Vertex Clique Covering Numbers Of r-Regular $(r-2)$-Edge-Connected Graphs

L. Caccetta and Purwanto

School of Mathematics and Statistics
Curtin University of Technology
GPO Box U 1987, Perth, WA 6001
AUSTRALIA

Abstract. Let G be a finite simple graph. The **vertex clique covering number** $vcc(G)$ of G is the smallest number of cliques (complete subgraphs) needed to cover the vertex set of G. In this paper we study the function $vcc(G)$ for the case when G is r-regular and $(r-2)$-edge-connected. A sharp upper bound for $vcc(G)$ is determined. Further, the set of possible values of $vcc(G)$ when G is a 4-regular connected graph is determined.

1. Introduction

All graphs considered in this paper are finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus G is a graph with vertex set $V(G)$, edge set $E(G)$, $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$, maximum degree $\Delta(G)$ and edge-connectivity $\kappa'(G)$. However, $\overline{G}$ denotes the complement of G.

A **clique** of G is a complete subgraph of G. A **vertex clique covering** C of G is a set of cliques such that every vertex of G belongs to at least one of the cliques in C. A **minimum vertex clique covering** of G is one having the fewest elements. The **vertex clique covering number** $vcc(G)$ of G is the cardinality of a minimum vertex clique covering of G. Thus the complete graph K_n on n vertices has $vcc(K_n) = 1$, the complete bipartite graph $K_{m,n}$ with bipartitioning sets of order m and n has $vcc(K_{m,n}) = \max\{m, n\}$.

We have studied the function $vcc(G)$ for: the case when G is a tree with each vertex having degree 1 or k in [2]; and for the case when G is a cubic graph in[4]. In this paper we consider the case when G is r-regular and $(r-2)$-edge-connected. For such a graph we prove, in Section 3, that

$$vcc(G) \leq \begin{cases} \frac{1}{2}(\nu(G)+1), & \text{for odd } \nu(G) < 3(r^2+r-1) \\ \frac{(r^2+r)\nu(G)}{2(r^2+r-1)}, & \text{otherwise.} \end{cases}$$

Further, the above bound is sharp for even $\nu(G) \geq 2r$ and for odd $\nu(G) \geq \frac{5}{2}r$.

An interesting problem that arises is that of determining the set of possible values of $vcc(G)$. We have resolved this problem for the case of connected cubic graphs [4]. In Section 4 we present the solution for the case of 4-regular connected graphs.

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2. Preliminaries

A matching M in G is a subset of $E(G)$ in which no two edges have a vertex in common. M is a **maximum matching** if $|M| \geq |M'|$ for any other matching M' of G. A vertex v is **saturated** by M if some edge of M is incident with v; otherwise v is said to be **unsaturated.** A matching M is **perfect** if it saturates every vertex of the graph. The **deficiency** $def(G)$ of G is the number of vertices unsaturated by a maximum matching M of G. Observe that $def(G) = \nu(G) - -2|M|$. Consequently, $def(G)$ has the same parity as $\nu(G)$, and $def(G) = 0$ if and only if G has a perfect matching.

The functions $vcc(G)$ and $def(G)$ are related as follows. When M is a maximum matching in G, it is clear that $vcc(G) \leq \nu(G) - |M|$ and hence

$$vcc(G) \leq \frac{1}{2}(\nu(G) + def(G)). \tag{2.1}$$

Observe that when G is triangle free $vcc(G) = \nu(G) - |M|$ and hence equality holds in (2.1).

Let $\mathcal{G}(n, r, k)$ denote the class of r-regular, k-edge-connected graphs on n vertices. Throughout this paper $k \geq 1$. The following result was proved in [3].

Theorem 2.1. *Let $G \in \mathcal{G}(n, r, k)$, with $1 \leq k \leq r-2$. Then*

(a) $def(G) \leq 2\lfloor \frac{r-k'}{2r} \lfloor \frac{rn}{rr^*+k'} \rfloor \rfloor$, *if n is even;*

(b) $def(G) = 1$, *if n is odd and* $n < \frac{r^2+r+k'}{r} \lceil \frac{3r}{r-k'} \rceil$;

(c) $def(G) \leq 1 + 2\lfloor \frac{r-k'}{2r} \lfloor \frac{rn}{r^2+r+k'} \rfloor - \frac{1}{2} \rfloor$, *otherwise;*

where k' is the least integer not less than k which has the same parity as r and r^* is the least odd integer greater than r. ∎

The bounds given in (a) and (c) above are sharp provided $k' \geq 2$.

Let

$$\mathcal{G}'(n, r, k) = \{G\colon G \in \mathcal{G}(n, r, k) \text{ and } G \text{ has no triangles}\}. \tag{2.2}$$

In [5] we studied the subclass $\mathcal{G}'(n, r, k)$ of $\mathcal{G}(n, r, k)$. We obtained bounds on $def(G), G \in \mathcal{G}'(n, r, k)$, which are better than those given in Theorem 2.1. However, the bounds obtained are sharp for $k = r-2$ or $r-3$, but not always sharp for $k < r-3$. One result from [5] that we make use of is the following.

Theorem 2.2. *For an integer $r \geq 4$, let*

$$D'(n, r, r-2) = \{def(G)\colon G \in \mathcal{G}'(n, r, r-2)\}.$$

Then

(a) $D'(n, r, r-2) = \phi$, *if n and r are odd, or $n < 2r$ or $n < \frac{5r}{2}$ is odd;*

(b) $D'(n, r, r-2) = \{1\}$, *if n is odd and* $\frac{5r}{2} \leq n < 3(r^2 + r - 1)$;

(c) $D'(n, r, r-2) = \{d\colon 0 \leq d \leq 2\lfloor \frac{n}{2(r^2+r-1)} \rfloor$, *$d$ is even*$\}$, *if n is even and* $n \geq 2r$;

(d) $D'(n, r, r-2) = \{d\colon 1 \leq d \leq 1 + 2\lfloor \frac{n}{2(r^2+r-1} - \frac{1}{2} \rfloor$, *$d$ is odd*$\}$, *otherwise.*

Since for a graph $G \in \mathcal{G}'(n, r, r-2)$ we have equality in (2.1), we have the following corollary to Theorem 2.2.

Corollary. *Let* $V'(n,r,r-2) = \{vcc(G): G \in \mathcal{G}'(n,r,r-2)\}$. *Then for* $r \geq 4$

(a) $V'(n,r,r-2) = \phi$, *if n and r are odd, or $n < 2r$ or $n < \frac{5}{2}r$ is odd;*
(b) $V'(n,r,r-2) = \{\frac{1}{2}(n+1)\}$, *if n is odd and $\frac{5}{2}r \leq n < 3(r^2+r-1)$;*
(C) $V'(n,r,r-2) = \{c \in \mathcal{N}: \lceil \frac{1}{2}n \rceil \leq c \leq \lfloor \frac{(r^2+r)n}{2(r^2+r-1)} \rfloor\}$, *otherwise.*

∎

Inequality (2.1) together with Theorem 2.1 yield an upper bound on $vcc(G)$ for $G \in \mathcal{G}(n,r,k)$, $1 \leq k \leq r-2$. Unfortunately, the resulting bound is generally not sharp. In this paper we provide a sharp upper bound on $vcc(G)$ for the case when $k = r - 2 \geq 2$. The case $r = 3$ was resolved in [4].

For $S \subset V(G)$, $G - S$ denotes the graph formed from G by deleting all the vertices in S together with their incident edges. We make use of the following lemma proved in [3].

Lemma 2.1. *Let $G \in \mathcal{G}(n,r,k)$, $1 \leq k \leq r-2$, be a graph with $def(G) \neq 1$. Then there exists a non-empty set $S \subset V(G)$ such that $G - S$ has*

$$\ell \geq \frac{r}{r-k'} def(G)$$

odd components each of which is joined to S by at most $r-2$ edges, where k' is the least integer not less that k having the same parity as r. ∎

Our next result makes use of the concept of graph closure. The **closure** $c(G)$ of a graph G is the graph formed from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least $\nu(G)$ until no such pair remains. A well known result (see [1] p. 57) states that G is hamiltonian if and only if $c(G)$ is hamiltonian. We use this result in the following lemma.

Lemma 2.2. *Let G be an odd order graph with $\nu(G) \leq 2\Delta(G) - 1$ and $\varepsilon(G) \geq \frac{1}{2}\Delta(G)(\nu(G)-1) + 1$. Then*

$$vcc(G) \leq \frac{1}{2}(\nu(G) - 1).$$

Proof: We first show that G is hamiltonian. Suppose to the contrary that G is non-hamiltonian. Then the closure $c(G)$ of G is also non-hamiltonian. Let m be the order of the largest clique in $c(G)$. The restriction on $\varepsilon(G)$ implies that $\delta(G) \geq 2$ and thus $\delta(c(G)) \geq 2$. Consequently $m \leq \nu(G) - 2$.

Let G have p vertices of degree at least $\Delta - 1$ (for convenience we omit the letter G from $\Delta(G)$ and $\nu(G)$). Then

$$\begin{aligned} p\Delta + (\nu - p)(\Delta - 2) &\geq 2\varepsilon(G) \\ &\geq \Delta(\nu - 1) + 2, \end{aligned}$$

and so

$$p \geq \nu - \frac{1}{2}(\Delta - 2).$$

When $\nu \leq 2\Delta - 2$, the vertices of G that have degree at least $\Delta - 1$ are adjacent in $c(G)$ and thus

$$m \geq \nu - \frac{1}{2}(\Delta - 2).$$

When $\nu = 2\Delta - 1$, G has at least $\Delta + 1$ vertices of degree Δ and hence the vertices of G having degree at least $\Delta - 1$ form a clique in $c(G)$. Thus we always have

$$m \geq \nu - \frac{1}{2}(\Delta - 2). \tag{2.3}$$

Let A denote the vertices in the maximum clique of $c(G)$. In $c(G)$, and hence in G, every vertex not in A has degree at most $\nu - m$. Hence

$$\begin{aligned} \sum_{v \in V(G)} d_G(v) &\leq m\Delta + (\nu - m)^2 \\ &\leq m\Delta + \frac{1}{2}(\Delta - 2)(\nu - m) \quad (\text{using}(2.3)) \\ &= \Delta\nu + \frac{1}{2}(\Delta + 2)(m - \nu) \\ &\leq \Delta\nu + \frac{1}{2}(\Delta + 2)(-2) \\ &= \Delta\nu - \Delta - 2, \end{aligned}$$

a contradiction. Thus G is hamiltonian.

Let $H = v_1 v_2 \ldots v_\nu v_1$ be a Hamilton cycle in G and suppose without loss of generality that $d_G(v_1) = \Delta$. Since $\Delta \geq \frac{1}{2}(\nu + 1)$, v_1 must be adjacent to an adjacent pair, v_{2t} and v_{2t+1} say, of vertices of H. Now

$$C = \{v_1 v_{2t} v_{2t+1}, v_2 v_3, \ldots, v_{2t-2} v_{2t-1}, v_{2t+2} v_{2t+3}, \ldots, v_{\nu-1} v_\nu\}$$

is a vertex clique covering of G of order $\frac{1}{2}(\nu - 1)$. This completes the proof of the lemma. ∎

We conclude this section by observing that for $n > r+1$ a graph $G \in \mathcal{G}(n, r, 1)$ has no clique of order greater than r, and hence

$$vcc(G) \geq \lceil n/r \rceil. \tag{2.4}$$

3. Upper Bound

In this section we determine an upper bound for $vcc(G)$, $G \in \mathcal{G}(n, r, r-2)$. The bound is sharp for even $n \geq 2r$ and for odd $n \geq \frac{5}{2}r$. We begin with some constructions.

For an integer $r \geq 4$ define the graph $A(r)$ as follows. Take the complete bipartite graph $K_{r,r-1}$ and add two new vertices, say x and y. Join x and y and join each vertex in the larger bipartitioning set of $K_{r,r-1}$ to exactly one of x and y in such a way that the degree of x and y differ by at most one. Call the resulting graph $A(r)$. This graph will become our basic building block. We assume without loss of generality that $d(x) \leq d(y)$. Observe that every vertex of $A(r)$ other than x and y has degree r and $d(x) = \lfloor \frac{1}{2}r \rfloor + 1$ and $d(y) = \lceil \frac{1}{2}r \rceil + 1$. Further, $A(r)$ has $2r+1$ vertices, is triangle free, is $\frac{1}{2}r$-edge-connected and $def(A(r)) = 1$.

Define a graph $H(r)$ as follows. Take an empty graph $\overline{K}_{r-2}$ with vertices $u_1, u_2, \ldots, u_{r-2}$ and $r-1$ copies $A_1, A_2, \ldots, A_{r-1}$ of $A(r)$. We relabel the x and y vertices of the ith copy A_i as x_i and y_i, respectively. Join each x_i to u_j for $1 \leq j \leq \lceil \frac{1}{2}r \rceil - 1$, and join y_i to u_j for $\lceil \frac{1}{2}r \rceil \leq j \leq r-2$. Call the resulting graph $H(r)$. Observe that $H(r)$ has $2r^2 - 3$ vertices and is triangle free. Further, all vertices of $H(r)$ except $u_1, u_2, \ldots, u_{r-2}$ have degree r: the u_i's all have degree $r-1$. The important properties of $H(r)$ which are needed to establish our main result of this section are given in the following lemma.

Lemma 3.1. *For $r \geq 4$ the graph $H(r)$ has the following properties:*

(a) *For each $U \subseteq \{u_1, u_2, \ldots, u_{r-2}\}$, $vcc(H(r) - U) = r^2 - 1$.*
(b) $\kappa'(H(r)) = r - 2$.

Proof: Since $def(A_i) = 1$, $1 \leq i \leq r-1$, we have $vcc(A_i) = r+1$. Consider the subgraph H_{ij} of $H(r)$ induced by the vertices $V(A_i) \cup \{u_j\}$. It is easily seen that

$$vcc(H_{ij} = vcc(A_i) = r + 1$$

for all i, j, $1 \leq i \leq r-1$ and $1 \leq j \leq r-2$. Consequently for each $U \subseteq \{u_1, u_2, \ldots, u_{r-2}\}$ we have

$$\begin{aligned} vcc(H(r) - U) &= (r-1)(r+1) \\ &= r^2 - 1, \end{aligned}$$

proving (a).

Suppose that $\kappa'(H(r)) = t$. Since the vertex partition

$$X = V(A_1), \overline{X} = V(H(r)) - V(A_1)$$

gives rise to an edge-cut set $(X, \overline{X})$ containing $r-2$ edges, we have $t \le r-2$. We now prove that $t = r-2$. Suppose to the contrary that $t \le r-3$ and let $(Y, \overline{Y})$ be an edge-cut set of size t. It is clear that for each i the vertices of $A_i - x_i - y_i$ are all in Y or all in $\overline{Y}$. Let $U_i = Y \cap \{u_1, u_2, \ldots, u_{r-2}\}$ and $U_2 = \overline{Y} \cap \{u_1, u_2, \ldots, u_{r-2}\}$.

We first prove that $U_1 \neq \phi$ and $U_2 \neq \phi$. For suppose without loss of generality that $U_1 = \phi$. Then $V(A_i) \not\subseteq Y$ for any i and further $V(A_i) \cap Y \neq \phi$ for some i. Suppose $V(A_j) \cap Y \neq \phi$. Since $t < r-2$ and the vertices of $A_j - x_j - y_j$ are either all in Y or all in $\overline{Y}$. Y contains exactly one of x_j or y_j. But each of these possibilities results in $t \ge r-2$. Thus we have $U_1 \neq \phi$ and $U_2 \neq \phi$.

Let p_1 and p_2 denote the number of A_i's that have all their vertices in Y and $\overline{Y}$, respectively. Then, since each A_i is $\frac{1}{2}r$-edge-connected we have by simple counting

$$t \ge p_1|U_2| + p_2|U_1| + \frac{1}{2}r(r-1-p_1-p_2).$$

Since $t \le r-3$ we must have $r-1-p_1-p_2 \le 1$ and hence

$$\begin{aligned} t &\ge p_1 + p_2 \\ &\ge r-2. \end{aligned}$$

This contradiction proves (b). ∎

We now establish an upper bound on $vcc(G)$ for $G \in \mathcal{G}(n, r, r-2)$.

Theorem 3.1. *Let* $G \in \mathcal{G}(n, r, r-2)$, $r \ge 4$. *Then*

(a) $vcc(G) \le \frac{1}{2}(n+1)$, *if n is odd and* $n < 3(r^2+r-1)$,

(b) $vcc(G) \le \frac{(r^2+r)n}{2(r^2+r-1)}$, *otherwise.*

Moreover, this bound is sharp for even $n \ge 2r$ *and for odd* $n \ge \frac{5}{2}r$.

Proof: In view of the corollary to Theorem 2.2 it suffices to establish the bounds. Since $vcc(G) \le \frac{1}{2}(n + def(G))$, the result is true when $def(G) \le 1$. So suppose that $def(G) \ge 2$. By Lemma 2.1, there is a vertex set $S \subset V(G)$ such that G-S has $\ell \ge \frac{1}{2}r\, def(G) (\ge r)$ odd components each of which is joined to S by exactly $r-2$ edges. It is easily established that each of the odd components of G-S has at least $r+1$ vertices. Since G is r-regular, simple counting implies that $r|S| \ge \ell(r-2)$ and hence $|S| \ge r-2$. Consequently, $n \ge |S| + \ell(r+1) \ge r^2 + 2r - 2$.

We define a maximal sequence of vertex disjoint subgraphs $G_1, G_2, \ldots, G_p$ of non-decreasing order as follows:

(i) each G_i is odd and $\nu(G_i) \le 2r-1$;

(ii) each G_i has $r-2$ edges going to the vertices of $G - V(G_i)$.

Observe that

$$\nu(G_i) = \begin{cases} r + 2m, & \text{if } r \text{ is odd} \\ r + 2m - 1, & \text{if } r \text{ is even} \end{cases} \tag{3.1}$$

for some positive integer $m \leq \frac{1}{2}r$. Let p_m denote the number of G_i's having order $r + 2m$ when r is odd and order $r + 2m - 1$ when r is even.

Now we define the sequence of graphs $G^{(0)}, G^{(1)}, \ldots, G^{(p)}$ as follows. $G^{(0)} = G$. For $1 \leq i \leq p$ we form $G^{(i)}$ from $G^{(i-1)}$ as follows. Take $G^{(i-1)} - V(G_i)$ and a copy H_i of $H(r)$. Recall that H_i has $r - 2$ vertices, say $u_{i1}, u_{i2}, \ldots, u_{i(r-2)}$, having degree $r - 1$. Let $v_{i1}, v_{i2}, \ldots, v_{iq}$ denote the vertices of $G^{(i-1)} - V(G_i)$ that are adjacent to the vertices of G_i. Note that $q \leq r - 2$. Our graph $G^{(i)}$ is obtained by adding $r - 2$ edges between the vertices $u_{i1}, u_{i2}, \ldots, u_{i(r-2)}$ and $v_{i1}, v_{i2}, \ldots, v_{iq}$ such that each vertex has degree r. Observe that if $G^{(i-1)}$ is $(r-2)$-edge-connected, then $G^{(i)}$ is also $(r-2)$-edge-connected, since by Lemma 3.1, $\kappa'(H_i) = r - 2$. Consequently, since $G \in \mathcal{G}(n, r, r-2)$, each $G^{(i)}$ is $(r-2)$-edge-connected.

Let $G^* = G^{(p)}$. Since $\nu(H(r)) = 2r^2 - 3$, we have

$$n^* = \nu(G^*) = n + \sum_{m=1}^{\lfloor r/2 \rfloor} p_m(2r^2 - r - 2m - 3 + \lambda(r)) \tag{3.2}$$

where $\lambda(r)$ is 1 or 0 according as r is even or odd. Further, since by Lemma 3.1 (a) $vcc(H_i - U) = vcc(H_i) = r^2 - 1$ for any set $U \subseteq \{u_{i1}, u_{i2}, \ldots, u_{i(r-2)}\}$, we have

$$\begin{aligned} vcc(G^*) &= vcc(G^* - \bigcup_{i=1}^{p} V(H_i)) + \sum_{i=1}^{p} vcc(H_i) \\ &= vcc(G - \bigcup_{i=1}^{p} V(G_i) + p(r^2 - 1). \end{aligned}$$

Now for G we have

$$\begin{aligned} vcc(G) &\leq vcc(G - \bigcup_{i=1}^{p} V(G_i)) + \sum_{i=1}^{p} vcc(G_i) \\ &= vcc(G^*) + \sum_{i=1}^{p}(vcc(G_i) - r^2 + 1) \\ &\leq \frac{1}{2}(n^* + def(G^*)) + \sum_{i=1}^{p}(vcc(G_i) - r^2 + 1). \end{aligned} \tag{3.3}$$

Since G_i satisfies the hypothesis of Lemma 2.2 we have

$$vcc(G_i) \leq \frac{1}{2}(\nu(G_i) - 1).$$

This together with (3.1), (3.2) and (3.3) yields

$$vcc(G) \leq \frac{1}{2}(n + def(G^*)) - p. \tag{3.4}$$

Consequently the result is true for $def(G^*) \leq 1$. So suppose that $def(G^*) \geq 2$.

Since G^* is $(r-2)$-edge-connected, Lemma 2.1 implies that there is a non-empty vertex set $S^* \subset V(G^*)$ such that G^*-S^* has $\ell^* \geq \frac{1}{2} r\, def(G^*)$ odd components each of which is joined to S^* by exactly $r-2$ edges. By the construction of G^*, each odd component of G^*-S has at least $2r+1$ vertices. Now since G^* is r-regular, simple counting implies that

$$r|S^*| \geq (r-2)\ell^*.$$

Hence

$$\begin{aligned} n^* &\geq |S^*| + (2r+1)\ell^* \\ &\geq \frac{2(r^2+r-1)\ell^*}{r}. \end{aligned}$$

Thus

$$\ell^* \leq \frac{rn^*}{2(r^2+r-1)},$$

and

$$def(G^*) \leq \frac{2\ell^*}{r} \leq \frac{n^*}{r^2+r-1}. \tag{3.5}$$

This together with (3.2) and (3.4) yields

$$\begin{aligned} vcc(G) &\leq \frac{1}{2}(n + \frac{n}{r^2+r-1}) \\ &= \frac{(r^2+r)n}{2(r^2+r-1)} \end{aligned}$$

This completes the proof. ∎

4. The Class $\mathcal{G}(n,4,2)$

In this section we determine the set of possible value of $vcc(G)$ as G ranges over the class $\mathcal{G}(n,4,2)$. Throughout this section $G \in \mathcal{G}(n,4,2)$. If G is triangle free then equality holds in (2.1). Further, the following construction shows the existence of a graph $G_0 \in \mathcal{G}(n,4,2)$ with $vcc(G_0) = \lceil \frac{1}{2}n \rceil$ for every $n \geq 10$. Let $V(G_0) = \{0,1,\ldots,n-1\}$. Start with a Hamilton cycle $C = 0\ 1\ 2\ \ldots (n-1)\ 0$. Then for each i, join vertex i to vertex $i+3$, where addition is modulo n. Call the resulting graphs G_0. Clearly G_0 is triangle free and has deficiency 0 or 1 according as n is even or odd.

Let

$$V(n,4) = \{vcc(G) : G \in \mathcal{G}(n,4,2)\}.$$

Theorem 3.1 gives a sharp upper bound for $vcc(G)$ when $n \geq 10$. The following lemma establishes $V(n,4)$ for small values of n.

Lemma 4.1.

(a) $V(n,4) = \{n-4\}$, *for* $5 \leq n \leq 7$.
(b) $V(8,4) = \{2,3,4\}$.
(c) $V(9,4) = \{3,4\}$.

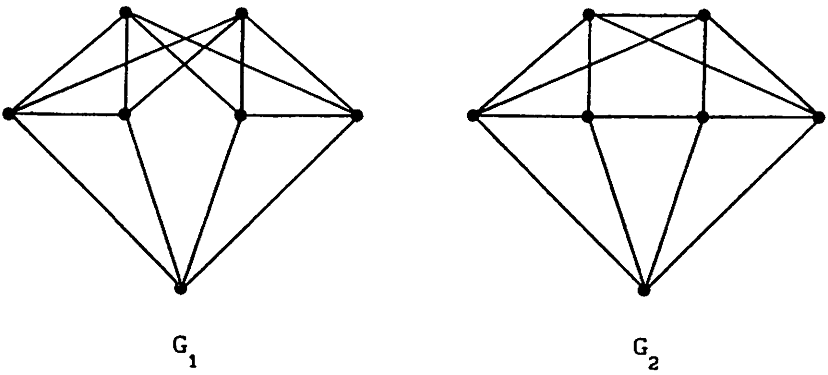
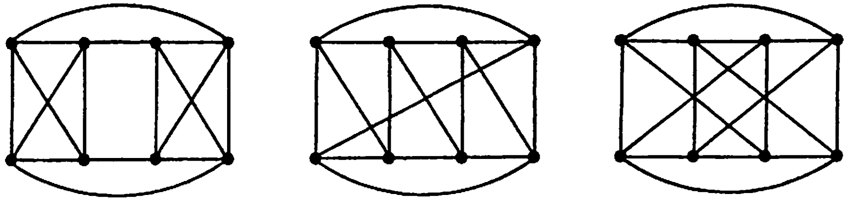


Figure 4.1

Proof: $\mathcal{G}(n,4,2)$ contains only one graph when $n = 5$ and $n = 6$, namely K_5 and the complement of a perfect matching of size 3, which has vertex clique covering number 2. The class $\mathcal{G}(7,4,2)$ consists of the two non-isomorphic graphs, see [6], pictured in Figure 4.1. Clearly $vcc(G_1) = vcc(G_2) = 3$. This proves (a). Part (b) follows from inequality (2.4), Theorem 3.1 and the graphs pictured in Figure 4.2.

Now consider a graph $G \in \mathcal{G}(9,4,2)$. Inequality (2.4) and Theorem 3.1 imply that $3 \leq vcc(G) \leq 5$. Since G cannot be bipartite it must have odd girth. It is

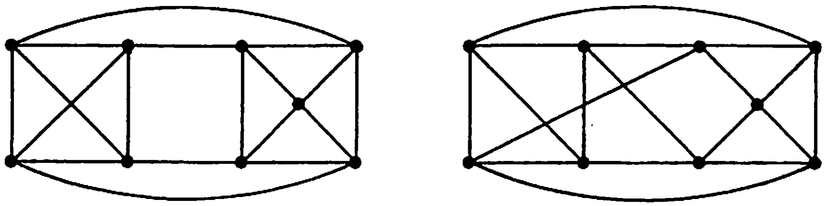


G_3 G_4 G_5

Figure 4.2

easy to observe that the girth of G must be 3. Let $u\ v\ w$ be a triangle in G and let $G' = G - \{u, v, w\}$. Then $\nu(G') = 6$, $\varepsilon(G') = 9$ and since G is 4-regular G' must be connected. Now if G' is bipartite, then $G' \cong K_{3,3}$ and hence $vcc(G) \leq 4$. If G' is not bipartite it contains a cycle of length 3 or 5. In either case it is easily seen that $vcc(G') \leq 4$. This proves that $5 \notin V(9,4)$.

The graphs given in Figure 4.3 establishes that 3 and 4 are in $V(9,4)$. This completes the proof. ∎



G_6 G_7

Figure 4.3

The corollary to Theorem 2.2 provides the members of $V(n,4,2)$ which are greater than or equal to $\lceil \frac{1}{2}n \rceil$ for each $n \geq 10$. Inequality (2.4) implies that $vcc(G) \geq \lceil \frac{1}{4}n \rceil$ for $n > 5$. We will establish that for each integer c, $\lceil \frac{1}{4}n \rceil \leq c \leq \lceil \frac{1}{2}n \rceil$, there exists a graph $G \in \mathcal{G}(n,4,2)$ with $vcc(G) = c$. The graphs displayed in Figure 4.4 will form the basic building blocks in our construction. Note that $G_c(n)$ denotes a member $\mathcal{G}(n,4,2)$ with vertex clique covering number c.

Let G be a graph with minimum vertex clique covering C. A pair e_1 and e_2 of edges of G is called **free** if e_1 and e_2 are independent and neither e_1 nor e_2 is contained in any member of C. It is easy to find a free pair in each graph of Figure

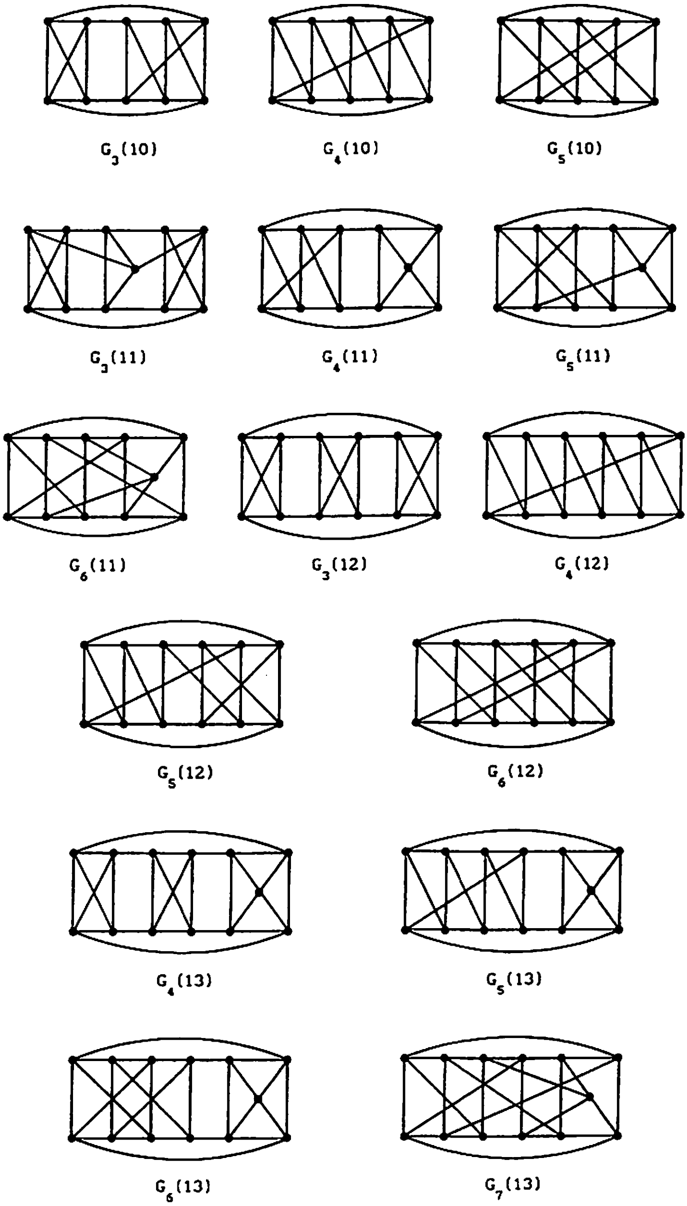

G3(10)
G4(10)
G5(10)
G3(11)
G4(11)
G5(11)
G6(11)
G3(12)
G4(12)
G5(12)
G6(12)
G4(13)
G5(13)
G6(13)
G7(13)

Figure 4.4

4.4. We define an operation α on a graph G with a free pair e_1, e_2 as follows. We replace e_1 and e_2 with a subgraph H as indicated in Figure 4.5. Call the resulting graph G'. Observe that $\nu(G') = \nu(G) + 4$, $vcc(G') = vcc(G) + 1$ and G' has a free pair.

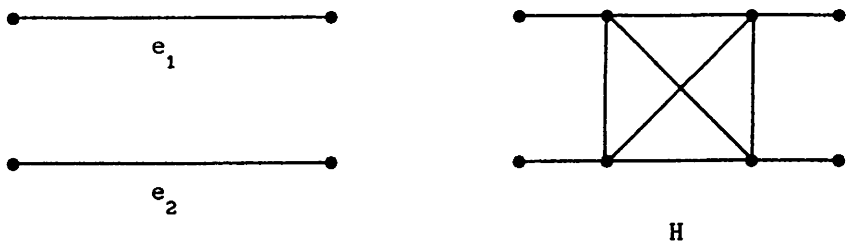


Figure 4.5

Lemma 4.2. *Let n and c be integers with $n \geq 10$ and*

$$\lceil \frac{1}{4} n \rceil \leq c \leq \lceil \frac{1}{2} n \rceil .$$

Then there exists a graph $G_c(n)$ having a free pair.

Proof: Let $n = 4p + q + 2$, where p and q are integers with $p \geq 2$ and $0 \leq q \leq 3$. We will establish the lemma using induction on p. The graphs given in Figure 4.4 establish the lemma for the cases $p = 2$. Suppose the lemma is true for $p \leq m$. Thus there exists a graph $G_c(4m + q + 2)$ having a free pair e_1, e_2 for each c

$$\lceil m + \frac{1}{4} q + \frac{1}{2} \rceil \leq c \leq \lceil 2m + \frac{1}{2} q + 1 \rceil .$$

Performing operation α on e_1 and e_2 yields a graph

$$G_{c+1}(4(m + 1) + q + 2)$$

with a free pair. Hence there exists a graph $G_c(4m + q + 6)$ with a free pair for each c

$$\lceil (m + 1) + \frac{1}{4} q + \frac{1}{2} \rceil \leq c \leq \lceil 2(m + 1) + \frac{1}{2} q \rceil .$$

Now the existence of a triangle free graph

$$G \in \mathcal{G}(4m + q + 6, 4, 2) \text{ with } vcc(G) = \lceil 2m + \frac{1}{2} q + 3 \rceil$$

was established at the beginning of this section. Thus there exists a graph $G_c(4m+q+6)$ for each c,

$$\lceil (m+1) + \frac{1}{4}q + \frac{1}{2} \rceil \leq c \leq \lceil 2(m+1) + \frac{1}{2}q + 1 \rceil,$$

completing the proof of the lemma. ∎

The Corollary to Theorem 2.2, Theorem 3.1 and lemmas 4.1 and 4.2 together gives

Theorem 4.1.

(a) $V(n,4) = \{n-4\}$, *for* $5 \leq n \leq 7$;

(b) $V(8,4) = \{2,3,4\}$;

(c) $V(9,4) = \{3,4\}$;

(d) $V(n,4) = \{c \in \mathcal{N} : \lceil \frac{1}{4}n \rceil \leq c \leq \frac{1}{2}(n+1)\}$, *if* n *is odd and* $11 \leq n \leq 55$;

(e) $V(n,4) = \{c \in \mathcal{N} : \lceil \frac{1}{4}n \rceil \leq c \leq \lceil \frac{10n}{19} \rceil\}$, *for even* $n \geq 10$ *or odd* $n \geq 57$.

∎

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