alpha - 1) \left(\Delta - \frac{\delta}{\rho}\right)
\end{aligned} \tag{4}$$

where (as earlier) $\Delta - \frac{\delta}{\rho} \alpha^2 + 1$. The groups of $\mathcal{G}(\mathcal{S})$ are the lines of $\mathcal{S}$.

To show this is quite straightforward, let $k^*$ be the blocksize of $\mathcal{D}(\mathcal{S})$. Each block $\mathcal{G}(\mathcal{S})$ has $k - k^*(\alpha + 1) = \left(\Delta - \frac{\delta}{\rho}\right) \alpha^2$ points of $\mathcal{G}(\mathcal{S})$ on it. Each point of $\mathcal{G}(\mathcal{S})$ is on $r - \lambda = \rho \left(\Delta - \frac{\delta}{\rho}\right) \alpha^2$ blocks of $\mathcal{G}(\mathcal{S})$. Clearly, each pair of points of $\mathcal{G}(\mathcal{S})$ in the same group are on no blocks of $\mathcal{G}(\mathcal{S})$. Consider two points $P$ and $Q$ on different lines $L_P$ and $L_Q$ of $\mathcal{S}$. The number of blocks of $\mathcal{D}$ containing both $L_P$ and $L_Q$ is the index of $\mathcal{D}(\mathcal{S})$ which is $\delta(\alpha + 1)$. The number of blocks of $\mathcal{D}$ containing $L_P$ and $Q$ is $\frac{\lambda k - r}{v - 1}$ (see [10]) and similarly for the number of blocks of $\mathcal{D}$ containing $L_Q$ and $P$. But $\frac{\lambda k - r}{v - 1} = \rho \left(\Delta + \frac{\delta}{\rho}\alpha\right)$. So the number of blocks of $\mathcal{G}(\mathcal{S})$ on $P$ and $Q$ is $\rho\Delta(\alpha + 1) - 2\rho \left(\Delta + \frac{\delta}{\rho}\alpha\right) + \delta(\alpha + 1) = \rho(\alpha - 1) \left(\Delta - \frac{\delta}{\rho}\right)$.

The GDDs $\mathcal{G}(\mathcal{S})$ are regular. First, $\overline{r} > 0 = \overline{\lambda}_1$ and so $\mathcal{G}(\mathcal{S})$ is not singular. To eliminate $\mathcal{G}(\mathcal{S})$ being semiregular it is sufficient, from (2), to show that $\overline{r} - \overline{\lambda}_1 > \overline{n}(\overline{\lambda}_2 - \overline{\lambda}_1)$, that is, to show that $\overline{r} > \overline{n}\overline{\lambda}_2$. But this can be easily verified.

From Section 3 we have that, for all prime powers $q$,

(i) there exist (symmetric) GDDs with parameters (4) with $\rho = 1$, $\alpha = q$ and $\delta = 0$ or $\sum_{i=0}^{d-2} q^{2i}$, $d \geq 2$, and

(ii) there exist (non-symmetric) GDDs with parameters (4) with $\rho = q + 1$, $\alpha = q^2$ and $\delta = 0$ or $\sum_{i=0}^{e-2} q^{4i}$, $e \geq 2$, which are not multiples of a symmetric GDD.

Suppose $\mathcal{D}$ is a Hadamard design with parameters (1). It is easily shown that we must have $\rho = 1$ and $\alpha = 2$. $\mathcal{D}$ is then a $(48\delta + 15, 24\delta + 7, 12\delta + 3)$-design. The following result (as well as its converse) was obtained in [10]. Here we give another proof of it.

**Theorem 1.** *If a* $(48\delta + 15, 24\delta + 7, 12\delta + 3)$*-design* $\mathcal{D}$ *has a* 1*-spread* $S$*, then there is an affine BIBD with four blocks in each affine resolution class and with each pair of non-disjoint blocks meeting in* $3\delta + 1$ *points.*

Proof: $\mathcal{G}(S)$ is a regular GDD with parameters $\overline{v} = \overline{b} = 3(16\delta + 5)$, $\overline{r} = \overline{k} = 4(3\delta + 1)$, $\overline{n} = 3$, $\overline{m} = 16\delta + 5$, $\overline{\lambda}_1 = 0$ and $\overline{\lambda}_2 = 3\delta + 1$. Clearly, the parameters of $\mathcal{G}(S)$ satisfy $\overline{k} = (\overline{n} + 1)\overline{\lambda}_2$. The result follows upon applying a result of Jungnickel and Vedder (see [7, p. 277]). ∎

Remarks: One way to obtain a GDD with the parameters of $\mathcal{G}(S)$ in the proof of Theorem 1 would be to sign ([3, p. 124]) a $(16\delta + 5, 12\delta + 4, 9\delta + 3)$-design over the cyclic group $Z_3$ of order three. A GDD $\mathcal{G}$ so obtained would have an automorphism of order three acting regularly on each of its groups ([6]). The affine BIBD $\mathcal{A}$ corresponding to $\mathcal{G}$ would possess an automorphism of order three fixing a point and each affine resolution class of $\mathcal{A}$. The smallest value of which yields an affine BIBD with "non-classical" parameters is $\delta = 2$. The problem as to whether any of the four $(37, 28, 21)$-designs can be signed over $Z_3$ is open ([3, p. 26]).

## 5. Dual properties.

Consider a $(v, b, r, k, \lambda)$-design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ whose parameters are given by (1) with $\rho = 1$ (that is, a $[\delta, 1, \alpha]$-design). Let $S$ be a 1-spread of $\mathcal{D}$.

**Lemma 2.**

(a) *If* $B$ *is a block of multiplicity* $\eta \geq 2$ *of* $\mathcal{D}(S)$*, then* $B$ *and the* $\eta - 1$ *other blocks of* $\mathcal{D}$ *containing the same lines of* $S$ *as* $B$ *form a line* ${}^{\star}_{B}$ *of* $\mathcal{D}^d$ *(the dual of* $\mathcal{D}$*).*

(b) *The multiplicity of a block of* $\mathcal{D}(S)$ *is less than or equal to* $\alpha + 1$.

(c) *If* $B$ *is a block of multiplicity* $\alpha + 1$ *of* $\mathcal{D}(S)$*, then* $L^{\star}_{B}$ *is a line of maximal length of* $\mathcal{D}^d$.

Proof:

(a) For any block $X$ of $\mathcal{D}$ denote the set of lines contained in $X$ by $S_X$. Let $A$ and $B$ be blocks of $\mathcal{D}$ such that $S_A = S_B$ and $A \neq B$. Now $\mathcal{D}^d$ is a BIBD with index $(\delta\alpha^2 + 1)(\alpha + 1)$. So the $(\delta\alpha^2 + 1)(\alpha + 1)$ points of $\mathcal{D}$ on the lines in $S_B = S_A$ are all of the blocks of $\mathcal{D}^d$ incident with $B$ and $A$. The line $L^{\star}_{B}$ which is the intersection of all the blocks of $\mathcal{D}^d$ on $A$ and $B$ contains precisely the blocks $E$ of $\mathcal{D}$ such that $S_E = S_B$.

(b) Suppose $B$ is a block of $\mathcal{D}$ of multiplicity $\eta$. If $\eta = 1$, then $\alpha + 1 > \eta$ since $\alpha \geq 1$. If $\eta \geq 2$, then $B$ is contained in a line $L^{\star}_{B}$ of $\mathcal{D}^d$ with $|L^{\star}_{B}| = \eta$. But $|L^{\star}_{B}| \leq \frac{b-\lambda}{r-\lambda} = \alpha + 1$.

(c) Immediate.

∎

Next we establish the following proposition.

**Proposition 1.** *Suppose $\mathcal{D}(S)$ is an $(\alpha+1)$-multiple and $S^* = \{L_B^*: B \in \mathcal{B}\}$.*

(a) *$S^*$ is a 1-spread of $\mathcal{D}^d$ and $\mathcal{D}^d(S^*)$ is an $(\alpha+1)$-multiple. In consequence, the lines $L_P^{**}$ of maximal length of $\mathcal{D}$ defined by points $P$ of $\mathcal{D}$ (as in Lemma 2(a) form a 1-spread $S^{**}$ of $\mathcal{D}$. Furthermore, $L_P^{**} = L_P$ for all $P \in \mathcal{P}$ and, in consequence, $S^{**} = S$*

(b) *$L_P \subseteq (B)$ if and only if $L_B^* \subseteq (P)$.*

Proof: (a) That $S^*$ is a 1-spread of $\mathcal{D}^d$ is immediate using Lemma 2(c).

Consider a line $L = \{P_1, \ldots, P_{\alpha+1}\}$ of $S$. The set of $(\delta\alpha^2+1)(\alpha+1)$ blocks of $\mathcal{D}$ containing $L$ is partitioned by $\delta\alpha^2+1$ sets of blocks of $\mathcal{D}$, each of which is a line of $S^*$. So $P_1, \ldots, P_{\alpha+1}$ as blocks of $\mathcal{D}^d$, contain $\delta\alpha^2+1$ common lines of $S^*$. So each of $P_1, \ldots, P_{\alpha+1}$ is a block of $\mathcal{D}^d(S^*)$ of multiplicity $\alpha+1$. But the lines of $S$ partition $\mathcal{P}$.

Consider a point $P \in \mathcal{P}$ and a block $B \in \mathcal{B}$ such that $L_P \subseteq (B)$. Now $P$ is in $L_P$ and $L_P$ is contained in all of the blocks of $\mathcal{D}$ which constitute $L_B^*$. So $P$, as a block of $\mathcal{D}^d$, contains $L_B^*$. Now $B$ is in $L_B^*$ and $\mathcal{D}^d(S^*)$ is an $(\alpha+1)$-multiple. So, by the same reasoning we have that $B$ contains $L_P^{**}$. Thus, all blocks of $\mathcal{D}$ containing $L_P$ contain $L_P^{**}$. But $L_P$ is the intersection of all the blocks of $\mathcal{D}$ containing $L_P$. We conclude that $L_P^{**} = L_P$ and then that $S^{**} = S$.

" (b)" That $L_P \subseteq (B) \Rightarrow L_B^* \subseteq (P)$ was established in Part(a). We then have

$$\begin{aligned} L_B^* \subseteq (P) &\Rightarrow L_P^{**} \subseteq (B) \\ &\Rightarrow L_P \subseteq (B) \quad (\text{since } L_P = L_P^{**}). \end{aligned}$$

∎

We are now in a position to establish the following theorem.

**Theorem 2.** *Suppose $\mathcal{D}$ is a $[\delta, \rho, \alpha]$-design and $S$ is a 1-spread of $\mathcal{D}$. $\mathcal{G}(S)^d$ is a GDD if and only if $\rho = 1$ and $\mathcal{D}(S)$ is an $(\alpha+1)$-multiple.*

Proof: Suppose $\rho = 1$ and $\mathcal{D}(S)$ is an $(\alpha+1)$-multiple. Let $\mathcal{D}^d = (\mathcal{B}, \mathcal{P}, \mathcal{J})$. Analogously to $\mathcal{G}(S)$ we define $\mathcal{G}(S^*) = (\mathcal{B}, \mathcal{P}, \mathcal{J} \setminus \mathcal{J}')$, where $(B, P) \in \mathcal{J}'$ if and only if $L_B^* \subseteq (P)$. Now

$$\begin{aligned} (P, B) \in I' &\Leftrightarrow L_P \subseteq (B) \\ &\Leftrightarrow L_B^* \subseteq (P) \qquad\qquad (\text{Proposition 1(b)}) \\ &\Leftrightarrow (B, P) \in J'. \end{aligned}$$

It follows that $\mathcal{G}(S^*) = \mathcal{G}(S)^d$. But $\mathcal{D}^d$ is a BIBD and $S^*$ is a 1-spread of $\mathcal{D}^d$. So $\mathcal{G}(S^*)$ is a GDD.

Conversely, suppose $\mathcal{G}(S^d)$ is a GDD. Let $A$ be an incidence matrix of $\mathcal{G}(S)$. Now $AA^t$ has three different non-zero eigenvalues and so $A^tA$ has three such eigenvalues. It follows that $\mathcal{G}(S)^d$ is regular. Then, using (3), we have that $\mathcal{G}(S)$ is symmetric and so $\rho = 1$. By Mitchell [8], $\mathcal{G}(S)$ is self-dual and the groups of $\mathcal{G}(S)$ and $\mathcal{G}(S)^d$ form a tactical decomposition ([2, p. 7]) of $\mathcal{G}(S)$. So, if a block $B$ of $\mathcal{G}(S)$ and a group $L$ of $\mathcal{G}(S)$ ($\doteq$ a line of $S$) are disjoint, then $L$ and each of the other blocks in the group of $\mathcal{G}(S)^d$ containing $B$ are also disjoint. But this means that these $\alpha + 1$ blocks, as blocks of $\mathcal{D}$, contain precisely the same lines of $S$. So the multiplicity of a block of $\mathcal{D}(S)$ is at least $\alpha + 1$. But the multiplicity of a block of $\mathcal{D}(S)$ is at most $\alpha + 1$ (Lemma 2(b)). We, thus, see that $\mathcal{D}(S)$ is an $(\alpha + 1)$-multiple. ∎

Let $d$ and $t$ be integers such that $d \geq 3$ and $1 \leq< d$. A $t$-spread $S$ of $PG(d, q)$ is said to be geometric if each component of $S$ is contained in or disjoint from each subspace of $PG(d, q)$ generated by two of the components of $S$ (or, equivalently, if $S$ induces a $t$-spread on each of the $(2t+1)$-dimensional subspaces of $PG(d, q)$ generated by a pair of components $S$). It is known (Segre [12]) that $PG(d, q)$ contains a geometric $t$-spread whenever $t + 1 \mid d + 1$. Also, the $t$-spread induced on the $(2t + 1)$-dimensional subspace generated by a pair of components of a geometric $t$-spread is a regular ([2, p. 221]) $t$-spread; see [12].

The following lemma will be of use.

**Lemma 3.** *If $S$ is a geometric $t$-spread of $PG(d,q)$ and $H$ is a hyperplane of $PG(d,q)$, then the components of $S$ contained in $H$ generate a $(d - t - 1)$-dimensional subspace of $H$. Furthermore, a $t$-spread $S$ of $PG(3t + 2, q)$ is geometric if and only if the components of $S$ contained in any hyperplane of $PG(3t + 2, q)$ generate a $(2t + 1)$-dimensional subspace.*

Proof: For a proof of the first statement see [11], Result 2. One half of the second statement follows from the first.

Suppose $S$ is a $t$-spread of $PG(3t + 2, q)$ such that the components of $S$ contained in any hyperplane of $PG(3t + 2, q)$ generate a $(2t + 1)$-dimensional subspace. Consider components $X$ and $Y$ of $S$ such that $X \neq Y$. Let $T$ be the subspace generated by $X$ and $Y$ and let $H$ be a hyperplane of $PG(3t + 2, q)$ containing $T$. There are $q^{t+1} + 1$ components of $S$ in $H$. These components of $S$ generate $T$ and form a 1-spread of $T$. So $S$ is geometric. ∎

Let $S$ be a geometric 1-spread of $PG(2d + 1, q)$ and $\mathcal{D} = PG_{2k}(2d + 1, q)$. From Lemma 3, the lines of $S$ in a hyperplane $H$ of $PG(2d + 1, q)$ generate a hyperplane of $H$. So $\mathcal{D}$ is a $(q + 1)$-multiple. Using Theorem 2, we infer the existence of self-dual regular GDDs which have parameters (4) with $\rho = 1, \alpha = q$, $\delta = 0$, or $\sum_{i=0}^{d-2} q^{2i}, d \geq 2$.

Next, let $S$ be a geometric 1-spread in $PG(5, q)$, $U$ be a 3-space generated by a pair of components of $S$ and $\mathcal{R}$ be a 1-regulus in the 1-spread $S_u$ induced on $U$ by $S$. Replacing $\mathcal{R}$ by its opposite 1-regulus yields a 1-spread $S'$ of $PG(5, q)$ which induces a 1-spread $S'_u$ on $U$. Provided $q > 2$, $S'_u$ is not regular (see Remark(a) below) and so $S'$ is not geometric. Using Lemma 3 we have that there is a hyperplane $H$ of $PG(5,q)$ such that the components of $S'$ in $H$ generate $H$. So $\mathcal{D}(S')$ is not a $(q+1)$-multiple. Using Theorem 2 we infer the existence of a symmetric regular GDD with parameters (4) with $\rho = 1 = \delta$ and $\alpha = q$ $(> 2)$ whose dual is not a GDD.

Remarks:

(a) The translation plane of order $q^2$ corresponding to the 1-spread $S'_u$ of $U$ is a Hall plane. For $q > 2$ a Hall plane is non-desarguesian and so $S'_u$ is not regular (see [2, p. 221]). For some details on the connection between 1-spreads in $PG(3,q)$ and translation planes see [2, Chapter 3, and 5].

(b) The only other infinite class of symmetric regular GDDs whose duals are not GDDs known to the author appears in [7].

(c) $\mathcal{D}(S')$ is a generalized symmetric design since a hyperplane of $PG(5,q)$ containing $U$ is a block of multiplicity $q+1$ of $\mathcal{D}(S')$.

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