Some Necessary Existence Conditions for Certain Arrays

D.V. Chopra

Wichita State University
Wichita, Kansas
U.S.A. 67208

Abstract. In this paper we obtain some necessary conditions for the existence of balanced arrays (B-arrays) of strength 4 and with two levels, and we state the usefulness of these conditions in obtaining an upper bound on the number of constraints for these B-arrays.

1. Introduction and Preliminaries

Balanced arrays (B-arrays) have been extensively used in the construction of asymmetrical and symmetrical fractional factorial designs of different resolutions which are also "optimal in some sense". An array T with m rows (constraints), N columns (runs, treatment-combinations), and levels is merely a matrix T $(m \times N)$ whose entries are from a set S with s elements (say; $0, 1, 2, \ldots, s-1$). For the sake of completeness, we recall the definition of a B-array. The symbols $\lambda(\underline{\alpha}; T)$ and $P(\underline{\alpha})$ stand respectively for the frequency of the $(m \times 1)$ vector $\underline{\alpha}$ in T and the vector obtained from $\underline{\alpha}$ by permuting its elements.

Definition 1.1: An array T $(m \times N)$ is said to be a balanced array (B-array) of strength t if in every $(t \times N)$ submatrix T^* of T, the following condition is satisfied:

$$\lambda(\underline{\alpha}; T^*) = \lambda(P(\underline{\alpha}); T^*).$$

In this paper we restrict ourselves to B-arrays with $t = 4$, and $s = 2$ i.e. arrays with two elements (say) 0 and 1. For this case, the above condition is reduced to the following: In every submatrix $T^*(4 \times N)$ of T, every vector $\underline{\alpha}$ of weight $i(0 \leq i \leq 4$; the weight of $\underline{\alpha}$, denoted by $w(\underline{\alpha})$, is the number of 1's in it) appears with the same frequency μ_i. The vector $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is called the index set of the array T, and the array is sometimes denoted by $(m, N, t = 4, s = 2; \underline{\mu}')$. It is quite obvious that N is known once we are given $\underline{\mu}'$, and we have the following result

$$N = \sum_{i=0}^{4} \binom{4}{i} \mu_i.$$

If μ_i is independent of i and $= \mu$ (say), then B-array T is reduced to an orthogonal array (0-array) and in this case $N = \mu 2^t$, μ being the index of the 0-arrays. Many other combinatorial structures arising in design of experiments are related to B-arrays, e.g. the incidence matrix of a BIB design with parameters (b, k, r, ν, λ) corresponds to a B-array $(\nu, b, t = 2; \underline{\mu}' = (\mu_0, \mu_1, \mu_2))$ with $\mu_2 = \lambda, \mu_1 = r - \lambda,$

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and $\mu_0 = b-2r+\lambda$. Thus the problems connected with the existence and construction of B-arrays, for a given $\underline{\mu}'$ are very important in combinatorics and statistical design of experiments. Another equally important problem both in combinatorial mathematics and statistical design of experiments is to construct B-arrays, for a given $\underline{\mu}'$ with the maximum possible number of constraints m. Such a problem for B-arrays and 0-arrays has been studied, among others, by Rao (1947), Bose and Bush (1952), Seiden and Zemach (1966), Rafter and Seiden (1974), Saha and Mukerjee and Kageyama (1988), etc. etc. The necessary existence conditions obtained in this paper should prove useful in both the problems mentioned above. To gain further insight into B-arrays and their importance to combinatorics and statistics, the interested reader may consult the list of references at the end, and further references given therein.

2. Main Results and Applications

Next we mention some results which are easy to establish.

Lemma 2.1. *A B-array T with $m = t = 4$ and index set $\underline{\mu}'$ always exists.*

Remark: It is not difficult to see that the problem of the existence of B-arrays is very much nontrivial for $m > t$.

Lemma 2.2. *Consider a B-array T of strength 4 with index set $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Then T is also of strength $4-i$ $(1 \le i \le 3)$ with index sets $\{\mu_i + \mu_{i+1}; i = 0,1,2,3\}$, $\{\mu_i + 2\mu_{i+1} + \mu_{i+2}; i = 0,1,2\}$, and $\{\mu_i + 3\mu_{i+1} + 3\mu_{i+2} + \mu_{i+3}; i = 0,1\}$ respectively.*

Remark: We use $m_r = m(m-1)(m-2)\dots(m-r+1)$.

Lemma 2.3. *Let x_j $(0 \le j \le m)$ denote the number of columns, with exactly j 1's in each, of a B-array T $(m \times N)$ with $t = 4$ and index set $\underline{\mu}'$. Then the following conditions must be satisfied:*

$$\sum x_j = N = \sum_{i=0}^{4} \binom{4}{i} \mu_i = A_0 \quad \textit{(say)} \tag{2.1}$$

$$\sum j x_j = m_1 \sum_{i=0}^{3} \binom{3}{i} \mu_{i+1} = A_1 \quad \textit{(say)} \tag{2.2}$$

$$\sum j^2 x_j = m_2 \sum_{i=0}^{2} \binom{2}{i} \mu_{i+2} + m_1 \sum_{i=0}^{3} \binom{3}{i} \mu_{i+1} = A_2 \quad \textit{(say)} \tag{2.3}$$

$$\sum j^3 x_j = m_3 \sum_{i=0}^{1} \binom{1}{i} \mu_{i+3} + 3m_2 \sum_{i=1}^{2} \binom{2}{i} \mu_{i+2}$$

$$+ m_1 \sum_{i=0}^{3} \binom{3}{i} \mu_{i+1} = A_2 \quad (say) \tag{2.4}$$

$$\sum j^4 x_j = m_4 \mu_4 + 6m_3 \sum_{i=0}^{1} \mu_{i+3} + 7m_2 \sum_{i=0}^{2} \binom{2}{i} \mu_{i+2}$$

$$+ m_1 \sum_{i=0}^{3} \binom{3}{i} \mu_{i+1} = A_4 \quad (say) \tag{2.5}$$

Next we state, without proofs, the Minkowski's inequality and Hölder inequality for later use.

Minkowski's Inequality. *For $x_i, y_i \geq 0$ $p > 1$, we have*

$$\left(\sum_{i=1}^{n}(x_i + y_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{1}^{n} y_i^p\right)^{\frac{1}{p}}.$$

The equality holds when either $p = 1$ or the sets $\{x\}$ and $\{y\}$ are proportional.

The above inequality can be easily extended.

$$\left(\sum_{i=1}^{n}(x_i + y_i + z_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{1}^{n} y_i^p\right)^{\frac{1}{p}} + \left(\sum_{1}^{n} z_i^p\right)^{\frac{1}{p}}.$$

Hölder Inequality. *If $x_i, y_i \geq 0, p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sum_{i=1}^{n} x_i y_i \leq (\sum x_i^p)^{\frac{1}{p}} (\sum y_i^q)^{\frac{1}{q}} \text{ or } \sum x_i^{\frac{1}{p}} y_i^{\frac{1}{q}} \leq (\sum x_i)^{\frac{1}{p}} (\sum y_i)^{\frac{1}{q}}.$$

Remark on Hölder Inequality: The inequality is reversed if $p < 1$ $(p \neq 0)$ (For $p < 0$, assume that $x_i, y_i > 0$) In each case, the sign of equality holds if and only if the sets $\{x\}$ and $\{y\}$ are proportional.

Theorem 2.1. *Consider the B-array $T\,(m, N, t = 4, s = 2; \underline{\alpha}')$. Then we have*

$$A_1 + A_2 + A_3 \leq \sqrt{A_0 A_2} + \sqrt{A_0 A_4} + \sqrt{A_2 A_4} \tag{2.6}$$

Proof: In Minkowski's Inequality, choose $p = 2$, $x_i = \sqrt{x_j}$, $y_i = j\sqrt{x_j}$, $z_i = j^2\sqrt{x_j}$, then we have

$$\left(\sum x_j(1 + j + +j^2)^2\right)^{\frac{1}{2}} \leq \left(\sum x_j\right)^{\frac{1}{2}} + \left(\sum j^2 x_j\right)^{\frac{1}{2}} + \left(\sum j^4 x_j\right)^{\frac{1}{2}}$$

Squaring both sides of the inequality, and simplifying leads us to the above result.

Theorem 2.2. *Let T be a B-array of size $(m \times N)$ with $t = 4, s = 2$, and index sets $\underline{\mu}'$. Then the following results hold:*

$$(a) \qquad A_3^3 \leq A_1 A_4^2 \qquad (2.7a)$$

$$(b) \qquad A_3^4 \leq A_0 A_4^3 \qquad (2.7b)$$

Proof: Here we use the Hölder Inequality.

(a) Take $p = 3$, then $q = \frac{3}{2}$. Choose $x_i = jx_j$ and $y_i = j^4 y_j$, then we have

$$\sum \sqrt[3]{(jx_j)(j^8 x_j^2)} \leq \sqrt[3]{\sum jx_j} \sqrt[3]{(\sum j^4 x_j)^2},$$

$$\sum j^3 x_j \leq \sqrt[3]{(\sum jx_j)(\sum j^4 x_j)^2}$$

(b) Here we take $p = 4$, and therefore $q = 3/4$. Set $x_i = x_j$ and $y_i = j^4 x_j$ in Hölder inequality. The result follows after some simplification.

Next, we give an example to illustrate the applications of the above results.

Example: Consider a B-array with $\underline{\mu}' = (6, 4, 1, 0, 0)$, and thus $N = 28$. It is not difficult to check that such an array exists with $m \leq 8$, and can be obtained by writing all the distinct (7×1) columns of weight 2 and weight 1. If we use (2.7a) above with different values of m starting with $m = 5$, we observe that the first value of m to contradict (2.7a) is $m = 9$, and consequently all $m > 9$. It can be easily seen that the upper bound $m = 8$ can be achieved by placing a 1 under every vector of weight 1, and a zero under each vector of weight 2.

To use the results in this paper, a computer program can be written for values of $\underline{\mu}'$ and $m \geq 5$. For a given $\underline{\mu}'$, if one of the above conditions is contradicted for (say) $m = m^* + 1$, then an upper bound for m is m^*. Since these are necessary conditions, an array may exist for the given $\underline{\mu}'$ and $m = m^*$ where $m^* \geq 5$. Thus the conditions given here check the existence of some B-arrays for a given $\underline{\mu}'$ and m, and also provide an upper bound on m.

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