A Brief History of the Embedding of Partial Odd Cycle Systems[1]

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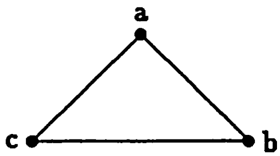
1. Introduction

A Steiner triple system (more simply, triple system) is a pair (S,T), where T is a collection of edge disjoint triangles (or triples) which partition the edge set of the complete undirected graph K_n with vertex set S. The *number* $n = |S|$ is called the *order* of the triple system (S,T) and it has been known forever (= since 1847 [4]) that the spectrum of triple systems (= the set of all n such that a triple system of order n exists) is precisely the set of all $n \equiv 1$ or $3 \pmod 6$. It is trivial to see that if (S,T) is a triple system of order n then $|T| = n(n-1)/6$.

A *partial triple system* of order n is a pair (S,P), where P is a collection of edge disjoint triangles of the edge set of K_n with vertex set S. The difference between a (complete) triple system and a partial triple system is that the edge disjoint triangles belonging to a partial triple system do not necessarily include all of the edges of K_n.

In what follows we will denote the triangle



by $\{a,b,c\}$.

Example 1.1. (S,P) *is a partial triple system of order 6, where* $S = \{1,2,3,4,5,6\}$ *and* $P = \{\{1,2,4\},\{1,5,6\},\{2,3,5\},\{3,4,6\}\}$.

Now given a partial triple system (S,P) we can ask whether or not it is possible to decompose $E(K_n)\backslash E(P)$ into edge disjoint triangles. The above example shows that this cannot be done in general, since the deficiency graph $D = E(K_6)\backslash E(P) = \{\{1,3\},\{2,6\},\{4,5\}\}$, is triangle free, not to mention the fact that 6 is *not* the order of a triple system! Since a partial triple system cannot necessarily be *completed* to a triple system, the problem of embedding a partial triple system in a (complete) triple system is immediate. The partial triple system (S,P) is said to be *embedded* in the triple system (S^*,T) provided that $S \subseteq S^*$ and $P \subseteq T$. Naturally, we would like $|S^*|$ to be as small as possible.

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Example 1.2. *Let* (S^*, T) *be the triple system of order 7 defined by* $S = \{1,2,3,4,5,6,7\}$ *and* $T = \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{1,5,6\},\{2,6,7\},\{1,3,7\}\}$. *Then the partial triple system* (S, P) *in Example 1.1 is embedded in* (S^*, T).

In 1971 Christine Treash [11] gave the first embedding of partial triple systems. Treash's embedding gives an extremely large containing system, guaranteeing only that a partial triple system of order n can be embedded in a triple system of order $< 2^{2n}$. Subsequently, this bound was improved in 1975 by C. C. Lindner to $6n+3$ [5] and finally in 1980 (the best so far) to the smallest admissible order $\geq 4n+1$ by L. D. Andersen, A. J. W. Hilton, and E. Mendelsohn [1]. The best possible bound is the smallest admissible order $\geq 2n+1$. An extremely difficult problem!

Now a triangle is also a 3-cycle and so a Steiner triple system (S, T) can be described as an edge disjoint collection of 3-cycles which partition the edge set of K_n (based on S). Since there is nothing particularly sacred about the number 3, we can certainly ask the same questions for m-cycle systems that are asked for triple systems. In particular, for a given $m \geq 4$, we can ask for the spectrum of m-cycle systems as well as for an embedding (as small as possible) of partial m-cycle systems. An obvious definition here: an m-cycle system of order n is a pair (S, C), where C is an edge disjoint collection of m-cycles which partition the edge set of K_n based on S. If the edge disjoint m-cycles belonging to C do not necessarily partition the entire edge set of K_n then we have the definition of a *partial* m-cycle system.

The obvious necessary conditions for the existence of an m-cycle system (S, C) of order $|S| = n$ are

$$\begin{cases} (1) & n \geq m, \text{if } n > 1, \\ (2) & n \text{ is odd, and} \\ (3) & n(n-1)\backslash 2m \text{ is an integer.} \end{cases}$$

Although these necessary conditions are sufficient for all $m \leq 50$ [2] the existence problem is far from settled. However, in contrast to block designs, it may well be the case that these obvious necessary conditions are also sufficient.

In [8, 9] it is shown that a partial m-cycle system of order n can be embedded in an m-cycle system of order $2nm + 1$ when m is EVEN and embedded in an m-cycle system of order $m((m-2)n(n-1) + 2n+1)$ when m is ODD. Quite a disparity!

Recently, C. C. Lindner and C. A. Rodger [6] removed this disparity by reducing the bound for m = ODD to $m(2n+1)$. The principal ingredient in the construction used to obtain this bound is a generalization of Allan Cruse's Theorem [3] on embedding partial idempotent commutative quasigroups to embedding partial idempotent commutative groupoids.

What follows is an elementary account of the struggle to obtain this small embedding for partial odd-cycle systems. Rather than obscure the essence of the constructions with Professor Backwards type details, the author has chosen to illustrate everything using triple systems and pentagon (= 5-cycle) systems. Pentagon systems are just large enough to illustrate the fact that the techniques used for triple systems do not necessarily work for larger odd-cycle systems, but small enough to illustrate the general constructions without cluttering things up with mind boggling details. The object of this paper is an attempt to popularize embedding theorems for partial cycle systems. One way NOT to do this is to destroy the reader's will to resist with excruciating details (in full generality). The interested reader who wishes to pursue the subject can, of course, go straight to the original papers for a large dose of tedium. So be it!

2. Embedding triple systems.

As mentioned in the introduction the serious history of embedding partial odd-cycle sytems began in 1971 with Christine Treash's result that a partial triple system of order n can always be embedded in a triple system of order $< 2^{2^n}$. In 1975 this was dramatically improved to $6n+3$ by C. C. Lindner and subsequently to the smallest admissible order $\geq 4n+1$ by L. D. Andersen, A. J. W. Hilton, and E. Mendelsohn. Although this is the best result to date, we will content ourselves here with a description of the $6n+3$ embedding. The principal reason being that the general embedding result for pentagon systems (as well as odd-cycle systems in general) is a *generalization* of the $6n+3$ embedding and *not* the $\geq 4n+1$ embedding. When you come to think about it, that's a pretty good reason!

The $6n+3$ embedding is based on the following (by now well-known) construction for Steiner triple systems.

The $6n+3$ Construction. Let $(Q, \circ)$ be an idempotent ($x^2 = x$) commutative ($xy = yx$) quasigroup of order $2n+1$, set $S = Q \times \{1,2,3\}$, and define a collection of triples T as follows:

(1) $\{(x,1),(x,2),(x,3)\} \in T$, for all $x \in Q$, and

(2) if $x \neq y$, $\{(x,1),(y,1),(x\circ y,2)\}$, $\{(x,2),(y,2),(x\circ y,3)\}$, and $\{(x,3),(y,3),(x\circ y,1)\}$ belong to T.

It is a trivial matter to see that, in fact, (S,T) is a triple system of order $6n+3$.

The $6n+3$ embedding for partial triple systems is a modification of the $6n+3$ Construction and is based on a remarkable result due to Allan Cruse on embedding partial idempotent commutative quasigroups. A few preliminaries are in order.

A *partial* idempotent quasigroup is a partial quasigroup $(P, \circ)$ with the additional requirement that $x \circ x$ is *defined* for every $x \in P$ and $x \circ x = x$. In other words, the word "partial" quantifies products of the form $x \circ y$ where $x \neq y$. A partial idempotent commutative quasigroup is a partial idempotent quasigroup $(P, \circ)$ with the additional requirement that if $x \circ y$ is defined then so is $y \circ x$ and furthermore $x \circ y = y \circ x$.

Example 2.1. *Partial* $x^2 = x, xy = yx$ *quasigroup of order 4.*

$\circ$	1	2	3	4
1	1	4		2
2	4	2		1
3			3	
4	2	1		4

Now we can ask the same questions for partial $x^2 = x, xy = yx$ quasigroups that were asked for partial triple systems. Namely, can a partial $x^2 = x, xy = yx$ quasigroup be "completed" to an $x^2 = x, xy = yx$ quasigroup? And if not, can it be embedded in an $x^2 = x, xy = yx$ quasigroup? Example 2.1 shows that the answer to the first question is NO, since we cannot define $3 \circ 4 = 4 \circ 3$ without violating the cancellation law. Since a partial $x^2 = x, xy = yx$ quasigroup cannot necessarily be completed to an $x^2 = x, xy = yx$ quasigroup the problem of embedding becomes paramount.

Example 2.2. *The partial* $x^2 = x, xy = yx$ *quasigroup* $(P, \circ)$ *of order 4 in Example 2.1 is embedded in the* $x^2 = x, xy = yx$ *quasigroup* $(Q, \circ)$ *of order 9.*

$\circ$	1	2	3	4
1	1	4		2
2	4	2		1
3			3	
4	2	1		4

$(P, \circ)$

$\circ$	1	2	3	4	5	6	7	8	9
1	1	4	7	2	9	8	3	6	5
2	4	2	9	1	8	7	6	5	3
3	7	9	3	8	6	5	1	4	2
4	2	1	8	4	7	9	5	3	6
5	9	8	6	7	5	3	4	2	1
6	8	7	5	9	3	6	2	1	4
7	3	6	1	5	4	2	7	9	8
8	6	5	4	3	2	1	9	8	7
9	5	3	2	6	1	4	8	7	9

$(Q, \circ)$

In 1974 Allan Cruse obtained the best possible bound for embedding partial $x^2 = x, xy = yx$ quasigroups.

Theorem 2.3 (Allan Cruse [3]). *A partial $x^2 = x, xy = yx$ quasigroup of order n can be embedded in an $x^2 = x, xy = yx$ quasigroup of order t for every ODD $t \geq 2n+1$.* ∎

Cruse's Theorem is the best possible result in that it is always possible to construct a partial $x^2 = x, xy = yx$ quasigroup of order n which *cannot* be embedded in an $x^2 = x, xy = yx$ quasigroup of order $< 2n+1$, for every $n \geq 4$.

The $6n+3$ Embedding. Let (X, P) be a partial triple system of order n and define a *partial* $x^2 = x, xy = yx$ groupoid $(X, \circ)$ as follows: (i) $x \circ x = x$ for all $x \in X$, and (ii) if $x \neq y$, $x \circ y$ and $y \circ x$ are defined and $x \circ y = y \circ x = z$ if and only if $\{x, y, z\} \in P$.

Example 2.4. *Let (X^*, P^*) be the partial triple system defined by $X^* = \{1,2,3,4,5,6\}$ and $P^* = \{\{1,2,4\},\{1,5,6\},\{2,3,5\},\{3,4,6\}\}$ (Example 1.1). Then $(X^*, \circ)$ is given by the accompanying table.*

$\circ$	1	2	3	4	5	6
1	1	4		2	6	5
2	4	2	5	1	3	
3		5	3	6	2	4
4	2	1	6	4		3
5	6	3	2		5	1
6	5		4	3	1	6

Inspection shows that the partial groupoid in Example 2.4 is, in fact, a partial quasigroup. It is a trivial matter to see that this is ALWAYS the case. So the partial groupoid $(X, \circ)$ defined from the partial triple system (X, P) is a partial $x^2 = x$, $xy = yx$ quasigroup. Hence by Allan Cruse's Theorem we can embed $(X, \circ)$ in an $x^2 = x, xy = yx$ quasigroup $(Q, \circ)$ of order $2n+1$. Let $S = Q \times \{1,2,3\}$ and define a collection of triples T as follows:

(1) $\{(x,1),(x,2),(x,3)\} \in T$, for all $x \in Q$,

(2) if $\{x,y,z\} \in P$ take *exactly one* of (x,y,z), (x,z,y), (y,x,z), (y,z,x), (z,x,y), or (z,y,x), say (x,y,z), and define a collection of 9 triples as follows: Let $(\{1,2,3\}, \otimes)$ be the idempotent quasigroup given by

$\otimes$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

and for each ordered pair $(i,j) \in \{1,2,3\}$ (i and j not necessarily distinct) place the triple $\{(x,i),(y,j),(z,i \otimes j)\}$ in T, and

(3) if $x \neq y$ and $\{x, y\}$ doesn't belong to a triple of P, place the three triples $\{(x,1),(y,1),(x\circ y,2)\}$, $\{(x,2),(y,2),(x\circ y,3)\}$, and $\{(x,3),(y,3),(x\circ y,1)\}$ in T.

As with the $6n+3$ Construction it is easy to see that (S,T) is a triple system of order $6n+3$.

Since the quasigroup $(\{1,2,3\},\otimes)$ is idempotent, if $\{x,y,z\} \in P$, then $\{(x,1),(y,1),(z,1\otimes 1 = 1)\}$, $\{(x,2),(y,2),(z,2\otimes 2 = 2)\}$, and $\{(x,3),(y,3),(z,3\otimes 3 = 3)\} \in T$ and so three disjoint copies of P belong to T.

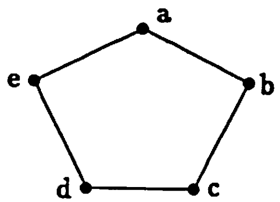
We have the following theorem.

Theorem 2.5. **(C. C. Lindner [5]).** *A partial triple system of order n can always be embedded in a triple sytem of order* $6n+3$. ∎

Remark 2.6. *In view of Cruse's Theorem we can embed a partial triple sytem of order n in a triple system of order $3t$ for every odd $t \geq 2n+1$. However, we are interested here in a small embedding and not a general result, so $6n+3$ will do quite nicely for our purposes.*

3. Pentagon (= 5-cycle) systems.

In what follows we will denote the pentagon



by any cyclic shift of (a,b,c,d,e) or (a,e,d,c,b).

Example 3.1.

(1) $(S_1,P_1), P_1 = \{(1,2,3,4,5),(1,3,5,2,4)\}$,

(2) $(S_2,P_2), P_2 = \{(1,3,9,5,4),(2,4,10,6,5),(3,5,11,7,6),(4,6,1,8,7),(5,7,2,9,8),(6,8,3,10,9),(7,9,4,11,10),(8,10,5,1,11),(9,11,6,2,1),(10,1,7,3,2),(11,2,8,4,3)\}$, *and*

(3) $(S_3,P_3), P_3 = \{(1,3,10,5,4),(2,4,11,6,5),(3,5,1,7,6),(4,6,2,8,7),(5,7,3,9,8),(6,8,4,10,9),(7,9,5,11,10),(8,10,6,1,11),(9,11,7,2,1),(10,1,8,3,2),(11,2,9,4,3)\}$.

The two pentagon systems of order 11 will be used for illustrative purposes later.

In [10] Alex Rosa showed that the spectrum for pentagon systems is precisely the set of all $n \equiv 1$ or $5 \pmod{10}$. Of course if (S,P) is a pentagon system of order n, $|P| = n(n-1)/10$.

The obvious thing to do here is to try to generalize the $6n+3$ embedding of triple systems to pentagon systems. A bit of reflection shows that the place to start is with a $10n+5$ Construction.

The $10n+5$ Construction. Let $(Q,\circ)$ be a $x^2 = x$, $xy = yx$ quasigroup of order $2n+1$, set $S = Q \times \{1,2,3,4,5\}$, and define a collection of pentagons (= 5-cycles) C as follows:

(1) $((x,1),(x,2),(x,3),(x,4),(x,5))$ and $((x,1),(x,3),(x,5),(x,2),(x,4)) \in C$ for all $x \in Q$ (in other words, place a copy of the pentagon system of order 5 (Example 3.1) on $\{x\} \times \{1,2,3,4,5\}$ for each $x \in Q$), and

(2) if $x \neq y$, $((x,i),(y,i),(x,j),(x \circ y,k),(y,j)) \in C$, for all $(i,i,j,k,j) \in I = \{(1,1,2,4,2),(2,2,3,5,3),(3,3,4,1,4),(4,4,5,2,5),(5,5,1,3,1)\}$.

It is straightforward to see that (S,C) is a pentagon system of order $10n+5$. ∎

So far so good! But now the trouble begins. Given a (partial) pentagon system (S,C) there are *two reasonable* ways to define a binary operation from the pentagons belonging to C.

(1) $a \circ a = a$, for all $a \in Q$, and if $a \neq b$, $a \circ b = c$ and $b \circ a = e$ if and only if $(a,b,c,d,e) \in C$, OR

(2) $a \circ a = a$, for all $a \in Q$, and if $a \neq b$, $a \circ b = b \circ a = d$ if and only if $(a,b,c,d,e) \in C$.

Example 3.2. *Let (X,P) be the partial pentagon system given by $X = \{1,2,3,4,5,6,7,8\}$ and $P = \{(1,2,3,4,5),(2,6,7,4,8)\}$. Let $\circ_1$ be the binary operation defined by (1) and $\circ_2$ the binary opeartion defined by (2).*

$\circ_1$	1	2	3	4	5	6	7	8
1	1	3			4			
2	5	2	4			7		4
3		1	3	5				
4			2	4	1		6	2
5	2			3	5			
6		8				6	4	
7				8		2	7	
8		6		7				8

$\circ_2$	1	2	3	4	5	6	7	8
1	1	4			3			
2	4	2	5			4		7
3		5	3	1				
4			1	4	2		2	6
5	3			2	5			
6		4				6	8	
7				2		8	7	
8		7		6				8

The above example shows that, unlike the case for (partial) triple systems, it is not always possible to define a (partial) quasigroup from a (partial) pentagon system. So a straightforward extrapolation of the $6n+3$ embedding for triple systems applied to the $10n+5$ Construction for pentagon systems won't work.

The reason for the trouble in Example 3.2 is because the vertices 2 and 4 are joined by a path of length 2 in two different pentagons of P. This forces such things as $2 \circ_1 3 = 2 \circ_1 8 = 4$ and $4 \circ_2 5 = 4 \circ_2 7 = 2$, which guarantees that neither operation gives a quasigroup. A bit of reflection reveals, however, that if (S, C) is a (partial) pentagon system with the additional property that each pair of vertices are joined by (at most) *exactly* one path of length two belonging to a pentagon of C then both $\circ_1$ and $\circ_2$ produce quasigroups. Such a (partial) pentagon system is said to be 2-perfect. For example, in Example 3.1 (S_1, P_1) and (S_2, P_2) are 2-perfect, whereas (S_3, P_3) is NOT, since the vertices 1 and 10 are joined by a path of length 2 in the pentagons (1,3,10,5,4) and (8,10,6,1,11) belonging to P_3.

The march toward a small embedding for partial pentagon systems began in 1974 when R. M. Wilson [12] proved that all partial m-cycle systems, and not just pentagon systems, can be finitely embedded. However the order of the containing system is an exponential function of the order of the partial system. Since Wilson's result, the sequence of events is the following: (1) A partial 2-perfect pentagon system of order n can be embedded in a *not necessarily 2-perfect* pentagon system of order $10n+5$, (2) A partial pentagon system of order n can be embedded in a pentagon sytem of order $\leq 15n^2 - 5n + 1$, and finally (3) a partial pentagon system of order n can be embedded in a pentagon system of order $10n+5$ (the best result to date).

We will present each of these results in sequence. (1) and (2) are closely related and so will be presented together.

4. Embedding partial 2-perfect pentagon systems.

We begin this section with the following embedding for partial 2-perfect pentagon systems.

The 2-perfect $10n+5$ Embedding. Let (X, P) be a partial 2-perfect pentagon system of order n and define a partial $x^2 = x$, $xy = yx$ quasigroup $(X, \circ)$ as follows: (i) $x \circ x = x$ for all $x \in X$, and (ii) if $x \neq y$, $x \circ y$ and $y \circ x$ are defined and $x \circ y = y \circ x = z$ if and only if $(x, y, u, z, w) \in P$. By Allan Cruse's Theorem we can embed $(X, \circ)$ in an $x^2 = x$, $xy = yx$ quasigroup $(Q, \circ)$ of order $2n+1$. Let $S = Q \times \{1,2,3,4,5\}$ and define a collection of pentagons C as follows:

(1) $((x,1),(x,2),(x,3),(x,4),(x,5))$ and $((x,1),(x,3),(x,5),(x,2),(x,4)) \in C$ for all $x \in Q$,

(2) for each pentagon $c \in P$ take a fixed representation $(x_1, x_2, x_3, x_4, x_5)$ of c and define a collection of 25 pentagons as follows: Let $(\{1,2,3,4,5\}, \otimes)$ be the idempotent quasigroup given by

⊗	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

(any idempotent quasigroup will do) and for each ordered pair $(i,j) \in \{1,2,3,4,5\}$ (i and j not necessarily distinct) place the pentagon $((x_1,i), (x_2,j), (x_3, i\otimes j), (x_4,j), (x_5, i\otimes j))$ in C, and

(3) if $x \neq y$ and $\{x,y\}$ doesn't belong to a pentagon of P, place the 5 pentagons $(x,i), (y,i), (x,j), (x\circ y, k), (y,j))$ in C, for all $(i,i,j,k,j) \in I = \{(1,1,2,4,2),(2,2,3,5,3),(3,3,4,1,4),(4,4,5,2,5), (5,5,1,3,1)\}$.

It is straightforward to see that (S, C) is a *not necessarily 2-perfect* pentagon system of order $10n+5$ and since the quasigroup $(\{1,2,3,4,5\}, \otimes)$ is idempotent, five disjoint copies of P are embedded in C. ∎

Theorem 4.1 (C.C. Lindner, C.A. Rodger, D.R. Stinson [8]). *A partial 2-perfect pentagon system of order n can be embedded in a not necessarily 2-perfect pentagon system of order $10n+5$.*

Of course, if the partial pentagon system is not 2-perfect we're out of luck. However, the following trick will help us remove the 2-perfect requirement if we don't mind enlarging the size of the containing system a bit.

Two partial pentagon systems (X, P_1) and (X, P_2) are said to be *mutually balanced* provided the pentagons in P_1 and P_2 contain precisely the same edges.

Example 4.2. *Let* $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$, $P_1 = \{(1,2,3,4,5),(7,8,2,9,10),(10,11,3,12,13),(13,14,4,15,16),(16,17,5,18,19),(19,20,1,6,7)\}$, *and* $P_2 = \{(1,2,8,7,6),$

$(2,3,11,10,9)$, $(3,4,14,13,12)$, $(4,5,17,16,15)$, $(5,1,20,19,18)$, $(7,10,13,16,19)\}$. *Then* (X,P_1) *and* (X,P_2) *are mutually balanced. For example,* $\{7,8\} \in (7,8,2,9,10) \in P_1$ *while* $\{7,8\} \in (1,2,8,7,6) \in P_2$.

Pentagon systems have the so-called "replacement property".

The replacement property. Let (S,C) be a pentagon system and $\{P_1, P_2, P_3, \ldots, P_k\}$ a collection of pairwise edge disjoint partial pentagon systems such that $P_i \subseteq C$. If $\{P_i^*, P_2^*, \ldots, P_k^*\}$ is any co]lection of partial pentagon systems such that P_i and P_i^* are mutually balanced, then $(S,(C\backslash(\cup_{i=1}^k P_i)) \cup (\cup_{i=1}^k P_i^*))$ is a pentagon system.

The $15n^2 - 5n + 1$ Embedding. Let (X,P) be a partial pentagon sytem of order n. For each pentagon $p = (a,b,c,d,e) \in P$ let $X(p)$ be a set of size 15 such that $X \cap X(p) = \emptyset$ and $X(p_1) \cap X(p_2) = \emptyset$ for $p_1 \neq p_2 \in P$. For each $p = (a,b,c,d,e)$ let $P_1(p)$ and $P_2(p)$ be the pair of mutually balanced partial pentagon systems of order 20 based on $\{a,b,c,d,e\} \cup X(p)$ given by Example 4.2 with $p = (a,b,c,d,e) \in P_1(p)$. Let $S = X \cup (\cup_{p\in P} X(p))$ and $P(p) = \cup_{p\in P} P_2(p)$. Then $(S,P(p))$ is not only a partial pentagon system but a partial 2-perfect pentagon system as well. Hence by Theorem 4.1, $(S,P(p))$ can be embedded in a pentagon system of order $10|S| + 5$, say (S^*,C). By the replacement property $(S^*,(C\backslash(\cup_{p\in P} P_2(p))) \cup (\cup_{p\in P} P_1(p)))$ is a pentagon system, and of course contains a copy of (X,P). In fact, *five disjoint copies*, since Theorem 4.1 guarantees five disjoint copies of $(S,P(p))$ are contained in (S^*,C). Since (X,P) is a partial pentagon system of order n, $|P| \leq n(n-1)/10$, $|S| \leq n + 3n(n-1)/2$, and finally $|S^*| \leq 15n^2 - 5n + 5$.

Corollary 4.3. (C.C. Lindner, C.A. Rodger, D.R. Stinson [8]). *A partial pentagon system of order n can be embedded in a pentagon system of order* $\leq 15n^2 - 5n + 5$. ∎

While the bound of $\leq 15n^2 - 5n + 5$ is not particularly small, it is certainly a lot better than exponential! Finally, after a lot of struggling, the problem of twisting a partial pentagon system into a partial 2-perfect pentagon system (and thereby enlarging the size) was overcome. The key was a generalization of Cruse's Theorem. We now address this generalization and the accompanying embedding in the next section.

5. Embedding partial pentagon systems.

A partial groupoid $(X,\circ)$ with the following properties will be called a partial *embedding groupoid*:

(1) $x \circ x = x$, for all $x \in X$ (idempotent),
(2) $x \circ y$ is defined if and only if $y \circ x$ is defined (but $x \circ y$ and $y \circ x$ are *not neccessarily equal*),
(3) $(X,\circ)$ is ROW latin $= (a \circ x = a \circ y$ implies $x = y)$, and

(4) each $x \in X$ occurs as a product an ODD number of times.

Example 5.1. *A partial embedding groupoid of order* 8.

∘	1	2	3	4	5	6	7	8
1	1	2			5			
2	1	2	3			6		8
3		2	3	4				
4			3	4	5		7	8
5	1			4	5			
6		2				6	7	
7				4		6	7	
8		2		4				8

The following generalization of Cruse's Theorem is proved in [6].

Theorem 5.2. (C.C. Lindner and C.A. Rodger [6]). *A partial embedding groupoid of order n can allays be embedded in an idempotent groupoid of order $2n+1$ which is row latin and such that the partial groupoid consisting of the main diagonal plus all products not defined by the embedding groupoid is a partial idempotent commutative quasigroup (of order $2n+1$).* ∎

Example 5.3. *The groupoid of order 17 given below contains the partial embedding groupoid of order 8 in Example 5.1 and has all of the properties stated in Theorem 5.2.*

$\circ_1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	2	15	10	5	16	11	14	3	4	17	7	8	9	13	12	6
2	1	2	3	9	16	6	14	18	12	5	7	10	17	11	4	15	13
3	15	2	3	4	1	13	9	12	5	6	1	8	16	10	7	17	11
4	10	9	3	4	5	17	7	8	6	11	2	1	12	13	16	14	15
5	1	16	14	4	5	11	12	10	7	17	15	2	6	8	3	13	9
6	16	2	13	17	11	6	7	9	8	14	4	15	10	3	1	4	12
7	11	14	9	4	12	6	7	17	15	13	5	16	3	1	10	2	8
8	14	2	12	4	10	9	17	8	1	15	13	11	5	16	6	7	3
9	3	12	5	6	7	8	15	1	9	16	10	13	2	17	14	11	4
10	4	5	6	11	17	14	13	15	16	10	13	3	9	7	2	8	1
11	17	7	1	2	15	4	5	13	10	12	11	9	14	6	8	3	16
12	7	10	8	1	2	15	16	11	13	3	9	12	4	5	17	6	4
13	8	17	16	21	6	10	3	5	2	9	14	4	13	15	11	1	7
14	9	11	10	13	8	3	1	16	17	7	6	5	15	14	12	4	2
15	13	4	7	16	3	1	10	6	14	2	8	17	11	12	15	9	5
16	12	15	17	14	13	5	2	7	11	8	3	6	1	4	9	16	10
17	6	13	11	15	9	12	8	3	4	1	16	4	7	2	5	10	17

Theorem 5.2 is exactly what is needed to amend the $10n+5$ Construction to obtain a $10n+5$ embedding a la the $6n+3$ embedding for triple systems. Here goes!

The $10n+5$ Embedding. Let (X, P) be a partial pentagon system of order n and define a partial groupoid $(X, \circ)$ as follows: (i) $x \circ x = x$ for all $x \in X$, and (ii) if $x \neq y$, $x \circ y = y$ and $y \circ x = x$ if and only if $\{x, y\}$ belongs to a pentagon of P. It is straightforward to see that $(X, \circ)$ is a partial embedding groupoid. (For example, if $X = \{1,2,3,4,5,6,7,8\}$, and $P = \{(1,2,3,4,5),(2,6,7,4,8)\}$, then $(X, \circ)$ is the partial embedding groupoid given in Example 5.1.) By Theorem 5.2 we can embed $(X, \circ)$ in a groupoid $(Q, \circ)$ of order $2n+1$ having the properties guaranteed by Theorem 5.2. Let $S = Q \times \{1,2,3,4,5\}$ and define a collection of pentagons C EXACTLY as in the 2-perfect $10n+5$ Embedding. It is IMPORTANT to note here that if $\{x, y\}$ doesn't belong to a pentagon of P, then $x \circ y = y \circ x$ is computed in a partial idempotent commutative quasigroup. As with the 2-perfect embedding, C contains 5 disjoint copies of P. It remains to show that (S, C) is a pentagon system.

Claim: (S, C) is a pentagon system. The proof consists in showing that $|C| \leq (10n+5)(10n+4)/10$ and that each edge $\{(x,i),(y,j)\}$ belongs to at least one pentagon of the type described in the construction. A simple counting argument shows that the number of type (1) pentagons = $2(2n+1)$, type (2) pentagons = $25|P|$, and type (3) pentagons = $5\left(\binom{2n+1}{2} - 5|P|\right)$. The sum of these numbers is $(10n+5)(10n+4)/10 \geq |C|$ taking care of the first part of the proof. Now let $\{(x,i),(y,j)\}$ be any edge. There are several cases to consider.

(i) $x = y$. Then $\{(x,i),(x,j)\} \in \{x\} \times \{1,2,3,4,5\}$ and so belongs to a type (1) pentagon.

(ii) $x \neq y$ and $i = j$. If $\{x, y\}$ belongs to a pentagon of P, say (x, y, a, b, c), then $((x,i),(y,i),(a,i),(b,i),(c,i)) \in C$ (since $(\{1,2,3,4,5\}, \otimes)$ is idempotent). If $\{x, y\}$ does not belong to a pentagon of P, then $x \circ y = y \circ x$ is computed in a partial idempotent commutative quasigroup and so $((x,i),(y,i),(x,j),(x \circ y,k),(y,j)) \in C$, where $(i,i,j,k,j) \in I$.

(iii) $x \neq y$ and $i \neq j$ and $\{x, y\}$ belongs to a pentagon of P. There are essentialy five different ways the edge $\{x, y\}$ can'sit inside of the representation of this pentagon: (x,y,a,b,c), (c,x,y,a,b), (b,c,x,y,a), (a,b,c,x,y), or (y,a,b,c,x).

  (i) If (x,y,a,b,c), then $((x,i),(y,j),(a,i\otimes j),(b,j),(c,i\otimes j)) \in C$.

  (ii) If (c,x,y,a,b), let $k \otimes i = j$. Then $((c,k),(x,i),(y,j),(a,i),(b,j)) \in C$.

  (iii) If (b,c,x,y,a), let $k \otimes j = i$. Then $((b,k),(c,j),(x,i),(y,j),(a,i)) \in C$.

  (iv) If (a,b,c,x,y), let $k \otimes i = j$. Then $((a,k),(b,i),(c,j),(x,i),(y,j)) \in C$.

(v) If (y,a,b,c,x), let $j \otimes k = i$. Then $((y,j),(a,k),(b,i),(c,k),(x,i)) \in C$.

(iv) $x \neq y$ and $i \neq j$ and $\{x,y\}$ does NOT belong to a pentagon of P. It is *extremely important* to note here that the groupoid $(Q,\circ)$ has the property that if we delete all products corresponding to edges belonging to a pentagon of P, the result is a *partial idempotent* commutative QUASIGROUP. Having said this, one of two things is true: $|i-j| = 1$ or $|i-j| = 2$. We will handle each case in turn.

$|i-j| = 1$ There is no loss in generality in assumming $i = 1$ and $j = 2$. Then $((x,1),(y,1),(x,2),(x \circ y,4),(y,2)) \in C$, where $x \circ y = y \circ x$ is computed in the partial $x^2 = x, xy = yx$ quasigroup part of $(Q,\circ)$.

$|i-j| = 2$ Again we can take $i = 2$ and $j = 4$. Since $(Q,\circ)$ is row latin $x \circ z = y$ for some $z \in Q$. Now $z \neq x$, since $(Q,\circ)$ is idempotent. Further $\{x,z\}$ does NOT belong to a pentagon of P, since $x \circ z = z = y$ in the embedding groupoid $(X,\circ)$ implies that $\{x,y\}$ belongs to a pentagon of P. Hence $x \circ z = z \circ x$ is computed in the partial $x^2 = x, xy = yx$ quasigroup part of $(Q,\circ)$ and so by construction $((x,1),(z,1),(x,2),(x \circ z = y,4),(z,2)) \in C$.

Combining all of the above cases shows that (S,C) is a pentagon system. Part (ii) shows that 5 disjoint copies of (X,P) are embedded in (S,C).

Theorm 5.4. (C.C. Lindner and C.A. Rodger [6]). *A partial pentagon system of order n can be embedded in a pentagon system of order $10n+5$.* ∎

6. Concluding remarks.

An obvious generalization of the $10n+5$ Embedding to partial odd-cycle systems in general gives the following theorem.

Theorem 6.1 (C.C. Lindner and C.A. Rodger [6]). *If m is ODD, a partial m-cycle system of order n can be embedded in an m-cycle system of order $m(2n+1)$.* ∎

As mentioned in the introduction, we chose to illustrate the history and travails of obtaining this general result with partial triple and pentagon systems. And why not? This is a survey paper and the proof of Theorem 6.1 involves technicalities which obscure the essence of the construction. An understanding of the pentagon embedding will allow the interested reader to breeze through the general result.

It is worth mentioning that Theorem 6.1 gives a *unified treatment* of the embedding problem for partial odd-cycle systems, in that it does not distinguish between 2-perfect and non 2-perfect partial m-cycle systems. So for example, Theorem 6.1 gives the same result for triple systems (which are always 2-perfect) as Theorem 2.5.

Finally, we would be remiss if we didn't say something about embedding partial even-cycle systems. All we can say here is that the technique for such embeddings are completely different from the embedding techniques for odd-cycle systems. Since there is a limit to the length of this paper, the even-cycle case is best left for another day. The interested reader is referred to the survey paper [7] by the author and C.A. Rodger for further reading on this subject.

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