On Cocircuit Graphs

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Abstract. It is shown that the circuit polynomial of a graph when weighted by the number of nodes in the circuits, does not characterize the graph. i.e. non-isomorphic graphs can have the same circuit polynomial. Some general theorems are given for constructing graphs with the same circuit polynomial (cocircuit graphs). Analogous results can be deduced for characteristic polynomials.

1. Introduction

The idea of finding a polynomial which characterizes a graph, is an old one and several attempts have been made to achieve this. Many of the well known graph polynomials were, at one time or the other thought to be characterizing polynomials; that is two graphs were isomorphic if and only if they had the same polynomial. It was once conjectured that the characteristic polynomial of the adjacency matrix of a graph characterized the graph (see Harary et al [4]). However it turned out that none of these polynomials characterized the graph. The problem of finding such a polynomial is still essentially unsolved.

One interesting attempt to find a characterizing polynomial was made by Balasubramanian and Parthasarathy [1]. They have conjectured that their "permanent polynomial" is a complete invariant for graphs i.e. it will characterize the graphs. However this conjecture has not been proved.

In what follows, we show that the circuit polynomial of a graph when weighted in a certain way, does not characterize the graph. We then give theorems which serve to identify several families of non-isomorphic graphs which have the same circuit polynomial.

2. Preliminaries

Let G be a graph. By a *circuit cover* of G, we will mean a spanning subgraph of G whose components are circuits only. We take a circuit with one node to be an isolated node and a circuit with two nodes to be an edge. Circuits with more than two nodes will be called *proper circuits*. With every circuit α in G, let us associate a weight w_α and with every circuit cover C, the weight

$$W(C) = \Pi_\alpha w_\alpha,$$

where the product is taken over all components of the cover. Then the *circuit polynomial* of G is $\sum W(C)$, where the summation is taken over all the circuit covers in G. The circuit polynomial was introduced in Farrell [2].

JCMCC 12 (1992), pp. 7-15

If two graphs have the same circuit polynomial, we will say that they are *cocircuit* and call them *cocircuit graphs.* It is clear that in order for two graphs to have the same circuit polynomial, they must have not only the same types of circuit covers, but also the same number of the same types.

It would seem therefore that it would be difficult for two non-isomorphic general graphs to have the same circuit polynomial.

In order to obtain a practicable circuit polynomial of a graph G we must assign weights to the circuits, in a less general form. However, when we do so, some property of the relation of the circuits to the rest of the graph might be lost. This loss might be sufficient to make the circuit polynomial of G equal to that of some non-isomorphic graph H. Therefore the problem is this. How could we assign weights to the circuits so that the resulting polynomial would characterize the graph? Of course, we would be interested in the simplest assignment of weights that would achieve this. An interesting and simpler problem is this. How could we assign weights to the circuit so that particular kinds of non-trivial families of graphs are characterized?

In [2], we assigned weights in a simple way. Circuits with k nodes were given the weight w_k. We will assume in what follows, that weights are assigned in this way, when speaking about the circuit polynomial of a graph. One reason for assigning weights in this manner is because it yields an interesting connection between the circuit polynomial and the characteristic polynomial of a graph. This connection is given in the following theorem, which was proved in [2].

Theorem 1. *Let G be a graph. The characteristic polynomial of G is obtained from its circuit polynomial by putting $w_1 = x$, $w_2 = -1$ and $w_k = -2$, for $k > 2$.*

We will normally represent the circuit polynomial of a graph G by $C(G; \underline{w})$, where $\underline{w} = (w_1, w_2, \ldots, w_k, \ldots)$ is a vector of indeterminates. If we denote the characteristic polynomial of G by $\phi(G; x)$, then Theorem 1 can be restated as

$$\phi(G; x) = C(G; (x, -1, -2, -2, \ldots, -2)).$$

The following corollary is immediate from the theorem.

Corollary 1.1. *Let G_1 and G_2 be two graphs such that*

$$C(G_1; \underline{w}) = C(G_2; \underline{w}).$$

Then

$$\phi(G_1; x) = \phi(G_2; x).$$

3. Some Basic Properties of Circuit Polynomials

Let G be a graph containing an edge ab joining nodes a and b. We can partition the circuit covers in G into three classes, (i) those which do not contain ab, (ii) those in which ab is a component by itself and (iii) those in which ab is part of a proper circuit. The covers in Class (i) will be covers of the graph G' obtained from G by deleting ab. The covers in Class (ii) will be covers of the graph G'' obtained from G by removing nodes a and b, i.e. the graph $G - a - b$. The covers in Class (iii) will be covers of the graph G^* obtained from G by distinguishing ab in some way and requiring it to belong to every cover that we consider. We then say that ab is *incorporated* in G^* and call G^* the *restricted graph* in which ab is incorporated. Our discussion leads to the following fundamental theorem for circuit polynomials.

Theorem 2. *Let G be a graph containing an edge ab joining nodes a and b. Then*

$$C(G; \underline{w}) = C(G'; \underline{w}) + w_2 C(G''; \underline{w}) + C(G^*; \underline{w}),$$

where G', G'' and G^* are as defined above.

It is clear that Theorem 2 yields an algorithm for finding circuit polynomials smaller and smaller graphs of graphs. We simply apply the theorem recursively to smaller and smaller graphs, until we obtain graphs whose circuit polynomials are known. This algorithm, called the fundamental algorithm for circuit polynomials, will be referred to as the *reduction process.*

The following lemmas will be useful when using the reduction process. They can be easily established from the definition of G^*.

Lemma 1. *If G^* contains more than two incorporated edges incident to a node, then $C(G^*; \underline{w}) = 0$.*

Lemma 2. *If G^* is a tree, then $C(G^*; \underline{w}) = 0$.*

Lemma 3. *If G^* has an incorporated bridge, then $C(G^*; \underline{w}) = 0$.*

If G contains several components then the circuit covers of G can be obtained by independently taking covers in each component. This leads to the following theorem.

Theorem 3. *Let G be a graph consisting of k components $H_1, H_2, \ldots, H_k$. Then*

$$C(G; \underline{w}) = \Pi_{i=1}^{k} C(H_i; \underline{w}).$$

Let G be a graph and H a subgraph of G, We will use the notation $G - V(H)$, or simply $G - H$, to denote the graph obtained from G by removing the nodes of H. For simplicity of notation we will sometimes write $C(G)$ for $C(G; \underline{w})$ when it is convenient to do so.

The following theorem is the node analogue of Theorem 2 (u *adj* v means node u is adjacent to node v).

Theorem 4. *Let G be a graph with p nodes and v a node of G. Then*

$$C(G;\underline{w}) = w_1 C(G-v) + w_2 \sum_{u adj v} C(G-u-v) + \sum_{i=3}^{p} w_i \sum_{C_i} C(G-C_i),$$

where the final summation is taken over all the cycles C_i with i nodes and containing node v.

Proof: In any circuit cover of G, node v can either be (a) a component by itself, (b) incident with an edge which is a component or (c) part of a proper circuit. The result therefore follows.

The following corollary is immediate from Theorem 1.

Corollary 4.1. *For any node v of G,*

$$\phi(G;x) = x\phi(G-v) - \sum_{u adj v} \phi(G-u-v) - 2\sum_{i=3}^{p}\sum_{C_i} \phi(G-C_i).$$

This corollary is also given in Schwenk [10]. Also, some of the results given in this section are given in [2]. For other basic properties of circuit polynomials, the reader can consult [2].

4. Cocircuit Rooted Graphs

Let G and H be connected graphs. By *attaching* G to H (or H to G) we will mean that a specified node v of G is identified with a specified node x of H to yield a connected graph, denoted by $G_v + H_x$, in which G and H are subgraphs. The node formed by identification, denoted by v_x, will be called the *node of attachment*. When either G or H is rooted, we will assume that the root is used in the identification process, unless otherwise specified.

Theorem 5. *Let G be a graph rooted at a node v and H a graph rooted at a node x. Then*

$$C(G_v + H_x) = C(G)\, C(H-x) + C(G-v)C(H) - w_1 C(G-v)C(H-x).$$

Proof: In any cover of $G_v + H_x$ either

(i) v_x is isolated ;
(ii) v_x is incident with a component edge either totally in G or totally in H;
(iii) v_x belongs to a circuit either totally in G or totally in H.

By using Theorem 4 with v replaced by v_x, we get

$$C(G_v + H_x) = w_1\, C(G-v)C(H-x) + w_2\, C(H-x) \sum_{\binom{u \in G}{u\, adj\, v}} C(G-u-v)$$
$$+ w_2\, C(G-v) \sum_{\binom{y \in H}{y\, adj\, x}} C(H-x-y)$$
$$+ C(H-x) \sum_{i=3}^{m} w_i \sum_{v \in C_i \subseteq G} C(G-C_i)$$
$$+ C(G-v) \sum_{j=3}^{n} w_j \sum_{x \in C_j \subseteq H} C(H-C_j),$$

where m and n are the number of nodes in G and H respectively.

By simplifying the RHS, using the relation

$$w_2 \sum_{\binom{u \in G}{u\, adj\, v}} C(G-u-v) + \sum_{i=3}^{m} w_i \sum_{v \in C_i \subseteq G} C(G-C_i) = C(G) - w_1 C(G-v)$$

from Theorem 4, and the analogous relation for H, we obtain the desired result. ∎

The following definition is suggested by an analogous definition given by Schwenk [11].

Let G be a graph containing a node v and H a graph containing a node x, such that

(i) $C(G) = C(H)$ and
(ii) $C(G-v) = C(H-x)$.

Then G and H will be called *a pair of cocircuit rooted graphs* (with *root pair* (v, x)) or *a cocircuit rooted pair*. If $G = H$, we say that G is cocircuit rooted.

Theorem 6. *Any rooted graph may be attached to the roots of a pair of cocircuit rooted graphs to form another pair of cocircuit rooted graphs.*

Proof: Let G and H be a pair of cocircuit rooted graphs rooted at u and x respectively. Let A be a graph rooted at a. By Theorem 5, we get

(i) $C(G_u + A_a) = C(G)C(A-a) + C(A)C(G-u) - w_1 C(G-u)C(A-a)$,
(ii) $C(H_x + A_a) = C(H)C(A-a) + C(A)C(H-x) - w_1 C(H-x)C(A-a)$.

But G and H are cocircuit rooted (with root pair (u, x)). Therefore

$$C(G) = C(H) \quad \text{and} \quad C(G-u) = C(H-x). \qquad (1)$$

Hence

$$C(G_u + A_a) = C(H_x + A_a).$$

Let the cutnodes formed by identifying u and a in $G_u + A_a$ and x and a in $H_x + A_a$ be u_a and x_a respectively. Consider $G_u + A_a$ to be rooted at u_a and $H_x + A_a$ to be rooted at x_a. Then

$$C(G_u + A_a - u_a) = C(G - u)C(A - a)$$

and

$$C(H_x + A_a - x_a) = C(H - x)C(A - a). \qquad \text{(Theorem 3)}$$

Hence from Equation (1), we get

$$C(G_u + A_a - u_a) = C(H_x + A_a - x_a).$$

It follows that $G_u + A_a$ and $H_x + A_a$ are cocircuit rooted. ∎

The following corollary is immediate.

Corollary 6.1. *Any rooted graph may be attached to the roots of a cocircuit rooted graph to form a pair of cocircuit rooted graphs.*

5. Some Deductions For Pseudo-Similar Nodes

Let G be a graph containing two nodes u and v. We say that u and v are *pseudo-similar* if $G - u$ and $G - v$ are isomorphic, but no automorphism of G maps u onto v. Several articles have been written about graphs with pseudo-similar nodes and their properties. For example, see Herndon and Ellzey [5], Kimble, Schwenk and Stockmeyer [6], Kocay [7] and Krisnamoorthy and Parthasarathy [9]. A method for constructing graphs with pairs of pseudo-similar nodes is given in [5]. A technique for constructing infinite graphs with pseudo-similar nodes is given in Godsil and Kocay [3].

Lemma 4. *Let G be a 2-connected graph with a pair of pseudo-similar nodes u and v. Let H be a graph containing a node x. Then*

$$G_u + H_x \ncong G_v + H_x.$$

This result was recently proved by Kocay [8]. Since the condition of 2-connecte(ness is sufficient, it is possible that further classes of graphs with this property will be identified in the near future. In his paper [8], Kocay gave a technique for constructing a graph G with pseudo-similar nodes u and v, for which

$$G_u + H_x \cong G_v + H_x.$$

The following corollary is immediate from Theorem 6.

Corollary 6.2. *Let G be a graph containing a pair of pseudo-similar nodes u and v. Let H be a graph containing a node x. Then*

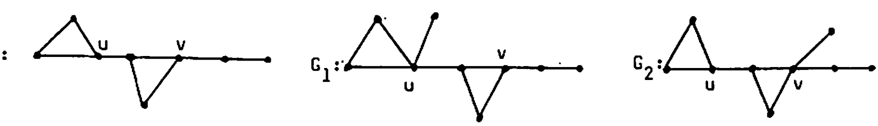
$$C(G_u + H_x; \underline{w}) = C(G_v + H_x; \underline{w}).$$

The following theorem can be obtained by combining the results of Corollaries 1.1 and 6.2. It is a well known result.

Theorem 7. *Let G be a graph containing a pair of pseudo-similar nodes u and v. Let H be a graph containing a node y. Then*

$$\phi(G_u + H_y; x) = \phi(G_v + H_y; x).$$

The smallest non-isomorphic graphs satisfying Corollary 6.2 and Theorem 7 are the graphs G_1 and G_2 shown below in Figures 1(ii) and (iii) respectively. The "parent" smallest graph with a pair of pseudo-similar nodes is shown in Figure 1(i) (This was taken from [3])



(i) (ii) (iii)

Figure 1

It can be easily confirmed that

$$C(G_1; \underline{w}) = C(G_2; \underline{W}) = w_1^9 + 10w_1^7w_2 + 29w_1^5w_2^2 + 25w_1^3w_2^3 + 5w_1w_2^4$$
$$+ 2w_1^6w_3 + 10w_1^4w_2w_3 + 9w_1^2w_2^2w_3 + w_2^3w_3 + w_1^3w_3^2 + w_1w_2w_3^2.$$

We note however that not all pairs of non-isomorphic cocircuit graphs arise in this way. The following graphs H_1 and H_2 are the smallest non-isomorphic cocircuit graphs.

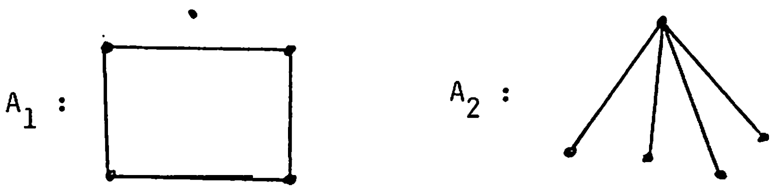
It can be easily verified that

$$C(H_1; \underline{w}) = C(H_2; \underline{w}) = w_1^6 + 4w_1^4w_2 + 3w_1^2w_2^2.$$

(i) (ii)

Figure 2

The converse of Corollary 1.1 is false. This is confirmed by the following cospectral graphs A_1 and A_2 taken from [4] (Figure 3).



(i) (ii)

Figure 3

$$\phi(A_1; x) = \phi(A_2; x) = x^6 - 7x^4 - 4x^3 + 7x^2 + 4x - 1.$$

However, it can be seen that A_1 has a circuit of length 4, while A_2 has no circuits at all. Therefore

$$C(A_1; \underline{w}) \neq C(A_2; \underline{w}).$$

It is clear that the circuit polynomial gives a greater degree of characterization than the characteristic polynomial.

7. Final Remarks

We have shown that there exist non-isomorphic cocircuit graphs, when weights are given to circuits according to the number of nodes that they contain. We also implicitly gave several methods for constructing pairs of non-isomorphic cocircuit graphs, and therefore pairs of cospectral graphs. It would be interesting to find other general techniques for constructing non-isomorphic cocircuit graphs. Any such technique will automatically be a technique for constructing non-isomorphic cospectral graphs.

Acknowledgements

The authors wish to thank Brendan D. McKay and W.L. Kocay for drawing our attention to some of the latest relevant results on pseudo-similar nodes.

References

1. K. Balasubramanian and K.R. Parthasarathy, *In Search of a Complete Invariant for Graphs*, Presented at Symposium in Combinatorics, ISI, Calcutta (1980).
2. E.J.. Farrell, *On a Class of Polynomials Obtained from the Circuits in a Graph and Its Application to Characteristic Polynomials of Graphs*, Discrete Math. **25** (1979), 121–133.
3. C.D. Godsil and W.L. Kocay, *Constructing Graphs with Pairs of Pseudo-Similar Vertices*, J. of Combinatorial Theory (B), to appear.
4. F. Harary, C. King, A. Mowshowitz and R.C. Read, *Cospectral Graphs and Digraphs*, Bull. London Math. Soc. **3** (1971), 321–328.
5. W.C. Herndon and M.L. Ellzey Jr., *Isospectral Graphs and Molecules*, Tetrahedron **31** (1975), 99–107.
6. R.J. Kimble, A.J. Schwenk and P.K. Stockmeyer, *Pseudo-Similar Vertices in a Graph*, J. Graph Theory **5** **No. 2** (1981), 171–181.
7. W.L. Kocay, *On Pseudo-Similar Vertices*, ARS Combinatoria **10** (1980), 147–163.
8. W.L. Kocay, *Attaching Graphs to PseudoSimilar Vertices*, privately communicated (1982).
9. V. Krishnamoorthy and K.R. Parthasarathy, *Cospectral Graphs and Digraphs with Given Automorphism Group*, J. Comb. Theory (B) **19** (1975), 204–273.
10. A.J. Schwenk, *Computing the Characteristic Polynomial of a Graph* in "Graphs and Combinatorics", R. Bari and F. Harary, eds, Springer-Verlag, Berlin, 1974, pp. 153–172.
11. A.J. Schwenk, *Removal-Cospectral Sets of Vertices in a Graph*, Proceedings of the Tenth Southeastern Conference in Combinatorics, Graph Theory and Computing 11, 849–860.