Bailey's Conjecture Holds Through 87 Except Possibly for 64

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Abstract. R.A. Bailey has conjectured that all finite groups except elementary Abelian 2-groups with more than one factor have 2-sequencings (i.e., terraces. She verified this for all groups of order n, $n \leq 9$. Results proved since the appearance of Bailey's paper make it possible to raise this bound to $n \leq 87$ with $n = 64$ omitted. Relatively few groups of order not 2^n, $n \in \{4, 5\}$ must be handled by machine computation.

1. Introduction

Bailey [10] defined 2-sequencings (she called them terraces) of finite groups. Her interest was in generalizing a construction of Gordon [12] to build quasi-complete Latin squares. Since then 2-sequencings have been used to find 1-factorizations of K_{2n} with interesting symmetry groups [2], [5] and to solve certain partitioning problems involving the edges of a complete graph [9].

A *sequencing* of a finite group G of order n with identity e is an ordering

$$S : e, s_2, s_3, \ldots, s_n$$

of all the elements of G such that the partial products

$$P : e, es_2, es_2 s_3, \ldots, es_2 \ldots s_n$$

are distinct and hence also all of G. A finite group G is a Λ-group if and only if G has a unique element of order 2. A sequencing S of a Λ-group G of order $2n$ with unique element z of order 2 is a *symmetric sequencing* if and only if $s_{n+1} = z$ and for $1 \leq i \leq n-1$, $s_{n+1+i} = (s_{n+1-i})^{-1}$. A symmetric sequencing S with associated partial product sequence

$$P : e, t_2, t_3, \ldots, t_{2n}$$

is a *symmetric d-sequencing* if and only if there is a j, $2 \leq j \leq 2n-1$ and there is a y in G such that

i) $y \neq e \neq y^2$
ii) $(t_j, t_{j+1}) \in \{(y, y^2), (y^2, y)\}$

JCMCC 12 (1992), pp. 187-195

Suppose G is a group of odd order $2n+1$ and identity e. Then $L = \{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$ is a *left starter* for G if and only if

i) every nonidentity element of G occurs in some pair of L,
ii) every nonidentity element of G occurs in $\{x_i^{-1}y_i, y_i^{-1}x_i : 1 \leq i \leq n\}$.

If $h \in G$, then $hL = \{\{hx_i, hy_i\} : 1 \leq i \leq n\}$ is a *left translate* of L.

Suppose H is a finite group of order n with identity e. A *2-sequencing* of H is an ordering

$$\sigma : e, s_2, s_3, \ldots, s_n$$

of certain elements of H (not necessarily distinct) such that

i) the associated partial products

$$\rho : e, es_2, es_2s_3, \ldots, es_2 \ldots s_n = e, t_2, t_3, \ldots, t_n$$

are distinct and hence all of H,

ii) if $y \in H$ and $y \neq y^{-1}$ then

$$|\{i : 2 \leq i \leq n \text{ and } (s_i = y \text{ or } s_i = y^{-1})\}| = 2,$$

iii) if $y \in H$ and $y = y^{-1}$ then

$$|\{i : 1 \leq i \leq n \text{ and } s_i = y\}| = 1.$$

If $|H|$ is odd, then the collection $\{\{x, x^{-1}\} : x \in H\backslash\{e\}\}$, the *patterned starter*, will be denoted PS_H. It is easily seen that PS_H is a left starter for H. The statement that the 2-sequencing a is a *starter-translate* 2-sequencing (st-2-sequencing) means that both

$$S_{\sigma(H)} = \{s_3, s_5, \ldots, s_n\} \text{ and } T_{\sigma(H)} = \{s_2, s_4, \ldots, s_{n-1}\}$$

are transversals of (the pairs of) PS_H. Note that if $S_{\sigma(H)}$ is a transversal of PS_H then

$$A = \{\{t_2, t_3\}, \{t_4, t_5\}, \ldots, \{t_{n-1}, t_n\}\}$$

is a left starter for H and if $T_{\sigma(H)}$ is a transversal of PS_H, then

$$B = \{\{e, t_2\}, \{t_3, t4\}, \ldots, \{t_{n-2}, t_{n-1}\}\}$$

is a left translate by t_n of a left starter for H.

2. Reductions

Excluding order 64, there are 493 groups of order ≤ 87. The goal is to show that all but 4 of these groups have 2-sequencings. Bailey [10], showed that the groups Z_2^n (where Z_j is the cyclic group of order j), $n \geq 2$, do not have 2-sequencings. Several recent theorems allow one to construct 2-sequencings on most of the 493 groups. The groups of odd order can be handled by the following

Theorem 1. [6] *All groups of odd order have st-2-sequencings.*

It is well-known that if p is an odd prime, then every group of order $2p$ is either cyclic or dihedral (e.g., [13, p. 62]). The next two results will control this situation.

Theorem 2. [1] *All Abelian Λ-groups have symmetric sequencings.*

Theorem 3. [5] *If $n \geq 3$, then D_n, the dihedral group of order $2n$, has a 2-sequencing.*

This disposes of the even orders

$$6, 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86.$$

Here is a very useful construction.

Theorem 4. [7] *If G is a finite group, C is a normal odd order subgroup and G/C has a 2-sequencing, then G has a 2-sequencing.*

Another result of that same paper is

Theorem 5. [7] *If p is an odd prime and n is a positive integer, then the semi-direct product $[(Z_p)^n](Z_2 \times Z_2)$ has a 2-sequencing.*

A few simple applications of the Sylow Theorems will pay nice dividends.

Lemma 6. *If $p \geq 5$ is an odd prime and G is a finite group of order $2^2 p$, then the Sylow p-subgroup Z_p of G is normal.*

If G is a group of order $4p$, p an odd prime ≥ 5, then by Lemma 6, G is a semi-direct product of Z_p by either Z_4 or $Z_2 \times Z_2$. Theorem 2 implies that Z_4 has a 2-sequencing. Thus G has a 2-sequencing by application of either Theorem 4 or Theorem 5. The orders 20, 28, 44, 52, 68 and 76 fall to this method.

Lemma 7. *If p is an odd prime, n is a positive integer and G is a group of order $2p^n$, then the Sylow p-subgroup of G is normal.*

This can be used with Theorem 4 to eliminate orders 18, 50 and 54.

If p and q are odd primes, $p < q$, then often Sylow type arguments will give useful information for groups of order $2pq$. In the cases

$$42 = 2 \cdot 3 \cdot 7,\ 66 = 2 \cdot 3 \cdot 11,\ 70 = 2 \cdot 5 \cdot 7 \text{ and } 78 = 2 \cdot 3 \cdot 13$$

it is easy to see that the Sylow q-subgroup must be normal. Since all groups of order $6 = 2 \cdot 3$ and $10 = 2 \cdot 5$ have 2-sequencings, all groups of orders 42, 66, 70 and 78 do also. One must be slightly more careful with $30 = 2 \cdot 3 \cdot 5$. A simple counting argument shows that every group of order 30 must have either a normal Z_3 or a normal Z_5. In fact all groups of order 30 have both [14], and this case can be settled like the others.

All groups of order 12 were shown 2-sequenceable in [2]. Actually, A_4 is the only group of order 12 that can't be 2-sequenced by the methods given here so far. A Sylow argument will then work for all groups of order 84.

Consider next the 14 groups of order 36. Four of these groups are Abelian and so they all have a normal Z_3. Since all groups of order 12 have 2-sequencings, so do these groups. The Supplementary Summary Sheet (SSS) of [14] gives much useful information. By SSS, 3 of the non-Abelian groups are of the form $[Z_3^2](Z_2^2)$ so that Theorem 5 applies; 3 are of the form $[Z_3^2]Z_4$ and one is of type $[Z_9]Z_4$ so that Theorem 4 works. One group is dihedral and the remaining 2 are of the form $[Z_2^2](Z_3^2)$ and $[Z_2^2]Z_9$. Fortunately both of these last two groups have a Z_3 center so they can be 2-sequenced in the same way as the Abelian groups.

There are 13 groups of order 60. By SSS, all of these groups except A_5 have a normal Z_5. Since A_5 has a sequencing [4], all groups of order 60 have 2-sequencings. The question has now been reduced to groups of order a multiple of 8.

3. Groups of Order $8N$

Currently some machine computation is required for an these orders except 8 and 72. First consider the orders 8, 16, 24 and 32. The following result gives 2-sequencings for several of these groups.

Theorem 8. [8] *If a finite Λ-group G has a symmetric d-sequencing, then $G \times Z_2$ has a 2-sequencing. All Abelian Λ-groups except Z_2 have symmetric d-sequencing.*

There are 5 groups of order 8. Theorems 2 and 8 yield 2-sequencings for Z_8 and $Z_4 \times Z_2$ respectively. By [10], Z_2^3 does not have a 2-sequencing. The dihedral group D_4 does have one by Theorem 3 and the quaternion group Q_8 does also [2].

The non-Abelian groups of orders 16, 24 and 32 are covered by the application of an algorithm developed in [3], [4].

Theorem 9. *All non-Abelian groups of order n, $10 \leq n \leq 32$ are sequenceable (and thus 2-sequenceable).*

There are 14 groups of order 16; 5 of them are Abelian. As with the groups of order 8, Z_{16} and $Z_8 \times Z_2$ have 2-sequencings by Theorems 2 and 8, respectively, and Z_2^4 does not have a 2-sequencing. Solutions for the 2 remaining Abelian groups, Z_4^2 and $Z_4 \times Z_2$ have been found by machine and will be listed shortly.

The order 24 is associated with 15 groups. Twelve of these groups have a normal Z_3 and the other 3 groups are all non-Abelian [14] and thus sequenceable. If G is one of the 12 with a normal Z_3, then Theorem 4 applies unless $G/Z_3 \approx Z_2^3$. This happens twice with $D_6 \times Z_2$ and $Z_3 \times Z_2^3$. But $D_6 \times Z_2$ is non-Abelian and a 2-sequencing for $Z_3 \times Z_2^3$ will be listed shortly.

There are 51 groups of order 32; 7 of them are Abelian. As before, Z_{32} and $Z_{16} \times Z_2$ are 2-sequenceable and Z_2^5 is not. The other 4 Abelian groups can be 2-sequenced by machine. At this point, then, there are 7 Abelian groups to be 2-sequenced.

The numbering system of [14] will be used here. In [14] the groups of order $2m$ are labelled $2m/1, 2m/2, \ldots$. The group $2m/1$ is the cyclic group of order $2m$ and the Abelian groups are always first in the ordering.

A 2-sequencing σ of a finite group H is a *minimal deficiency* 2-sequencing (see [3] for the reasons for this terminology) if and only if σ is a 2-sequencing and there is exactly one pair $\{x, x^{-1}\}$ of distinct elements of H such that x appears twice in σ. Thus for all other pairs $\{y, y^{-1}\}$ of H, each element appears exactly once in σ as do all self-inverse elements of H. The algorithm of [3] is easily modified to search for minimal deficiency 2-sequencings. Thus modified algorithm is successful on the 7 groups listed above. Here are solutions in the notation of the appropriate table of [14].

16/3: $Z_4 \times Z_4$

1 10 13 2 12 15 5 9 8 3 7 11 16 14 6 2

16/4: $Z_4 \times Z_2 \times Z_2$

1 12 10 13 8 14 15 16 9 3 5 11 14 4 2 7

24/3: $Z_3 \times Z_2 \times Z_2 \times Z_2$

1 22 4 9 21 16 5 14 2 7 20 12 15 3 17 8
18 20 13 11 23 6 19 10

32/3: $Z_8 \times Z_4$

1 11 4 31 5 21 8 11 17 13 22 24 10 15 20 32
12 14 19 3 6 26 18 2 9 23 16 25 29 30 28 7

32/4: $Z_8 \times Z_2 \times Z_2$

1 9 17 12 3 16 21 14 19 31 24 20 26 9 27 6
15 28 18 29 32 13 23 2 7 11 5 8 30 22 10 4

32/5: $Z_4 \times Z_4 \times Z_2$

1 23 10 3 9 7 21 16 27 18 15 28 20 25 6 29
2 11 22 17 5 24 4 26 32 8 12 19 30 31 14 29

32/6: $Z_4 \times Z_2 \times Z_2 \times Z_2$

1 25 27 6 2 11 4 22 5 24 3 23 15 19 17 16
28 29 14 20 12 18 9 32 10 13 8 31 14 26 7 21

Sylow theory shows that every one of the 14 groups of order 40 has a normal Z_5

subgroup. If G is such a group and $G/Z_5 \not\approx Z_2^3$. then Theorem 4 says that G has a 2-sequencing. Two groups $Z_5 \times Z_2^3$ and $[Z_5]Z_2^3$ must be considered separately.

The order 48 is the most troublesome. This order has 52 groups, 5 of which are Abelian. The Abelian groups all have a normal Z_3, so all are 2-sequenceable by Theorem 4 except $Z_3 \times Z_2^4$, which will be tackled later. Most of the non-Abelian groups of order 48 can be taken care of similarly. In fact, 36 of these 47 groups have a normal Z_3 with a complement that is not Z_2^4 see SSS of [14]). The 11 remaining non-Abelian groups and the single Abelian group make 12 special cases of order

There are 13 groups of order 56. By [SSS], 12 of them have a normal Z_7 and so problems arise only when such a group G has $G/Z_7 \approx Z_2^3$. This happens twice with $Z_7 \times Z_2^3$ and $[Z_7]Z_2^3$. The remaining group $[Z_2^3]Z_7$ must also be dealt with.

It is, perhaps, somewhat surprising that all 50 groups of order 72 can be 2-sequenced without resorting to machine testing. First, the 6 Abelian groups of order 72 all have a normal Z_3, and all groups of order 24 have 2-sequencings. The list in SSS of [14] shows that the groups listed 7–10, 15–17, 19–32, 36–46 all have either a normal Z_9 or a normal Z_3^2 and in all cases, a complement that is not Z_2^3. In the groups labelled 11–14, the center is always Z_6. But then Z_3 is a characteristic subgroup of Z_6 so each of these groups has a normal Z_3 and is 2-sequenceable. The group numbered 18 has a normal Z_9 and, as above, a normal Z_3. The commutator subgroup of group 34 is Z_3 so, again this group has a normal Z_3. The commutator subgroup of group 47 is $Z_3 \times Z_2$ and certainly Z_3 is a characteristic subgroup of it. Next, Z_3 is the center of group 48. The last two groups numbered 49 and 50 have as commutator subgroups the last two groups mentioned in the discussion of the order 36. Both these groups have a center of Z_3 and the center is a characteristic subgroup.

This leaves the two groups numbered 33 and 35. Both are $[Z_3^2]Z_2^3$ and an argument from [7] can be used. From the semi-direct product structure, each group has an (in these two cases nontrivial) associated homomorphism

$$\alpha : Z_2^3 \rightarrow \mathrm{Aut}(Z_3^2) \approx GL(2,3).$$

The idea is to show that each group has a normal subgroup Z_3. Since α is not trivial, Im(α) is an Abelian subgroup of GL(2,3) whose elements are the identity transformation and a collection of involutions. But all involutions in GL(2,3) are diagonalizable. Thus, the vector space Z_3^2 has a basis of simultaneous characteristic vectors for the transformations in Im(c) [11, p. 200] and clearly each of these vectors generates a normal subgroup Z_3. The argument is completed by noting that all groups of order 24 have 2-sequencings.

Finally, there are 52 groups of order 80. All but one of them has a normal Z_5 (see SSS of [14]) and so problems arise only when G is such a group with $G/Z_5 \approx Z_2^4$. This happens twice with $Z_5 \times Z_2^4$ and $[Z_5]Z_2^4$. The remaining group is $[Z_2^4]Z_5$. These 3 groups are handled by machine.

Here are sequencings for the 16 non-Abelian groups left with the group table constructed via the disc of [14].

40/11: $[Z_5]Z_2^3$

1 27 29 8 19 28 36 35 17 38 26 23 5 33 24 15 37 34 11 22 10
32 16 31 18 7 40 30 12 20 2 6 21 3 25 14 13 39 4 9

48/15: $[Z_4^2]Z_3$

1 3i 11 13 27 8 26 47 42 25 20 30 3 15 9 23 17 46 14 43 6
37 22 10 19 21 7 16 39 38 40 35 18 33 45 36 24 48 44 34 28 5
4 29 41 32 2 12

48/16: $[Z_4 \times Z_2^2]Z_3$

1 18 24 25 4 38 9 48 28 14 26 15 37 22 27 41 47 21 7 30 6
29 46 39 44 33 10 20 13 3 2 11 35 43 8 40 17 36 32 42 16 5
31 23 34 19 45 12

48/17: $[Z_2^4]Z_3$

1 32 14 37 19 48 34 25 11 43 18 39 41 35 33 26 21 2 6 3 28
23 27 16 36 9 22 4 12 40 15 5 38 7 10 31 13 17 20 46 8 24
47 30 45 42 29 44

48/18: $[Z_2^4]Z_3$

1 10 25 18 38 4 5 31 11 7 6 14 47 21 27 40 46 37 28 12 44
3 29 9 2 39 22 45 15 41 17 42 32 36 34 24 13 16 19 23 33 20
48 35 43 26 30 8

48/19: $[Q_8 \times Z_2]Z_3$

1 4 30 46 22 11 13 39 8 3 24 21 29 5 20 18 31 12 40 7 36
17 42 34 43 25 33 47 9 14 15 26 38 44 23 19 27 48 37 16 6 35
28 41 2 45 10 32

48/20: $[[Z_4 \times Z_2]Z_2]Z_3$

1 39 22 48 33 14 20 30 5 9 47 24 36 16 11 19 27 35 46 13 43
2 32 41 7 10 8 12 4 6 3 40 34 28 44 17 29 26 23 31 45 38
15 25 21 42 37 18

48/27: $[Z_3]Z_2^4$

22 39 43 3 23 42 5 17 12 16 45 28 31 20 15 38 40 44 41 24
35 29 48 13 34 10 36 32 46 1 33 2 27 9 18 30 8 26 25 11 47
14 4 6 7 37 19

The 4 remaining groups of order 48 are such that no Sylow subgroups Sy_p are normal. Clearly, Sy_3 must be Z_3.

48/49: $Sy_2 = D_4 \times Z_2$

1 4 43 15 38 36 31 46 10 23 5 9 3 22 27 29 32 12 14 18 44
13 20 11 45 48 6 37 35 34 33 30 40 41 17 19 47 39 2 21 25 16
7 24 28 42 8 26

48/50: $Sy_2 = [Z_4 \times Z_2]Z_2$
1 24 44 35 36 29 16 46 11 22 12 21 47 32 23 37 2 39 8 48 18
17 45 4 43 7 42 9 13 6 27 15 40 38 28 14 10 3 5 30 33 19
26 25 20 31 34 41

48/51: $Sy_2 = Q_{16}$
1 38 25 28 34 48 33 43 47 41 42 6 30 44 31 46 11 23 17 37 22
35 24 29 19 18 21 26 2 36 13 9 7 27 39 8 45 32 5 3 15 14
40 12 10 4 16 20

48/52: $Sy_2 = [Z_8]Z_2$
1 27 22 44 5 12 37 21 19 6 46 7 14 38 28 23 13 42 47 43 25
31 36 48 8 11 2 35 30 32 15 10 34 29 20 18 3 26 16 24 33 45
17 39 40 4 9 41

56/6: $[Z_2^3]Z_7$
1 28 22 9 47 20 12 6 13 27 54 24 33 21 36 15 30 26 25 5 43
56 2 51 14 29 23 3g 19 41 32 34 7 3 49 37 46 35 11 4 40 52
18 17 53 50 42 44 31 16 45 55 48 38 10

56/10: $[Z_7]Z_2^3$
1 26 56 25 36 44 11 14 2 31 15 46 22 49 53 41 9 4 30 6 55
23 24 19 8 27 17 42 5 43 7 16 50 38 34 3 32 52 48 29 12 45
51 20 54 37 2 13 10 35 21 33 18 47 39 40

80/15: $[Z_2^4]Z_5$
46 39 18 22 10 13 33 29 59 12 66 0 67 14 43 49 26 74 52 58
42 5 68 77 61 53 51 79 45 36 6 69 4 4 15 40 75 57 20 27 35
37 8 71 23 55 16 17 60 21 28 30 63 47 44 9 78 65 19 34 31 11
64 54 41 62 3 38 73 32 70 7 50 25 24 2 56 76 72

80/27: $[Z_5]Z_2^4$
1 41 29 37 38 60 34 57 7 66 53 77 24 43 16 36 65 26 10 35 8
2 52 45 72 22 31 73 51 59 32 25 49 48 17 63 14 56 39 33 42 55
79 6 62 78 3 18 74 19 27 64 75 13 58 12 71 21 68 15 9 30 67
50 46 20 40 76 80 23 47 11 4 28 5 69 61 70 54 44

Lastly, each of the 4 Abelian groups still to be 2-sequenced has a minimal deficiency 2-sequencing as follows.

40/3: $Z_5 \times Z_2^3$
1 23 18 35 7 28 31 38 9 22 17 20 21 25 4 37 26 11 34 27 13
36 30 24 10 5 19 12 14 40 39 6 15 16 33 3 10 8 29 32

48/5: $Z_3 \times Z_2^4$
1 2 5 39 30 44 14 28 9 15 38 34 22 36 17 37 47 25 4 33 12
10 18 26 45 35 43 11 41 13 20 7 27 21 40 24 46 19 16 38 29 8
6 3 32 31 48 23

56/3: $Z_7 \times Z_2^3$

1 31 39 42 48 24 7 <u>53</u> 29 21 14 20 22 16 49 25 6 15 <u>53</u> 35 52
36 5 38 55 13 44 54 11 4 9 30 8 3 37 17 51 46 45 18 40 33
41 23 27 10 56 12 26 50 32 2 43 19 28 34

80/5: $Z_5 \times Z_2^4$

1 <u>18</u> 71 49 45 28 21 25 24 79 70 76 3 12 37 40 73 59 77 36 43
38 9 47 7 66 80 6 39 69 75 10 60 65 8 33 62 64 32 11 2 44
15 58 67 56 16 53 20 5 26 48 74 17 35 52 68 30 63 27 42 41 <u>18</u>
4 31 78 22 19 61 72 55 13 50 34 23 51 57 54 46 29

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