Universal and Global Irredundancy in Graphs

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Abstract. A set S of vertices of a graph $G = (V, E)$ is a global dominating set if S dominates both G and its complement $\overline{G}$. The concept of global domination was first introduced by Sampathkumar. In this paper we extend this notion to irredundancy. A set S of vertices will be called universal irredundant if S is irredundant in both G and $\overline{G}$. A set S will be called global irredundant if for every x in S, x is an irredundant vertex in S either in G or in $\overline{G}$. We investigate the universal irredundance and global irredundance parameters of a graph. It is also shown that the determination of the upper universal irredundance number of graphs is NP-Complete.

1. Introduction

Let $G = (V, E)$ denote a simple graph. For any vertex v, the open neighborhood of v in G is the set $N(v) = \{x \mid xv \in E\}$ and the closed neighborhood of v in G is the set $N[v] = N(v) \cup \{v\}$. For a subset S of V, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The complement of G, $\overline{G}$ is the graph with vertex set V and edge set $\overline{E} = E(\overline{G}) = \{xy \mid x, y \in V$ and xy is not in $E(G)\}$. For any $x \in V(G)$, if $y \in N(x)$ in G, then we will say y is a G-neighbor of x and write $y \in N_G(x)$. If $y \in N(x)$ in $\overline{G}$, we will say y is a $\overline{G}$-neighbor of x and write $y \in N_{\overline{G}}(x)$. Similar notations for $N_G(S)$ and $N_{\overline{G}}(S)$ will be used. A subset S of V is a *dominating* set of G if $N[S] = V$, and S is a minimal dominating set of G if no proper subset of S dominates G. A vertex $x \in S$ is *irredundant* in S if $N[x] - N[S - x] \neq \phi$. A subset S of V is an *irrredundant set* if for all x in S, $N[x] - N[S - x] \neq \phi$. That is S is irredundant if for every x in S, there exists a vertex y in $N[x]$, which is not in $N[S - x]$. Such a vertex will be called a "private neighbor" of x, it could be x itself if x is an isolate in $\langle S \rangle$, the induced subgraph of S; if x is not an isolate of $\langle S \rangle$, then a "private neighbor" of x must be a vertex outside of S. The minimum number of vertices in a dominating set is called the *domination number of* G, denoted $\gamma(G)$. The upper domination number of G, $\Gamma(G)$ is the maximum number of vertices in a minimal dominating set. The *irredundance number* $\mathrm{ir}(G)$ and the *upper irredundance number* $\mathrm{IR}(G)$ are respectively the minimum cardinality and

JCMCC 12 (1992), pp. 179-185

maximum cardinality of a maximal irredundant set of G. The following string of inequalities connecting these parameters of any graph G is well known [2]

$$\mathrm{ir}(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq \mathrm{IR}(G),$$

where $i(G)$ and $\beta_0(G)$ are minimum and maximum cardinalities of a maximal independent set of vertices.

We will use expressions like IR-set for a set S of vertices of G if S is an irredundant set with $|S| = \mathrm{IR}$. Similar expressions will be used for other parameters.

Recently Sampathkumar introduced the concept of global domination [6]. A set $S \subseteq V(G)$ is a *global dominating* set if S dominates both G and $\overline{G}$. In other words, S is a global dominating set if for every x not in S, there exist vertices y and z in S such that $xy \in E(G)$ and $xz \in E(\overline{G})$. Sampathkumar showed in particular, that if T is a tree then $\gamma(T) \leq \gamma_g(T) \leq \gamma(T) + 1$, where $\gamma_g(T)$, the global denomination number of T is the minimum number of vertices in a global dominating set of G. In [5] Rall has shown that for trees (with two exceptional cases) and for graphs having diameter at least five, the global domination number and the upper global domination number are equal to the domination number and the upper domination number respectively where the upper global domination number $\Gamma_g(G)$ of any graph is the maximum number of vertices in a minimal global dominating set of G.

In this paper, we introduce two similar concepts of irredundancy. A set S of vertices is called a *universal irredundant set* if S is an irredundant set in both G and $\overline{G}$. In other words, S is a universal irredundant set if for every $x \in S$,

i) $N_G[x] - N_G[S - x] \neq \phi$, and
ii) $N_{\overline{G}}[x] - N_{\overline{G}}[S - x] \neq \phi$.

We will call a set $S \subseteq V(G)$ a *global irredundant* set if for every $x \in S$, either

i) $N_G[x] - N_G[S - x] \neq \phi$, or
ii) $N_{\overline{G}}[x] - N_{\overline{G}}[S - x] \neq \phi$.

If $S \subseteq V(G)$ is irredundant in G, we will say S is G-irredundant and if S is irredundant in $\overline{G}$, then S will be called $\overline{G}$-irredundant.

The *universal irredundance number*, $\mathrm{ir}_u(G)$ and the *upper universal irredundance* number $\mathrm{IR}_u(G)$ are respectively the minimum and maximum cardinalites of a maximal universal irredundant set. Similarly the *global irredundance number*, $\mathrm{ir}_g(G)$ and the upper global irredundance number $\mathrm{IR}_g(G)$ are defined.

In this paper we investigate parameters $\mathrm{IR}_u(G)$ and $\mathrm{IR}_g(G)$ of any graph and their relations with related parameters. We also show that determining $\mathrm{IR}_u(G)$ is NP-complete for any arbitrary graph G. In the conclusion we state some open problems.

2. Maximum Universal Irredundance

For any graph G it is clear that $\mathrm{IR}(G) \geq \mathrm{IR}_u(G)$ and $\mathrm{IR}(\overline{G}) \geq \mathrm{IR}_u(G)$. It is easy to verify that $\mathrm{IR}_u(P_n) = 2$, for $n > 3$; $\mathrm{IR}_u(K_{1,n}) = 1$; $\mathrm{IR}_u(K_{m,n}) = 2$, with $m, n > 1$; $\mathrm{IR}_u(K_n) = 1$, and $\mathrm{IR}_u(K_{n,n} - 1\text{-factor}) = n$.

Brewster, Cockayne and Mynhardt [1] showed the following result.

Lemma 1. *$\overline{G}$ has an irredundant set of size 3 if and only if G contains a K_3 or a $C_6 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ with the following edges in $\overline{G}$: v_1v_4, v_2v_5 and v_3v_6.*

An immediate consequence of the above lemma is the following:

Corollary 1. *For any tree T, $IR_u(T) \leq 2$.*

Lemma 2. *For any graph G if diameter $G \geq 3$ then $IR_u(G) \geq 2$.*

Proof: Let $x, y \in V(G)$ be such that $d(x, y) = 3$. Let x, v_1, v_2, y be a shortest distance path in G joining x and y. Consider the set $S = \{v_1, v_2\}$. Clearly S is independent in $\overline{G}$ and hence $\overline{G}$-irredundant. The set S is also G-irredundant, since both v_1 and v_2 have private G-neighbors x and y respectively. Thus S is a universal irredundant set, and hence $\mathrm{IR}_u(G) \geq 2$.

From Corollary 1 and Lemma 2 we have the following result:

Theorem 3. *For any tree $T \neq K_{1,n}$, $IR_u(T) = 2$.*

Our next result gives an upper bound of $\mathrm{IR}_u(G)$ for any G.

Theorem 4. *For any graph G, $IR_u(G) \leq 1+\gamma(G)$ where $\gamma(G)$ is the maximum degree of G. For a connected graph G, with more than two vertices equality holds if and only if $G = K_{n,n} - 1$-factor.*

Proof: Let S be an $\mathrm{IR}_u(G)$-set of any graph G. Let $x \in S$. Suppose, $\mathrm{IR}_u(G) > 1 + \gamma(G)$. Then there exists a vertex $y \neq x$ in S, such that xy is not in $E(G)$ and so $xy \in E(\overline{G})$. Since S is $\overline{G}$-irredundant, x must have a private $\overline{G}$-neighbor, outside of S. Thus, there exists a vertex z not in S, such that $xz \in E(\overline{G})$ and $zu \in E(G)$, for all $u \neq x$ in S. Hence $\gamma(G) \geq \deg_G z \geq \mathrm{IR}_u(G) - 1$, that is $\mathrm{IR}_u(G) \leq 1 + \gamma(G)$, a contradiction.

Clearly if $G = K_{n,n} - 1$-factor, G is connected and a partite set $\{v_1, v_2, \ldots, v_n\}$ of G is irredundant in both G and $\overline{G}$ and it follows $\mathrm{IR}_u(G) = n = 1 + \gamma(G)$.

Now, suppose for any connected graph G, $\mathrm{IR}_u(G) = 1 + \gamma(G)$. Let S be an $\mathrm{IR}_u(G)$-set. Let $x \in S$, suppose x is adjacent to all vertices $y \neq x$ in S. Then, since S is G-irredundant, x must have a private G-neighbor outside of S, but then $\deg_G x > \gamma(G)$. Therefore for every $x \in S$, there must exist a vertex $y \in S$, such that xy is not in $E(G)$, that is, $xy \in E(\overline{G})$. But then, since S is $\overline{G}$-irredundant and $xy \in E(\overline{G})$, there must exist a private $\overline{G}$-neighbor x' of x, such that $xx' \in E(\overline{G})$ and $x'y \in E(G)$ for all other vertices y of S. In other words, for every $x \in S$, there exists x' not in S, such that xx' is not in $E(G)$ but x' is adjacent to all other vertices of S in G. Let S' be the set of all such x' vertices. Clearly each vertex of S and S' is a max-degree vertex in G and both S and S' are independent sets of vertices in G. Also since G is connected and every vertex of S and S' is a max-degree vertex of G, it follows that $G = \langle S \cup S' \rangle$, so G is a $K_{n,n} - 1$-factor.

3. The Complexity of the Upper Universal Irredundance Number

Fellows, Fricke, Hedetniemi and Jacobs have recently established in [3] the NP-completeness of the problem of determination of $\mathrm{IR}(G)$ of any graph G. In this section we show that determining $\mathrm{IR}_u(G)$ for any G is also NP-complete. The decision problem we consider is as follows:

INSTANCE: $G = (V, E)$, positive integer k.
QUESTION: Is $\mathrm{IR}_u(G) \geq k$?

Clearly if $G = K_n$, $\mathrm{IR}_u(G) = 1$. Let $G \neq K_n$ and $G_1 = K_n$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_1) = \{v'_1, v'_2, \ldots, v'_n\}$. Construct G' as follows:

$$V(G') = V(G) \cup V(G_1)$$
$$E(G') = E(G) \cup E(G_1) \cup \{v_i v'_j, i \neq j\}$$

Theorem 5. *If $IR(G) \geq k$, then $IR_u(G') \geq k$.*

Proof: Let S be an $\mathrm{IR}(G)$ set with $\mathrm{IR}(G) \geq k$. Clearly S is an irredundant set in G'. Also, every $v_i \in S$ has a private neighbor v'_i in $\overline{G'}$. So S is irredundant in $\overline{G'}$ also. Thus S is universally irredundant in G' and hence $\mathrm{IR}_u(G') \geq k$.

Now we show the following:

Theorem 6. *If $IR_u(G') \geq k$, then $IR(G) \geq k$.*

Proof: Let S be an IR_u-set in G' with $|S| \geq k$. Let $S = X \cup Y$, where $X \subseteq V(G)$ and $Y \subseteq V(G_1)$.

Claim 1. *If $v_i \in X$ then v'_i is not in Y.*

Suppose on the contrary both $v_i \in X$ and $v'_i \in Y$. Then by our construction of G', $v_i v'_i$ is not in $E(G')$ and hence $v_i v'_i \in E(\overline{G'})$. Since S is universally irredundant in G', both v_i and v'_i must have private $\overline{G'}$-neighbors outside of S. But v_i is the only vertex adjacent to v'_i in $\overline{G'}$ and hence v'_i has no private $\overline{G'}$-neighbor, a contradiction.

Claim 2. $|Y| \leq 2$.

Suppose v'_i, v'_j, v'_k are distinct vertices of Y. Since G_1 is a complete graph, v'_i, v'_j, v'_k are mutually adjacent in G'. So each of these vertices must have G'-private neighbors and these private neighbors must be in G. Let x be a private neighbor of v'_i. Then x can not be adjacent to v'_j and v'_k. However, by our construction of G', no such vertex x in G exists. Thus $|Y| \leq 2$.

Claim 3. *If* $|Y| = 2$, *then* $|X| = 0$.

Suppose $Y = \{v_i'\}$ and let $v_j, v_k \in X$. By claim 1 v_k' is not in Y. So by construction of G', $v_i'v_k \in E(G')$ and $v_j'v_k \in E(G')$. Hence v_k must have a G'-private neighbor say x which can be adjacent neither to v_i' nor to v_j'. But by the construction of G' no such vertex x exists.

Claim 4. *If* $|Y| = 1$ *then* $|X| \leq 1$.

Suppose $Y = \{v_i'\}$ and $v_j, v_k \in X$. Then by claim 1, $i \neq j$ and $i \neq k$ and so both v_j and v_k are adjacent to v_i' in G'. Hence both v_j and v_k must have distinct G'-neighbors outside of S. Since G_1 is complete, these distinct G'-private neighbors must both lie in G and both be non-adjacent to v_i' in G'. But v_i' has only one vertex v_i non-adjacent in G' and hence $|X| \leq 1$.

Claim 5. *If* $|Y| = 0$, *then* $|X| \leq 2$ *or* X *is irredundant in* G.

If $|Y| = 0$ and X is not irredundant in G, then there exists $v_i \in X$ whose private G'-neighbor lie in G_1, say v_j'. But then there can be at most one other vertex in X, namely v_j, and thus $|X| \leq 2$.

Claim 6. $IR(G) \geq k$.

Claims 2, 3, 4, show that $|S| \leq 2$ when $|Y| > 0$. If $|Y| = 0$ then by claim 5 $|S| \leq 2$ or S is irredundant in G. If S is irredundant in G, then $\mathrm{IR}(G) \geq k$. If $|S| \leq 2$, then since G is not complete there exist two non-adjacent vertices v_i, v_j and $\{v_i, v_j\}$ is an irredundant set in G and so $\mathrm{IR}(G) \geq 2$. Hence for all cases $\mathrm{IR}(G) \geq k$. Thus we have the theorem.

Theorem 7. *Determination of* $IR_u(G)$ *for any graph* G *is NP-complete.*

Proof: Recently Fellows, Fricke, Hedetniemi and Jacobs [3] have shown that the determination of $\mathrm{IR}(G)$ for any graph G is NP-Complete. Theorems 5 and 6 together with their result establish the theorem.

4. Universal versus Global Irredundance.

Recall the definitions of a dominating set and an irredundant set. A set $S \subseteq V$ is a dominating set if

i) $N[S] = V$

A dominating set $S \subseteq V$ is a minimal dominating set if $\forall x \in S$, $S - x$ is not a dominating set. In other words

ii) $\forall x \in S, N[x] - N[S - x] \neq \phi$.

Note that for any set $S \subseteq V$, satisfying (ii) implies S is irredundant. As a matter of fact, any dominating set satisfying irredundancy condition (ii) is a minimal dominating set. More precisely, that a minimal dominating set is maximal irredundant is a well-known result [2].

Now consider a global dominating set S. What makes S a minimal global dominating set? A global dominating S set is a minimal global dominating set if for all $x \in S$, $S - x$ is no longer a global dominating set. In other words, a global dominating set S is a minimal global dominating set if

(iii) for $\forall x \in S$, either

$$N_G[x] - N_G[S - x] \neq \phi$$

or

$$N_{\overline{G}}[x] - N_{\overline{G}}[S - x] \neq \phi$$

Note that condition (iii) is exactly the definition of a global irredundant set. As a matter of fact it is easy to verify the following:

Lemma 8. *A minimal global dominating set is a maximal global irredundant set.*

The following string of inequalities is obvious.

$$\text{For any } G, \mathrm{ir}_g \leq \gamma_g \leq \Gamma_g \leq \mathrm{IR}_g.$$

Also note that $\mathrm{IR}_g \geq \mathrm{IR}$. Strict inequality can occur as shown below.

The universal irredundancy condition for a set is much stronger than global irredundancy. Each vertex in a universal irredundant set has to be irredundant in both G and $\overline{G}$. Thus $\mathrm{IR}_u \leq \mathrm{IR}_g$ for any graph G.

5. Concluding remarks and some open questions

In this paper we have introduced two new concepts of irredundancy which fit nicely with some of the well-studied dominating parameters. However, we have just started investigating some of their properties. We list some open problems here.

1. Determine the complexity of ir_g and ir_u and IR_g. Recently D. Jacobs has shown that determining Γ_g and γ_g for any graph G is NP-complete [4].
2. $\Gamma_g \leq \mathrm{IR}_g$ for any graph G. Determine for which graphs $\Gamma_g \leq \mathrm{IR}_g$.
3. Does there exist a graph G for which $\mathrm{IR}_g = \mathrm{IR}$?
4. Does there exist G for which $\mathrm{IR}_g = \mathrm{IR}_u$?
5. Find Nordhaus-Gaddum type results for these new parameters.

Acknowledgement

The authors wish to thank S.T. Hedetniemi and D. Rall for stimulating discussions during preparation of this paper.

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