

Enumeration of All Simple $t - (t + 7, t + 1, 2)$ Designs

Stanislaw P. Radziszowski¹

Department of Computer Science
Rochester Institute of Technology
Rochester, New York 14623

Abstract. We enumerate by computer algorithms all simple $t - (t + 7, t + 1, 2)$ designs for $1 \leq t \leq 5$, i.e. for all possible t , and this enumeration is new for $t \geq 3$. The number of nonisomorphic designs is equal to 3, 13, 27, 1 and 1 for $t = 1, 2, 3, 4$ and 5, respectively. We also present some properties of these designs including orders of their full automorphism groups and resolvability.

1. Introduction

A $t - (v, k, \lambda)$ design $D = (X, B)$ is a family B of k -subsets, called *blocks*, of a v -set X of *points*, such that every t -subset of X is contained in exactly λ blocks of B . If B has no repeated blocks then the design D is called *simple*. The family of $t - (t + 7, t + 1, \lambda)$ designs is perhaps one of the most investigated parameter situations, especially for $\lambda = 1$, in which case they are called Witt designs; their in depth presentation together with a number of references can be found in [1]. Only two other values of λ are of interest, namely 2 and 3, since for $\lambda \geq 4$ any such design is the complement of one with $\lambda \leq 3$. A $t - (v, k, \lambda)$ design is called *resolvable* if a part of its blocks forms a $t - (v, k, \lambda_r)$ design for some $1 \leq \lambda_r < \lambda$. Let $N(\lambda; t, k, v)$ denote the number of nonisomorphic simple $t - (v, k, \lambda)$ designs.

λ	t				
	1	2	3	4	5
1	1	1	1	1	1
2	3	13	27	1	1
3	6	332	≥ 539	≥ 18	≥ 13

Table I. Values and bounds for $N(\lambda; t, t + 1, t + 7)$

The table I summarizes enumeration results of simple $t - (t + 7, t + 1, \lambda)$ designs by listing known values and bounds for $N(\lambda; t, t + 1, t + 7)$. The entries in column $t = 1$ count $1 - (8, 2, \lambda)$ designs, which are just regular graphs of degree λ on 8 points. They are usually not considered in the design theory, but are included here for completeness. They also form starting points for our extension algorithms. The uniqueness of the designs with $\lambda = 1$ is discussed in [1], Gibbons calculated $N(2; 2, 3, 9) = 13$ [3], and Harms, Colbourn and Ivanov obtained the

¹Supported in part by a grant from the NSF CCR-8920692, and by Komitet Badań Naukowych 3 0465 91 01, Poland.

value $N(3; 2, 3, 9) = 332$ in [4]. Both proofs in [3] and [4] relied on computer algorithms. The remaining entries of table I are obtained in this paper. For $\lambda = 2$ we were able to enumerate all of them, and for $\lambda = 3$ we have found some such designs including all for $t = 2$ (as in [4]), and all with an automorphism of order 3 not fixing any block, for all t . We postpone their description until all such designs are enumerated, i.e. when the case of $t - (t + 7, t + 1, \lambda)$ designs is closed. These special $t - (t + 7, t + 1, 3)$ designs were necessary in the analysis of possible $4 - (12, 6, 6)$ designs [6], whose existence is still in question. The existence of $t - (t + 7, t + 1, \lambda)$ designs for all parameter situations was known already in 1977 (see Brouwer [2]).

2. Results

A cycle of length 8, two squares, and a triangle and a pentagon are the only 2-regular graphs on eight points, which can be treated as $1 - (8, 2, 2)$ designs. Obviously the first two are resolvable into $1 - (8, 2, 1)$ designs. All three of them extend to a $2 - (9, 3, 2)$ design.

The 13 $2 - (9, 3, 2)$ designs found by Gibbons [3] have the full automorphism groups of orders 80, 18, $8(2)$, $6(3)$, $2(3)$ and $1(3)$, where the number in parenthesis shows the number of corresponding designs (if larger than 1). Exactly two of them, with groups of order 18 and 6, are resolvable into two $2 - (9, 3, 1)$ designs. 11 out of them, including both resolvable ones, extend to a $3 - (10, 4, 2)$ design. The two nonextendible $2 - (9, 3, 2)$ designs have group orders 6 and 2.

We have found that there are exactly 27 nonisomorphic $3 - (10, 4, 2)$ designs; each of them has 60 blocks and each point appears in 24 blocks. Their full automorphism groups have orders 400, 40, 20, 16, $8(5)$, $4(6)$, $2(11)$ and 1. The $3 - (10, 4, 2)$ design with group of order 20 is the only one resolvable into two $3 - (10, 4, 1)$ designs, and also it is the only one extendible to a $4 - (11, 5, 2)$ design. Furthermore, up to isomorphisms, it extends uniquely to the well known $4 - (11, 5, 2)$ design. Consequently, by Alltop's extension theorem, there exists a unique $5 - (12, 6, 2)$ design. We note that one can easily see that any $5 - (12, 6, \lambda)$ design must be closed under block complementation.

Theorem 1. *There exist unique, up to isomorphism, $4 - (11, 5, 2)$ and $5 - (12, 6, 2)$ designs, and both of them are resolvable.*

Kramer and Mesner [5] gave the precise analysis of mutually disjoint $S(t, t + 1, t + 7)$ Steiner systems, in particular in the case of resolvable $4 - (11, 5, 2)$ designs. Since the unique $4 - (11, 5, 2)$ and $5 - (12, 6, 2)$ designs (both resolvable) that occur are the ones studied in [5], we refer the reader to this paper for more information, as well as to [1]. We remark that their automorphism groups have orders 110 and 1320, respectively, and recall that the orders of groups for the Witt designs $t - (t + 7, t + 1, 1)$ are 384, 432, 1440, 7920 and 95040, for $t = 1, 2, 3, 4$ and 5, respectively. We also observe that our enumeration implies

the uniqueness of $4 - (11, 5, 5)$ and $5 - (12, 6, 5)$ designs (the complements of unique designs), which permits an answer to a question posed by Kramer and Mesner [5] formulated in the next theorem.

Theorem 2. *The $4 - (11, 5, \lambda)$ and $5 - (12, 6, \lambda)$ designs are not resolvable for $\lambda = 3$ and $\lambda = 5$.*

Proof: Assume that a $4 - (11, 5, 5)$ design D can be partitioned into a $4 - (11, 5, \lambda_1)$ design D_{λ_1} and a $4 - (11, 5, \lambda_2)$ design D_{λ_2} , for some $1 \leq \lambda_1 < \lambda_2$, $\lambda_1 + \lambda_2 = 5$. If $\lambda_1 = 1$ then resolved \overline{D} and D_{λ_1} form three mutually disjoint $S(4, 5, 11)$'s. If $\lambda_1 = 2$ then both \overline{D} and D_{λ_1} can be resolved to a total of four mutually disjoint $S(4, 5, 11)$'s. Similarly, the uniqueness and resolvability of the $4 - (11, 5, 2)$ design imply that any resolvable $4 - (11, 5, 3)$ design is formed by three mutually disjoint $S(4, 5, 11)$'s. This contradicts a theorem of Kramer and Mesner [5] stating that there can be at most two mutually disjoint Steiner systems $S(4, 5, 11)$. The same reasoning is valid for the cases of $5 - (12, 6, 5)$ and $5 - (12, 6, 3)$ designs. ■

After a moment of thought one can easily see that theorems 1 and 2 leave open only one nontrivial resolvability question concerning these designs. We formulate this question in three equivalent forms as follows.

- (1) Does there exist a resolvable $4 - (11, 5, 4)$ design?
- (2) Do there exist a Steiner system $S(4, 5, 11)$ and a $4 - (11, 5, 3)$ design which are disjoint?
- (3) Do there exist two disjoint $4 - (11, 5, 3)$ designs?

In order to obtain the above enumeration we used natural algorithms starting from graphs ($t = 1$) and then performing consecutive extensions to $(t + 1)$ -designs. Each specific design extension and resolvability step was completed by solving appropriate systems of 0-1 integer linear equations. For automorphism groups and design isomorphism we did the calculations twice: with the software developed by the author, and independently with the program *nauty* written by B.D. McKay [7]. The completion of enumeration of all $t - (t + 7, t + 1, 3)$ designs with our current approach would require an extreme man/machine effort, unless a more efficient method is devised or some strong properties of such designs are discovered.

References

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