Toughness of Graphs and $[a, b]$-Factors with Prescribed Properties

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Abstract. Chvátal conjectured that if G is a k-tough graph and $k|V(G)|$ is even, then G has a k-factor. In [5] it was proved that Chvátal's conjecture is true. Katerinis [2] presented a toughness condition for a graph to have an $[a, b]$-factor. In this paper we prove a stronger result: every $(a - 1 + a/b)$-tough graph satisfying trivial necessary conditions has an $[a, b]$-factor containing any given edge and another $[a, b]$-factor excluding it. We also discuss some special cases of the above result.

1. Introduction

By a graph we mean a finite connected graph which has no multiple edges or loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a subset S of $V(G)$, we write $G[S]$ for the induced subgraph of G by S, $G-S = G[V(G)\setminus S]$ and $N_G(S) = \{v: uv \in E(G) \text{ and } u \in S\}$. For a vertex x of G, the degree of x in G is denoted by $d_G(x)$. The minimum degree of vertices of G is denoted by $\delta(G)$. Let a and b be integers such that $0 \leq a \leq b$. An $[a, b]$-*factor* of G is a spanning subgraph H of G satisfying $a \leq d_H(x) \leq b$ for every vertex $x \in V(G)$. A $[k, k]$-factor is called a k-*factor*.

A subset S of $V(G)$ is an *independent set* of G if no two elements of S are adjacent in G and a subset C of $V(G)$ is a *covering set* if every edge of G has at least one end-vertex in C. The number of connected components of G is denoted by $w(G)$. Let G be a non-complete graph and let t be a real number. If for every subset S of $V(G)$ with $w(G - S) > 1$, $tw(G - S) \leq |S|$, then we say that G is t-*tough*. The largest t such that G is t-tough is called the *toughness* of G and is denoted by $t(G)$. If $G = K_n$ is a complete graph with n vertices, we define $t(G) = n-2$, and G is said to be t-tough if and only if $n-2 \geq t$. Other graphical terminology in this paper can be found in [7].

In 1973 Chvátal [4] conjectured that if G is a graph and k is a positive integer such that $t(G) \geq k$ and $k|V(G)|$ is even, then G has a k-factor.

This conjecture was proved by Enomoto et al. [5]. Liu [8] proved that if G is a k-tough graph and $k|VG)|$ is even, then G has a k-factor that contains any given edge of G. Moreover, Chen [1] obtained the following result.

Theorem 1.1 [1]. *Let G be a graph and integer $k \geq 2$ such that $t(G) \geq k$ and $k|V(G)|$ is even. Then for every edge of G there is a k-factor containing it and another k-factor excluding it.*

Recently Katerinis showed the following theorem.

Theorem 1.2. [7]. *Let G be a graph and a, b be two positive integers such that $b \geq a$. If $t(G) \geq a - 1 + a/b$ and $a|V(G)|$ is even when $a = b$, then G has an $[a, b]$-factor.*

In addition to Theorem 1.1 and Theorem 1.2 we have the following result.

Theorem 1.3. [[2],[8], [11]]. *Let G be a graph of even order and $t(G) > 1$. Then G has a 1-factor containing any given edge.*

Chen [2] discussed the binding number and toughness conditions for a graph to have a $[1, b]$-factor and a $[2, b]$-factor which contains a give edge, respectively.

The main purpose of this paper is to present some toughness conditions for a graph to have an $[a, b]$-factor containing any given edge and another $[a, b]$-factor excluding it, extending and improving the above theorems.

2. Preliminary results

In order to prove the main result we shall need some lemmas. In [[3], [9]] Chen and Liu gave a necessary and sufficient condition for a graph to have a (g, f)-factor containing a given edge. In [10] Liu presented a simple existence criterion for an $[a, b]$-factor that contains a given edge. Let $p_j(G)$ denote the number of vertices of degree j in graph G.

Lemma 2.1. [4]. *If a graph is not complete, then $t(G) \leq \delta(G)/2$.*

Lemma 2.2. [10]. *Let G be a graph and let $b > a \geq 1$ be integers.*

Then for every edge of G there is an $[a, b]$-factor containing it if and only if for all $S \subseteq V(G)$

$$\sum_{j=0}^{a-1}(a - j)p_j(G - S) \leq b|S| - \epsilon(S) \tag{2.1}$$

where $\epsilon(s) = 2$ if S is not independent; $\epsilon(S) = 1$ if S is independent and there is an edge xy such that $x \in S$, $y \in V(G)\backslash S$ and $d_{G-S}(y) \geq a$; $\epsilon(S) = 0$ otherwise.

Heinrich et al. [6] proved the following result.

Lemma 2.3. [6]. *Let $b > a \geq 1$ be integers. Then the graph G has an $[a, b]$-factor if and only if for all $S \subseteq V(G)$*

$$\sum_{j=0}^{a-1}(a - j)p_j(G - S) \leq b|S|.$$

Lemma 2.4.. *Let G be a graph and let $b > a \geq 1$ be integers. Then for every edge of G there is an $[a, b]$-factor excluding it if and only if $\delta(G) \geq a + 1$ and for all $S \subseteq V(G)$ and $S \neq \emptyset$*

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leq b|S| - \epsilon_1(S) \tag{2.2}$$

where $\epsilon_1(S) = 2$ if there is an edge $e = uv$ in $G - S$ such that $d_{G-S}(u) \leq a$ and $d_{G-S}(v) \leq a$; $\epsilon_1(S) = 1$ if $T = \{t: 1 \leq d_{G-S}(t) \leq a\}$ is independent and there is an edge $e = uv$ in $G - S$ such that $d_{G-S}(u) \leq a$ and $d_{G-S}(v) > a$; $\epsilon_1(S) = 0$ otherwise.

Proof:. For any edge e of G, let $G' = G - e$ be a subgraph of G obtained by deleting edge e. Clearly G has an $[a, b]$-factor excluding e if and only if G' has an $[a, b]$-factor, by Lemma 2.3, if and only if for all $S \subseteq V(G')$

$$\sum_{j=0}^{a-1}(a-j)p_j(G'-S) \leq b|S|. \tag{2.3}$$

If $e = uv$ in $G - S$ such that $d_{G-S}(u) \leq a$ and $d_{G-S}(v) \leq a$, then

$$\sum_{j=0}^{a-1}(a-j)p_j(G'-S) = \sum_{j=0}^{a-1}(a-j)p_j(G-S) + 2.$$

If $e = uv$ in $G - S$ such that $d_{G-S}(u) \leq a$ and $d_{G-S}(v) > a$, then

$$\sum_{j=0}^{a-1}(a-j)p_j(G'-S) = \sum_{j=0}^{a-1}(a-j)p_j(G-S) + 1.$$

Otherwise,

$$\sum_{j=0}^{a-1}(a-j)p_j)G'-S) = \sum_{j=0}^{a-1}(a-j)p_j(G-S).$$

It is easy to see that inequality (2.2) holds if and only if condition (2.3) holds. ∎

The next result follows immediately from Lemma 2.2 and Lemma 2.4.

Lemma 2.5. *Let G be a graph and let $b > a \geq 1$ be integers. If $\delta(G) \geq a + 1$ and for all $S \subseteq V(G)$ and $S \neq \emptyset$*

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leq b|S| - 2,$$

then for every edge of G there is an $[a,b]$-factor containing it and another $[a,b]$-factor excluding it.

Lemma 2.6 [7]. *Let H be a graph and $d_H(t) \le j$ for each $t \in T_j$, $1 \le j \le a-1$, where $T_1, T_2, \ldots, T_{a-1}$ is a partition of $V(H)$ (we allow $T_j = \emptyset$). Then there exists a covering set C of H and an independent set I such that*

$$\sum_{j=1}^{a-1}(a-j)|C \cap T_j| \le (a-1)\sum_{j=1}^{a-1}(a-j)|I \cap T_j|.$$

3. Main results

Now we are ready to prove the main theorem.

Theorem 3.1. *Let G be a graph and a, b be two integers such that $2 \le a \le b$. If $tG) \ge a-1+a/b$ and $a\,|V(G)|$ is even when $a=b$, then for every edge of G there is an $[a,b]$-factor containing it and another $[a,b]$-factor excluding it.*

Proof: By Theorem 1.1, we may assume that $b > a$. By the definition of toughness, when G is complete, we have $\delta(G) = t(G)+1 \ge a-1+a/b+1 = a+a/b$. Since $\delta(G)$ is an integer, $\delta(G) \ge a+1$. By Lemma 2.1 when G is not complete and $a \ge 2$, we have $\delta(G) \ge 2t(G) \ge 2(a-1+a/b) \ge a+a/b$, or, $\delta(G) \ge a+1$. In order to prove the theorem, by Lemma 2.5 it suffices to prove for all $S \subseteq V(G)$ and $S \ne \emptyset$

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S|-2. \tag{3.1}$$

For $S \ne \emptyset$ and $S \subseteq V(G)$, set $T = \{t : t \in V(G)\backslash S \text{ and } 1 \le d_{G-S}(t) \le a-1\}$ and $H = G[T]$. Let $T_j = \{t \in T : d_{G-S}(t) = j\}$, $1 \le j \le a-1$. Since $d_H(t) \le j$ for each $t \in T_j$, by Lemma 2.6 we can find a covering set C and an independent set I of H such that

$$\sum_{j=1}^{a-1}(a-j)c_j \le (a-1)\sum_{j=1}^{a-1}(a-j)i_j \tag{3.2}$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for all $1 \le j \le a-1$. Clearly, we may assume that I is a maximal independent set of H and $C = V(H)\backslash I$. Thus by

(3.2) we have

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) = ap_0(G-S) + \sum_{j=1}^{a-1}(a-j)(c_j+i_j)$$

$$\leq ap_0(G-S) + (a-1)\sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)i_j \tag{3.3}$$

$$= ap_0(G-S) + a\sum_{j=1}^{a-1}(a-j)i_j$$

Let $p_0 := p_0(G-S)$. We consider two cases.

Case 1. $I = \emptyset$.

In this case $T = \emptyset$. When $w(G-S) > 1$, $|S| \geq t(G)w(G-S) \geq (a-1+a/b)p_0 \geq (1+a/b)p_0$. So $|S| \geq p_0 + 1$. When $w(G-S) = 1$, we have $p_0 = 0$ or 1. Clearly $|S| \geq p_0 + 1$. Thus $b|S| - 2 \geq b(p_0+1) - 2 = bp_0 + b - 2$. Therefore $\sum_{j=0}^{a-1}(a-j)p_j(G-S) = ap_0 \leq bp_0 + b - 2 \leq b|S| - 2$. (3.1) holds.

Case 2. $I \neq \emptyset$.

If $|I| = 1$ or $N_{G-S}(x) \cap N_{G-S}(y) = \emptyset$ for any $x, y \in I$ and $x \neq y$, let $X = S \cup (N_{G-S}(I) \backslash \{z\})$ where $z \in N_{G-S}(I)$. Otherwise, let $X = S \cup N_{G-S}(I)$. Then we have

$$|X| \leq |S| + \sum_{j=1}^{a-1} j i_j - 1 \tag{3.4}$$

and

$$w(G-X) \geq \sum_{j=1}^{a-1} i_j + p_0. \tag{3.5}$$

By the definition of toughness, we have

$$|X| \geq t(G)w(G-X) \geq (a-1+a/b)w(G-X) \tag{3.6}$$

if $w(G-X) \geq 1$. Moreover, for every element $x \in T$ $|X| \geq d_{G-S}(x) + |S| - 1 \geq d_G(x) - 1 \geq \delta(G) - 1 \geq t(G)$. So by Lemma 2.1 and the definition of toughness, (3.6) still holds when $w(G-X) = 1$. Thus by (3.4), (3.5) and (3.6) we obtain that

$$|S| + \sum_{j=1}^{a-1} j i_j - 1 \geq (a-1+a/b)\Big(\sum_{j=1}^{a-1} i_j + p_0\Big).$$

Thus

$$b|S| \geq \sum_{j=1}^{a-1}(ab-b+a-bj)i_j + (ab-b+a)p_0 + b.$$

By (3.3) to prove (3.1) it suffices to prove

$$ap_0 + a\sum_{j=1}^{a-1}(a-j)i_j \leq \sum_{j=1}^{a-1}(ab-b+a-bj)i_j + (ab-b+a)p_0 + b - 2,$$

namely,

$$\sum_{j=1}^{a-1}(ab-b+a-bj-a^2+aj)i_j \geq 2-b-b(a-1)p_0. \tag{3.7}$$

Since, for all $1 \leq j \leq a-1$, $ab-b+a-bj-a^2+aj = (b-a)(a-1-j) \geq 0$ and $2-b-b(a-1)p_0 \leq 2-b < 0$, (3.7) holds. ∎

Note that Theorem 3.1 is not true for $a = 1$. For example, let $G_1 = K_2 + (2b-1)K_1$ where $b > 1$ and let G_2 be defined as follows: $V(G_2) = \{v_1, v_2, \ldots, v_{2n}, u, v\}$, $n \geq 2$, where $G_2[\{v_1, \ldots, v_{2n}\}] = K_{2n}$ and $E(G_2) = E(K_{2n}) \cup \{uv_1, uv_2, vv_{2n-1}, vv_{2n}\}$. Then G_1 has no $[1,b]$-factor containing the edge of K_2 and G_2 has no 1-factor containing the edge $e = v_1v_2$ either. But $t(G_1) = 2/(2b-1) > 1/b$ and $t(G_2) = 1$. In fact when $a = 1$, we have the following results which are best possible by the above examples.

Theorem 3.2. *Let G be a graph of even order. If $t(G) > 1$, then for every edge of G there is a 1-factor containing it and another 1-factor excluding it.*

Proof: By Theorem 1.3 for every edge e of G there is a 1-factor containing it. We shall prove that $G' = G - e$ has a 1-factor. Since $t(G) > 1$, by Lemma 2.1 and the definition of toughness, $\delta(G) \geq tG) + 1 > 2$. Let e' be an edge adjacent to e. G has a 1-factor F containing e'. Obviously F excludes e. ∎

Theorem 3.3. *Let G be a graph and $b > 1$. If $\delta(G) \geq 2$ and $t(G) > 2/(2b-1)$, then for every edge of G there is a $[1,b]$-factor containing it and another $[1,b]$-factor excluding it.*

Proof: Let $S \subseteq V(G)$ and $S \neq \emptyset$ and let $p_0 = p_0(G-S)$. If $w(G-S) > 1$, then $p_0 \leq w(G-S) \leq |S|/t(G) < b|S| - |S|/2$. When $|S| \geq 2$, we have $p_0 \leq b|S| - 2$. When $|S| = 1$, we have $p_0 < b|S| - 1/2$, thus, $p_0 \leq b|S| - 1$. If there is an edge $e = uv$ in $G - S$ such that $d_{G-S}(u) = d_{G-S}(v) = 1$, then we have $p_0 \leq w(G-S) - 1 \leq |S|/t(G) - 1 < b|S| - |S|/2 - 1$, or, $p_0 \leq b|S| - 2$. If $w(G-S) = 1$, we have $p_0 = 0$ or 1. Clearly $p_0 \leq b|S| - 2$ or $p_0 \leq b|S| - 1$ according to $p_0 = 0$ or 1. When $S = \emptyset$, clearly, $p_0 \leq b|S|$ by the hypothesis

$\delta(G) \geq 2$. Thus by Lemma 2.2 and Lemma 2.4 we have known that for every edge of G there is a $[1, b]$-factor containing it and another $[1, b]$-factor excluding it. ∎

Theorem 3.1, in the case when $a = b = k$, is best possible. This can be seen from the graph given in [5], which has no k-factor and whose toughness is arbitrarily close to k. Unfortunately, we do not know if Theorem 3.1 is best possible when $a \geq 2$ and $b > a$.

Although we only consider simple graphs in this paper, the theorems in section 3 hold also for the graphs with multiple edges, since a graph with multiple edges has the same toughness as its underlying graph.

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